Helical phase of chiral nematic liquid crystals as the Bianchi VII$_0$ group manifold

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We show that the optical structure of the helical phase of a chiral nematic is naturally associated with the Bianchi VII$_0$ group manifold, of which we give a full account. The Joets-Ribotta metric governing propagation of the extraordinary rays is invariant under the simply transitive action of the universal cover $\tilde{E}(2)$ of the three-dimensional Euclidean group of two dimensions. Thus extraordinary light rays are geodesics of a left-invariant metric on this Bianchi type VII$_0$ group. We are able to solve, by separation of variables, both the wave equation and the Hamilton-Jacobi equation for this metric. The former reduces to Mathieu's equation, and the latter to the quadrantal pendulum equation. We discuss Maxwell's equations for uniaxial optical materials where the configuration is invariant under a group action and develop a formalism to take advantage of these symmetries. The material is not assumed to be impedance matched, thus going beyond the usual scope of transformation optics. We show that for a chiral nematic in its helical phase Maxwell's equations reduce to a generalized Mathieu equation. Our results may also be relevant to helical phases of some magnetic materials and to light propagation in certain cosmological models.

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I. INTRODUCTION

Recent years have seen a growing interest in the application of the geometrical ideas originally developed for studying Einstein's theory of general relativity to other areas of physics, such as condensed matter physics. The motivation is both the theoretical aim of developing the mathematical tools capable of dealing with as wide a range of physical problems as possible and the desire to construct laboratory analogues of the exotic conditions which general relativity allows, but which are likely ever to remain inaccessible to direct experimental investigation. This in turn may provide a stimulus for further laboratory investigations, possibly resulting in the discovery of new physical effects.

In the present article we pursue this direction by demonstrating how the mathematical formalism of Lie groups, which is of widespread use in general relativity and high-energy physics [1,2], can be harnessed to study the optical properties of symmetrical phases of matter. We develop a formalism allowing the symmetries of an electromagnetic medium to be directly exploited in solving Maxwell's equations. When considering symmetries in classical mechanics, the discussion is simplified by passing to the Lagrangian or Hamiltonian picture. We present a formalism which similarly makes symmetries manifest for Maxwell's equations. In order to motivate and illuminate the development of this formalism, we consider the example of light propagation in chiral nematic liquid crystals. We believe this to be the first application of Lie-group techniques to such a problem. The tools we develop, however, are more widely applicable to media with a continuous symmetry group. They may also be considered a generalization of "transformation optics," extending those ideas to allow for the possibility that the dielectric tensor and magnetic susceptibility differ.

We begin in Sec. II with a brief discussion of the helical ground state for a chiral nematic liquid crystal. In Sec. III we introduce the geometry of the optical metric of Joets and Ribotta [3], describing the propagation of the extraordinary ray in a uniaxial birefringent material. In particular, we study this metric and its geodesics for the helical ground state of a chiral nematic. The resulting metric is invariant under the simply transitive action of a three-dimensional group of isometries which is locally isometric to the Euclidean group $E(2)$ of the plane, which we discuss in detail in Sec. III A. In fact the isometry is the universal cover $\tilde{E}(2)$ and the Lie algebra is of type VII$_0$ in Bianchi's classification. We extend the discussion to VII$_h$ in the Appendix. The identification of the symmetry group for the chiral phase permits a fully geometrical discussion of electromagnetic phenomena, an approach we exploit. The high degree of symmetry allows us to solve the Hamilton-Jacobi and wave equations up to one quadrature in the former and up to solutions of Mathieu's equation in the latter case. This opens up the possibility of a detailed analytic investigation of the type of caustics and optical singularities which should be observable in such systems [4–7]. We then go beyond the geometric optics approximation to consider the full Maxwell equations in Sec. IV. For a uniaxial material whose director field takes on a helical configuration we show how the theory of Lie groups leads to separation of variables for these equations. The resulting equations take the form of a generalized Mathieu equation. This result is similar to others in the literature [8,9], but the derivation is fully motivated by the inherent symmetries of the problem.

Throughout the paper, we make use of the machinery of differential geometry, in particular, tangent vectors, differential forms, and the Lie derivative. References [1] and [2] provide readable accounts of these concepts.

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II. CHIRAL NEMATICS AND THEIR HELICAL GROUND STATE

A nematic liquid crystal has an order parameter given by a *director* or direction field specified by a unit vector $\mathbf{n} = (n_1,n_2,n_3)$, defined up to a sign $\mathbf{n} \sim -\mathbf{n}$, with $\mathbf{n}\cdot\mathbf{n} = 1$. A *chiral* nematic has a built-in twist, specified by a parameter q, which can be realized as a *torsion*, which alters the usual derivative operator acting on a vector by

$$\nabla^q_i n_j = \nabla_i n_j + q\epsilon_{ijk} n_k. \tag{1}$$

The Frank-Oseen free energy functional in the one-constant approximation is equivalent, up to a boundary term, to

$$F[\mathbf{n}] = \frac{1}{2}\int (|\nabla^q \mathbf{n}|^2 - \lambda(\mathbf{n}\cdot\mathbf{n} - 1))d^3x, \tag{2}$$

where we have added a Lagrange multiplier field λ to enforce the constraint that $\mathbf{n}\cdot\mathbf{n} = 1$. The free energy would be minimized if

$$\nabla^q_i n_j = 0. \tag{3}$$

However, as can be seen by taking another ∇^q derivative and skew symmetrizing, this is not possible over an extended region so the system is *frustrated* and must adopt some compromise configuration [10–12].

One such configuration is the helical phase, for which

$$\mathbf{n} = \mathbf{i}\cos(pz) + \mathbf{j}\sin(pz). \tag{4}$$

For more details about the liquid crystals the reader may consult [13]– [18]. For the helical phase,

$$\nabla\cdot\mathbf{n} = 0, \quad \nabla\times\mathbf{n} = -p\mathbf{n}, \quad \nabla^2\mathbf{n} = -p^2\mathbf{n}. \tag{5}$$

This configuration is a stationary point of the free energy, satisfying the second-order Euler-Lagrange equation resulting from extremizing $F[\mathbf{n}]$:

$$-\nabla^2\mathbf{n} + 2q\nabla\times\mathbf{n} = (\lambda - 2q^2)\mathbf{n}, \tag{6}$$

provided we choose

$$\lambda = p^2 - 2pq + 2q^2. \tag{7}$$

Among these solutions of the Euler-Lagrange equations, the one minimizing the free energy density has $p = q$, $\lambda = q^2$. The only nonvanishing components of $\nabla^q_i n_j$ are

$$\nabla^q_2 n_3 = q n_1, \quad \nabla^q_1 n_3 = -q n_2. \tag{8}$$

It follows that the helical ground state is not a solution of the first-order frustrated "Bogomolnyi equation,"

$$\nabla^q_i n_j = 0. \tag{9}$$

Another means of relieving the frustration is the "double twist" structure, given in cylindrical polar coordinates (ρ,z,ϕ) by

$$\mathbf{n} = \mathbf{e}_z\cos q\rho - \mathbf{e}_\phi \sin q\rho. \tag{10}$$

Along the z axis, this configuration has $\nabla^q_i n_j = 0$, so inside a sufficiently small cylinder, the free energy density is in fact *lower* than that for the helical phase. A structure composed of these tubes can fill space, but there will necessarily be defects where the tubes meet [12]. Such a configuration gives the so-called "blue phase." Whether the blue or the helical phase is thermodynamically preferred depends on the energetic cost associated with accommodating the defects of the blue phase.

III. OPTICAL METRICS FOR NEMATICS

If $\mathbf{n}$ is the director, and $\mathbf{t} = \frac{d\mathbf{x}}{ds}$, where $ds^2 = d\mathbf{x}^2$ is the unit tangent vector, then the inverse speed or slowness of an extraordinary ray is given by [3]

$$n = \sqrt{n_o^2(\mathbf{t}.\mathbf{n})^2 + n_e^2(\mathbf{t} - \mathbf{n}(\mathbf{n}\cdot\mathbf{t}))^2}, \tag{11}$$

where n_o is the refractive index of the *ordinary ray* and n_e that of the *extraordinary ray*. Fermat's principle reads

$$\delta\int n ds = 0. \tag{12}$$

Thus the rays are geodesics of the *Joets-Ribotta metric*,

$$ds_o^2 = n_e^2 d\mathbf{x}^2 + (n_o^2 - n_e^2)(\mathbf{n}.d\mathbf{x})^2. \tag{13}$$

Assuming that the refractive indices are constants, we can write down the metric for the helical ground state given above:

$$ds_o^2 = (n_o^2\cos^2(pz) + n_e^2\sin^2(pz))dx^2 + (n_o^2\sin^2(pz) + n_e^2\cos^2(pz))dy^2 + (n_o^2 - n_e^2)\sin(2pz)dxdy + n_e^2 dz^2. \tag{14}$$

As another example, consider a particular case where the director field, $\mathbf{n}$, is in a "hedgehog" configuration (cf. [19–21]) and where, in addition, the refractive indices n_o,n_e vary with position inside the ball $r = |\mathbf{x}| < 1$ according to

$$\mathbf{n} = \frac{\mathbf{x}}{r}, \quad n_e = \frac{1}{\sqrt{1-r^2}}, \quad n_o = \frac{1}{1-r^2}. \tag{15}$$

The resulting Joets-Ribotta metric is

$$ds_o^2 = \frac{d\mathbf{x}^2}{1-r^2} + \frac{(\mathbf{x}.d\mathbf{x})^2}{(1-r^2)^2}, \tag{16}$$

which one recognizes as that of hyperbolic space H^3 in Beltrami coordinates. Remarkably, because hyperbolic space is projectively flat in these coordinates, the light rays are straight lines in this case. One could consider the 3-sphere S^3, by changing the sign in front of r^2. Then one has a model related to Maxwell's fish-eye lens by a coordinate transformation, but whose rays are straight lines. Other examples may be found in [19–21]. Optical properties such as (15) may appear unnatural, but modern metamaterials are increasingly able to mimic such refractive indices, at least within a certain range of frequencies for the electromagnetic field.

A. $E(3)$ and left-invariant metrics

The aim of the present section is to obtain the isometry group of the apparently quite complicated metric (14). An enormous simplification results if we use the formalism of metrics on Lie groups. We start with a brief discussion, tailored to the Euclidean group $E(2)$ of isometries of the plane, consisting of translations and rotations in two dimensions. Those familiar with the construction of left- and right-invariant forms on Lie groups may wish to skip to the summary at the end of this subsection.

We can realize elements of $E(2)$ as matrices as follows. We first fix a number p. To any point (X,Y) in the plane, we associate the column vector

$$(X,Y) \sim \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}. \tag{17}$$

We first note that a general isometry of the plane can be decomposed into a clockwise rotation of angle pz about the origin, followed by a translation of (x,y). We can write down a matrix $M(x,y,z)$ depending on parameters (x,y,z) which performs this operation as follows:

$$\begin{pmatrix} X' \\ Y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos pz & -\sin pz & x \\ \sin pz & \cos pz & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = M(x,y,z) \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}. \tag{18}$$

Corresponding to any three numbers (x,y,z), we have a unique isometry, and conversely each isometry corresponds to a unique (x,y,z), provided that we regard z and $z + 2\pi/p$ as the same.[1] We refer to (x,y,z) as coordinates on group $E(2)$.

Our aim is to construct a set of vector fields which are invariant under an action of group $E(2)$. We will consider for the moment a matrix Lie group $\mathcal{G}$, i.e., a group whose elements are $n \times n$ matrices for some appropriate n where the group action is by matrix multiplication.[2] We note that any element A of $\mathcal{G}$ gives rise to two natural transformations on the group itself. Acting on M, a general element of $\mathcal{G}$, they give

$$\Lambda_A(M) = AM, \quad P_A(M) = MA \tag{19}$$

and are known, respectively, as left and right translation, or the left and right action of A. We note that the left action and the right action commute:

$$\Lambda_A(P_B(M)) = \Lambda_A(MB) = AMB = (\Lambda_A(M))B = P_B(\Lambda_A(M)). \tag{20}$$

Now suppose that A is infinitesimally close to the identity matrix $A = I + \epsilon\delta A$ for an infinitesimal parameter ϵ. We have

$$\Lambda_A(M) = (I + \epsilon\delta A)M = M + \epsilon\delta_\Lambda M, \tag{21}$$

where the infinitesimal generator $\delta_\Lambda M = (\delta A)M$ can be interpreted as a tangent vector to $\mathcal{G}$ at the point M, where we think of $\mathcal{G}$ as a submanifold, i.e., a surface[3] in the space of all $n \times n$ matrices. This is a *right-invariant* vector field, since $\delta_\Lambda(P_B(M)) = P_B(\delta_\Lambda M)$ because of Eq. (20). Following a similar procedure, we find that the vector fields $\delta_P M = M(\delta A)$ are *left-invariant*.

In order to construct the left- and right-invariant vector fields of $E(2)$, we must first find all suitable δA such that $I + \epsilon\delta A$ is an element of $E(2)$. We can do this by using the coordinate representation $M(x,y,z)$ above. Since $M(0,0,0) = I$, we can find the most general δA by Taylor expanding $M(x,y,z)$ for small (x,y,z). We find that the general δA is a linear combination of the matrices:

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

We would like to express the vector fields in terms of the (x,y,z) coordinates. To do this, we define a basis of vector fields for $E(2)$ as follows:

$$\frac{\partial}{\partial x} = \frac{\partial M(x,y,z)}{\partial x}, \quad \frac{\partial}{\partial y} = \frac{\partial M(x,y,z)}{\partial y}, \quad \frac{\partial}{\partial z} = \frac{\partial M(x,y,z)}{\partial z}. \tag{23}$$

We can think of $\partial/\partial x$ either as a concrete matrix tangent to $E(2)$ as a surface in the space of 3×3 matrices, or, more abstractly, as the vector field which generates a shift from (x,y,z) to $(x + \delta x, y, z)$ in the coordinate space. This notation captures the fact that under a change of variables for the coordinate space, $(x,y,z) \to (x',y',z')$, the vector fields transform in the same way as differential operators following the chain rule. We can readily calculate the left-invariant vector field corresponding to M_1:

$$L_1 = MM_1 = \begin{pmatrix} 0 & 0 & \cos pz \\ 0 & 0 & \sin pz \\ 0 & 0 & 0 \end{pmatrix} = \cos pz \frac{\partial}{\partial x} + \sin pz \frac{\partial}{\partial y}. \tag{24}$$

In a similar way, we can find the rest of the left- and right-invariant vector fields, $L_i = MM_i, R_i = M_i M$:

$$\begin{aligned} L_1 &= \cos pz \frac{\partial}{\partial x} + \sin pz \frac{\partial}{\partial y}, \quad R_1 = \frac{\partial}{\partial x}, \\ L_2 &= \cos pz \frac{\partial}{\partial y} - \sin pz \frac{\partial}{\partial x}, \quad R_2 = \frac{\partial}{\partial y}, \\ L_3 &= \frac{1}{p}\frac{\partial}{\partial z}, \quad R_3 = \frac{1}{p}\frac{\partial}{\partial z} + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \end{aligned} \tag{25}$$

Now that we have the left- and right-invariant vector fields, we can construct the left- and right-invariant 1-forms which are dual to them. Taking dx, dy, dz to be the 1-forms dual to $\partial/\partial x, \partial/\partial y, \partial/\partial z$, the left-invariant forms λ^i and right-invariant forms ρ^i are

$$\begin{aligned} \lambda^1 &= \cos(pz)dx + \sin(pz)dy, \quad \rho^1 = dx + pydz, \\ \lambda^2 &= \cos(pz)dy - \sin(pz)dx, \quad \rho^2 = dy - pxdz \\ \lambda^3 &= pdz, \quad \rho^3 = pdz. \end{aligned} \tag{26}$$

[1]For most of the rest of the paper, we in fact work with the covering group $\tilde{E}(2)$ obtained by dropping the identification of the z coordinate. This is a slightly technical point which we do not labor.

[2]There are some further assumptions on the smoothness of the group, but we take these as given.

[3]We do not assume that a surface is necessarily two dimensional, merely that it is of lower dimension than the space in which it lives.

The matrices M_i have a natural commutator algebra, the Lie algebra $\mathfrak{e}(2)$. This determines in a natural way the Lie algebra of the vector fields L_i and R_i,

$$\begin{aligned} [L_1,L_2] &= 0, \quad [R_1,R_2] = 0, \\ [L_3,L_1] &= +L_2, \quad [R_3,R_1] = -R_2, \\ [L_3,L_2] &= -L_1, \quad [R_3,R_2] = +R_1, \end{aligned} \tag{27}$$

and the Maurer-Cartan algebra of the 1-forms,

$$\begin{aligned} d\lambda^1 &= +\lambda^3 \wedge \lambda^2, \quad d\rho^1 = -\rho^3 \wedge \rho^2, \\ d\lambda^2 &= -\lambda^3 \wedge \lambda^1, \quad d\rho^2 = +\rho^3 \wedge \rho^1, \\ d\lambda^3 &= 0, \quad d\rho^3 = 0. \end{aligned} \tag{28}$$

The fact that left and right actions commute (20) is reflected in the fact that $[L_i,R_j] = 0$.

We can check the claimed invariance explicitly. First, we note the matrix identity,

$$\begin{aligned} M(x,y,z)M(\nu,\eta,\zeta) = M(x + \nu\cos pz - \eta \sin pz, y \\ + \eta\cos pz + \nu \sin pz, z + \zeta), \end{aligned} \tag{29}$$

from which we deduce that the element $M(\nu,\eta,\zeta)$ acting by right translation takes (x,y,z) to (x',y',z'), where

$$x' = x + \nu\cos pz - \eta \sin pz, \tag{30}$$
$$y' = y + \eta\cos pz + \nu \sin pz, \tag{31}$$
$$z' = z + \zeta. \tag{32}$$

Making these substitutions into ρ^i, treating ν,η,ζ as constants, we find

$$dx + pydz = dx' + py'dz', \tag{33}$$
$$dy - pxdz = dy' - px'dz', \tag{34}$$
$$pdz = pdz', \tag{35}$$

so that the ρ^i are indeed invariant under right translations. Interchanging the roles of x,y,z and ν,η,ζ in (29), we deduce that the element $M(\nu,\eta,\zeta)$ acting by left translation takes (x,y,z) to (x',y',z'), where

$$x' = x\cos p\zeta - y \sin p\zeta + \nu, \tag{36}$$
$$y' = y\cos p\zeta + x \sin p\zeta + \eta, \tag{37}$$
$$z' = z + \zeta. \tag{38}$$

Substituting into λ^i, again treating ν,η,ζ as constants, we can check that

$$\cos(pz)dx + \sin(pz)dy = \cos(pz')dx' + \sin(pz')dy', \tag{39}$$
$$\cos(pz)dy - \sin(pz)dx = \cos(pz')dy' - \sin(pz')dx', \tag{40}$$
$$pdz = pdz', \tag{41}$$

so the λ^i are invariant under left translations.

Armed with these invariant 1-forms, we are now in a position to construct metrics which are invariant under an action of $E(2)$. For example, the flat metric can be written in terms of the left-invariant 1-forms as

$$ds^2 = p^{-2}(\lambda^3)^2 + (\lambda^1)^2 + (\lambda^2)^2 \tag{42}$$

and is hence manifestly left-invariant. Now for the helical ground state of the nematic liquid crystal,

$$\mathbf{n} \cdot d\mathbf{x} = \lambda^1; \tag{43}$$

thus the *Joets-Ribotta metric* of the helical phase may be written as

$$\begin{aligned} ds_o^2 &= n_e^2(p^{-2}(\lambda^3)^2 + (\lambda^1)^2 + (\lambda^2)^2) + \left(n_o^2 - n_e^2\right)(\lambda^1)^2 \\ &= n_o^2(\lambda^1)^2 + n_e^2(\lambda^2)^2 + \frac{n_e^2}{p^2}(\lambda^3)^2, \end{aligned} \tag{44}$$

which is a left-invariant metric on $\tilde{E}(2)$. In fact, any left-invariant metric may be brought into this form by a global right action of $E(2)$. Cartan's formula for a p-form reads

$$\mathcal{L}_X\, \omega = i_X d\omega + d(i_X\omega), \tag{45}$$

and so the nonvanishing Lie derivatives are

$$\mathcal{L}_{L_3}\, \lambda^1 = -\lambda^2, \tag{46}$$
$$\mathcal{L}_{L_3}\, \lambda^2 = \lambda^1, \tag{47}$$
$$\mathcal{L}_{L_1}\, \lambda^2 = -\lambda^3, \tag{48}$$
$$\mathcal{L}_{L_2}\, \lambda^1 = \lambda^3. \tag{49}$$

Thus while L_3 is an additional symmetry of the flat metric, none of the L_i are symmetries of the Joets-Ribotta metric.

To summarize, then, $(x,y,z) \in R^3$ may be considered as coordinates on $\tilde{E}(2)$, the universal cover of the two-dimensional Euclidean group $E(2)$ with λ^i left-invariant 1-forms and L_i left-invariant vector fields. If we were to identify the coordinate modulo $\frac{2\pi n}{p}$, $n = 1,2,\ldots$, the group would be the n-fold cover of the Euclidean group $E(2)$, which corresponds to $n = 1$. With this identification, the Joets-Ribotta metric of the helical phase is a left-invariant metric. Its symmetry algebra, the Lie algebra $\mathfrak{e}(2)$, is of Bianchi type VII_0. As an aside, this may be obtained from the rotation-group algebra $\mathfrak{so}(3)$ by means of a Wigner-Inönü contraction. If $\tilde{M}_i$ are the generators of $\mathfrak{so}(3)$, one sets $\tilde{M}_1 = \frac{1}{\epsilon}M_1$, $\tilde{M}_2 = \frac{1}{\epsilon}M_2$, $\tilde{M}_3 = M_3$ and takes the limit $\epsilon \to 0$. Under this contraction, the direction field $\mathbf{n} \cdot d\mathbf{x} = L_1$ arises as the image of the Hopf fibration generated by the right action of $\tilde{L}_1$ and the Joets-Ribotta metric as the image of the Berger-Sphere. For the relevance of the Hopf fibration to chiral nematics, see [10–12,16], and [22–25].

B. The ray approximation

In the ray approximation we are looking at geodesics with respect to a left-invariant metric on the universal cover of the Euclidean group $\tilde{E}(2)$. Now the Euclidean group $E(2)$ is the configuration space for a rigid body in two-dimensional Euclidean space $\mathbb{E}^2$. The motion of a rigid body moving in a homogeneous, incompressible, inviscid fluid [26] is known to correspond to geodesic motion with respect to a left-invariant metric on the Euclidean group [27,28]. The present situation corresponds to a cylinder with its axis in a plane [26,29] which may be reduced to the quadrantal pendulum.

To see this in detail note that the Eikonal equation is

$$\frac{p^2}{n_e^2}(L_3 W)^2 + \frac{1}{n_e^2}(L_2 W)^2 + \frac{1}{n_o^2}(L_1)^2 = \omega^2, \tag{50}$$

and it separates. If $W = k_x x + k_y y + G(z)$, then

$$\frac{1}{n_e^2}\left(\frac{dG}{dz}\right)^2 + \frac{1}{n_o^2}(k_x\cos(pz) + k_y\sin(pz))^2 + \frac{1}{n_e^2}(k_x\sin(pz) - k_y\cos(pz))^2 = \omega^2. \tag{51}$$

The Killing vectors R_i give rise to three constants of the motion of the form

$$p_i = g_{\mu\nu}\frac{dx^\mu}{dt}R_i^\mu, \tag{52}$$

of which two, p_1 and p_2, mutually commute.

We may immediately find the equations for rays in a first-order form by making use of the relation

$$\frac{dx^\mu}{dt} = g^{\mu\nu}\frac{\partial W}{\partial x^\mu}. \tag{53}$$

We find

$$n_e^2\dot{x} = \frac{k_x}{2}\left(1 + \frac{n_e^2}{n_o^2}\right) - \frac{k}{2}\left(1 - \frac{n_e^2}{n_o^2}\right)\cos(2pz - \psi), \tag{54}$$

$$n_e^2\dot{y} = \frac{k_y}{2}\left(1 + \frac{n_e^2}{n_o^2}\right) - \frac{k}{2}\left(1 - \frac{n_e^2}{n_o^2}\right)\sin(2pz - \psi), \tag{55}$$

$$n_e^4\dot{z}^2 = \omega^2 n_e^2 - \frac{k^2}{2}\left(1 + \frac{n_e^2}{n_o^2}\right) + \frac{k^2}{2}\left(1 - \frac{n_e^2}{n_o^2}\right)\cos(2pz - \theta). \tag{56}$$

Here $\tan\psi = k_y/k_x$, $k = \sqrt{k_x^2 + k_y^2}$ and $\tan\theta = 2k_x k_y/(k_x^2 - k_y^2)$. Let us first consider (56). Introducing new constants α,β and defining $\zeta = pz - \theta/2$, we find

$$\dot{\zeta}^2 - \frac{p^2}{n_e^4}(\alpha + \beta\cos(2\zeta)) = 0, \tag{57}$$

which is the so-called *quadrantal pendulum* equation [29]. The pendulum has two different types of behavior, depending on the constants α and β. If $\alpha > |\beta|$, then $|\zeta|$ and hence $|z|$ will increase without bound. This corresponds to a pendulum swinging through complete revolutions. If $\alpha < |\beta|$, then ζ will oscillate about $2n\pi$ for some integer n. This corresponds to the standard libratory motion of a pendulum. Thus we find two behaviors for the rays. Either the rays can penetrate in the z direction or they are trapped to move between two planes perpendicular to the z axis. Finally, we can consider the other equations of motion, (54) and (55). We can interpret these as saying that the tangent vector of the ray oscillates around an average direction. For rays which are not bounded in z, the result is a "corkscrew" curve, similar to a helix. The "tightness" of the spiral is determined by how close n_e^2/n_o^2 is to 1. Figure 1 shows some examples.

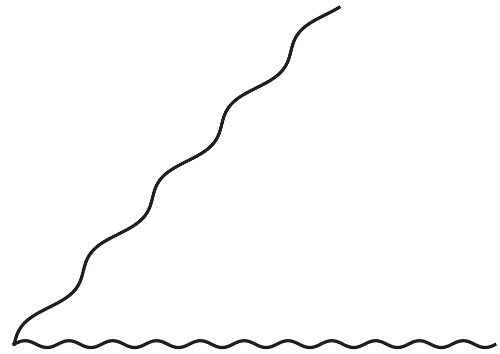

FIG. 1. Two geodesics of the Joets-Ribotta metric (14). Curves are shown projected into the x-z plane. One is unbounded in z, whereas the other is bounded. Not shown is the motion in the y direction which gives both these curves a corkscrew motion.

C. The wave equation

For an approximate description of light propagation beyond the ray approximation, one may use the scalar wave equation. The scalar wave equation captures some of the wave aspects of light, while ignoring the complications relating to polarization which arise in the full Maxwell equations. We may expect the wave equation on the Joets-Ribotta metric to share some features with one of the two polarizations of the full Maxwell equations. It takes the form

$$0 = -\frac{\partial^2\Psi}{\partial t^2} + \frac{p^2}{n_e^2}L_3L_3\Psi + \frac{1}{n_e^2}L_2L_2\Psi + \frac{1}{n_o^2}L_1L_1\Psi. \tag{58}$$

It separates. That is, if $\Psi = e^{i(+k_x x + k_y y - \omega t)}F(z)$, then

$$\frac{1}{n_e^2}\frac{d^2F}{dz^2} + \left(\omega^2 - \frac{1}{n_o^2}(k_x\cos(pz) + k_y\sin(pz))^2 - \frac{1}{n_e^2}(k_x\sin(pz) - k_y\cos(pz))^2\right)F = 0. \tag{59}$$

Recalling α,β,ζ from the previous section, this is of the form

$$\frac{d^2F}{d\zeta^2} + (\alpha + \beta\cos(2\zeta))F = 0, \tag{60}$$

which is Mathieu's equation.

By the Floquet-Bloch theorem, the general solution of (60) is of the form

$$F = c_1 e^{i\mu\zeta}f(\zeta) + c_2 e^{-i\mu\zeta}f(-\zeta), \tag{61}$$

where $f(\zeta) = f(\zeta + 2\pi)$ and μ depends on α and β. Expanding $f(\zeta)$ as a Fourier series, we deduce the Laue-Bragg conditions that an incoming wave with wave vector $\mathbf{k}_{\text{in}}$ incident on some region where propagation is described by (58) is reflected/diffracted with wave vector $\mathbf{k}_{\text{out}}$, where

$$p(\mathbf{k}_{\text{out}} - \mathbf{k}_{\text{in}})_z = m \in Z. \tag{62}$$

We may think of $\mu = \mu(k_x,k_y,\omega)$ as defining a dispersion relation, averaged over the period in the vertical direction. When μ is real we expect propagating waves, whereas when μ has an imaginary component the solutions either decay or grow exponentially in ζ. It can be shown that the marginal cases between propagation and damping occur when $\mu = 0,\pi$ (note that μ is only defined up to multiples of 2π). This defines a set of surfaces in the (k_x,k_y,ω) space which separate out the regions where the wave propagates and where it is damped. To determine these surfaces, we can (for $\mu = 0$, the other case follows similarly) expand F in a Fourier series,

$$F = \sum_{-\infty}^{\infty} c_n e^{in\zeta}, \tag{63}$$

and obtain a three-term recurrence relation,

$$-n^2 c_n + \alpha c_n + 2\beta(c_{n-2} + c_{n+2}) = 0. \tag{64}$$

The condition that this relation admits a nontrivial solution may be related to the vanishing of an infinite determinant, a procedure known as Hill's method (see, e.g., [30]).

IV. MAXWELL'S EQUATIONS

Before we discuss Maxwell's equations for the helical phase of a nematic liquid crystal, we first formulate Maxwell's equations for a general medium in the language of differential forms. This is the most convenient language in which to discuss how to apply the machinery of Lie groups to the problem in hand. We work directly with the fields rather than introducing potentials, as this avoids tackling the issue of gauge invariance. We work on a four-dimensional manifold M, but this restriction is not necessary. We begin by noting that we can define two 2-forms:

$$F = \tfrac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = E_i dx^i \wedge dt + \tfrac{1}{2}\epsilon_{ijk}B^i dx^j \wedge dx^k,$$
$$G = \tfrac{1}{2}G_{\mu\nu}dx^\mu \wedge dx^\nu = H_i dx^i \wedge dt - \tfrac{1}{2}\epsilon_{ijk}D^i dx^j \wedge dx^k.$$

These forms encode the fields contained in the antisymmetric 4-tensors with components

$$F_{i0} = E_i, \quad F_{ij} = \epsilon_{ijk}B^k, \qquad G_{i0} = H_i, \quad G_{ij} = -\epsilon_{ijk}D^k, \tag{65}$$

where E_i and B^i are the electric field and magnetic displacement, D^i is the electric displacement field, and H_i is the magnetic field. The advantage of packaging the fields as 2-forms is that Maxwell's equations become the simple pair of relations

$$dF = 0, \quad dG = J,, \tag{66}$$

with J the current 3-form. Of course, to close this system of equations for a prescribed J, we must specify a relation between F and G, the *constitutive relation*. This is nothing more than the usual relations one requires relating (**E**,**B**) and (**D**,**H**). In the language of forms, we require a map from the space of sections of $\Omega^2(M)$ to itself.[4] In many materials, the constitutive relation is local and linear so may be represented by a section of the bundle $End(\Omega^2(M))$, i.e., a possibly space-dependent linear map C such that

$$G = CF, \quad \text{i.e.,} \quad G_{\mu\nu} = C_{\mu\nu}{}^{\kappa\tau}F_{\kappa\tau}, \tag{67}$$

where C acts pointwise. This tensor C, together with the differentiable structure of the manifold, is the minimal datum required to define Maxwell's equations—it has not thus far been necessary to introduce a metric or other structure to M. In index notation the Maxwell equations take the form

$$\partial_{[\mu}F_{\nu\sigma]} = 0, \quad \partial_{[\mu}\left(C_{\nu\sigma]}{}^{\kappa\tau}F_{\kappa\tau}\right) = 0. \tag{68}$$

We note that defining C as an endomorphism, i.e., with two indices up and two down, ensures that it is not necessary to define a connection in order to take derivatives covariantly.

[4] In higher dimensions, G will be an $n-2$ form, but similar considerations apply.

In order for (66) to define a suitable hyperbolic system of partial differential equations, restrictions are required on C. We assume that C satisfies some such suitable conditions, without specifying what those might be. As a simple example, we may take C to be the Hodge map induced by a Lorentzian metric g, i.e., we take $C = \star_g$. If g is the flat metric, this gives the classical Maxwell equations in the vacuum. If g is not flat, we may interpret the field as an electromagnetic field propagating in a gravitational background. In the case that g is static, we can alternatively interpret the field as propagating through some material with a position-dependent dielectric tensor ε_{ij} and magnetic susceptibility μ_{ij}. This is the basis of *transformation optics* [31–33]. For a material in which Maxwell's equations have a gravitational interpretation it must be the case that $\varepsilon_{ij} = \mu_{ij}$ in suitable units, i.e., the material is *impedance matched* [33]. This need not be the case for a general material. We take C to have the following form:

$$C(dx^i \wedge dt) = -\tfrac{1}{2}\varepsilon_{ij}\epsilon_{jkl}dx^k \wedge dx^l, \qquad C(dx^i \wedge dx^j) = \epsilon_{ijk}(\mu^{-1})_{kl}dx^l \wedge dt. \tag{69}$$

Note that if $C = \star_g$ for some Lorentzian metric, we have $C^2 = -1$, so that $\varepsilon_{ij}(\mu^{-1})_{jk} = \delta_{jk}$, justifying our assertion that materials with a gravitational analog are impedance matched.

The liquid crystals in which we are interested are *uniaxial*, so that at each point, we can assume that the quadrics defined by ϵ and μ are *spheroidal* (i.e., ellipsoids with an axis of symmetry) with a common axis. In other words, there is locally a basis in which the tensors have the form

$$\varepsilon = \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\perp \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_{||} & 0 & 0 \\ 0 & \mu_\perp & 0 \\ 0 & 0 & \mu_\perp \end{pmatrix}. \tag{70}$$

If we assume that the axis of the material lies along **n**, this can be written in a more covariant form as

$$\varepsilon_{ij} = \varepsilon_\perp\delta_{ij} + (\varepsilon_{||} - \varepsilon_\perp)n_i n_j, \quad \mu_{ij} = \mu_\perp\delta_{ij} + (\mu_{||} - \mu_\perp)n_i n_j. \tag{71}$$

Before we discuss the consequences of such a constitutive relation in the case of a nematic liquid crystal in the helical ground state, let us first consider for a moment the geometric optics approximation.

A. Geometric optics

Let us consider the Maxwell equations described above in a geometric optics limit. We consider a field which takes the form

$$F = e^{\frac{iS}{\alpha}}(F_0 + \alpha F_1 + \cdots), \tag{72}$$

where, by assumption, F_i are $O(1)$ as $\alpha \to 0$. We assume that there are no currents or charges, so that Maxwell's equations become $dF = dG = 0$. We also assume that C varies slowly by comparison to the wavelength of the field. Inserting our ansatz and collecting terms in α, we find

$$0 = \frac{i}{\alpha} dS \wedge F_0 + \sum_{k=1}^{\infty} (dF_{k-1} + i dS \wedge F_k)\alpha^{k-1},$$
$$0 = \frac{i}{\alpha} dS \wedge CF_0 + \sum_{k=1}^{\infty} (dCF_{k-1} + i dS \wedge CF_k)\alpha^{k-1}. \tag{73}$$

Let us first consider the $O(\alpha^{-1})$ terms. This is a system which asserts that F_0 is in the kernel of a linear operator which maps from one six-dimensional space to another six-dimensional space. The condition that a nontrivial F_0 exists gives a differential condition on S involving C which we interpret as the eikonal equation. Associated with a solution of the eikonal equation is a 2-form F_0 which gives the polarization of the wave. In general, there will be only one polarization associated with each solution of the eikonal equation. Once we have solved for S and F_0, we can inductively construct F_k by solving the equations

$$0 = dF_{k-1} + i dS \wedge F_k,$$
$$0 = dCF_{k-1} + i dS \wedge CF_k. \tag{74}$$

Presumably, the well posedness of this system is a necessary condition that C be an acceptable constitutive map.

In the case where $C = \star_g$, the eikonal equation can be shown to reduce to

$$dS \wedge \star_g dS = 0, \tag{75}$$

which is the Hamilton-Jacobi equation for geodesics of the metric. In this case, there is a two-dimensional space of possible polarization tensors. They take the form

$$F_0 = dS \wedge f_0, \quad g(dS, f_0) = 0. \tag{76}$$

The Hamilton-Jacobi equation requires that dS be null. Suppose, for example, that at a point, dS is parallel to $dt - dx$, then the space of polarizations at that point is spanned by $dS \wedge dy$ and $dS \wedge dz$.

In the case where C has the uniaxial form introduced above, the eikonal equation reduces to the form

$$\left(-\mu_\perp S_t^2 + \frac{1}{\varepsilon_{||}}\nabla S^2 + \left(\frac{1}{\varepsilon_\perp} - \frac{1}{\varepsilon_{||}}\right)(\mathbf{n}\cdot\nabla S)^2\right) \times \left(-\varepsilon_\perp S_t^2 + \frac{1}{\mu_{||}}\nabla S^2 + \left(\frac{1}{\mu_\perp} - \frac{1}{\mu_{||}}\right)(\mathbf{n}\cdot\nabla S)^2\right) = 0. \tag{77}$$

The medium is thus *birefringent*. We see straight away that the condition on S factors into two separate Hamilton-Jacobi equations associated with the two metrics

$$g_B = -\frac{dt^2}{\mu_\perp} + \varepsilon_{||} d\mathbf{x}^2 + (\varepsilon_\perp - \varepsilon_{||})(\mathbf{n}\cdot d\mathbf{x})^2, \tag{78}$$

$$g_E = -\frac{dt^2}{\varepsilon_\perp} + \mu_{||} d\mathbf{x}^2 + (\mu_\perp - \mu_{||})(\mathbf{n}\cdot d\mathbf{x})^2. \tag{79}$$

These are both of the Joets-Ribotta form we have previously considered. It can be checked that the polarization tensor associated with a solution of the Hamilton-Jacobi equation of g_B has $\epsilon_{ijk}F_{ij}n_k = B_{\mathbf{n}} = 0$, whereas for a solution of the Hamilton-Jacobi equation of g_E, the polarization tensor has $n_i F_{it} = E_{\mathbf{n}} = 0$. Note that we do not require that $\mathbf{n}$ remains constant for this derivation, provided that it varies slowly compared to the wavelength of the light. In the case that $\mathbf{n}$ varies from point to point, the polarization will also change, so that to leading order in α, either the magnetic or the electric field parallel to the director will vanish, depending on which type of ray we consider. Often, one takes $\mu_\perp = \mu_{||}$, in which case, g_E is simply the Minkowski metric and its geodesics are the *ordinary rays*. The rays of the metric g_B are the *extraordinary rays* and g_B is the Joets-Ribotta metric, where we identify $\varepsilon_\perp\mu_\perp = n_o^2$ and $\varepsilon_{||}\mu_\perp = n_e^2$.

If C is of the form (69), but with no uniaxial assumption, then the rays will typically be geodesics of a Finsler geometry.

B. Symmetries

So far, we have recast familiar results into the notation of differential forms. While this is a satisfying exercise, it is not clear that it introduces any benefits beyond putting the equations in a manifestly coordinate invariant form. For our purposes, the great advantage is that this form of the equations permits a concise discussion of the symmetries of the system and allows the machinery Lie groups to be brought to bear. We start by defining a Killing vector K to be a vector which satisfies

$$\mathcal{L}_K C = 0. \tag{80}$$

Recall that C is simply a tensor, so the Lie derivative is defined as a consequence of the differentiable structure of M. Making use of this and Cartan's relation, we deduce that if a 2-form F obeys Maxwell's equations,

$$dF = 0, \quad d(CF) = 0, \tag{81}$$

then so will $\mathcal{L}_K F$, and in particular, the diffeomorphism induced by K will map solutions of the equations into solutions of the equations. An important example occurs when $C = \star_g$ and K is a Killing vector of g.

Suppose that we have a group which acts simply transitively on M by left actions and which preserves the material configuration, as is the case for the $\tilde{E}(2) \times \mathbb{R}_t$ symmetry of the helical ground state of the nematic liquid crystal. Then it must be that C may be written in terms of the left-invariant 1-forms and their duals as

$$C = \tfrac{1}{4} C_{ab}{}^{cd} (\lambda^a \wedge \lambda^b) \otimes (L_c \wedge L_d), \tag{82}$$

where $C_{ab}{}^{cd}$ are some *constant* coefficients. Here, indices run over $0, \ldots, 3$. We can make use of this to write down Maxwell's equations for a nematic liquid crystal in its helical state. We take

$$F = E_i \lambda^i \wedge dt + \tfrac{1}{2}\epsilon_{ijk} B_i \lambda^j \wedge \lambda^k. \tag{83}$$

This choice of basis is very similar to the rotating basis chosen by Peterson, who investigated the electromagnetic field propagating through a nematic liquid crystal in its ground state [8]. In our case, this choice of basis arises naturally from the group structure of underlying symmetries. We assume further that

$$C(\lambda^i \wedge dt) = -\tfrac{1}{2}\varepsilon_{ij}\epsilon_{jkl}\lambda^k \wedge \lambda^l,$$
$$C(\lambda^i \wedge \lambda^j) = \epsilon_{ijk}(\mu^{-1})_{kl}\lambda^l \wedge dt, \tag{84}$$

where ε, μ have the uniaxial form we previously assumed, (70). Maxwell's equations for the electric and magnetic fields take the form

$$L_i(B_i)=0,\ \ \epsilon_{ijk}L_j(E_k)+\frac{\partial B_i}{\partial t}-P_{ij}E_j=0,\ \ \varepsilon_{ij}L_i(E_j)=\rho,$$
$$(\mu^{-1})_{kl}\epsilon_{ijk}L_j(B_l)-\varepsilon_{ij}\frac{\partial E_j}{\partial t}-P_{ij}(\mu^{-1})_{jk}B_k=J_i. \tag{85}$$

The matrix P_{ij} has nonzero components

$$P_{11} = P_{22} = 1. \tag{86}$$

These equations can be separated with the ansatz

$$E_i = e^{i(k_x x + k_y y - \omega t)} f_i(z), \quad B_i = e^{i(k_x x + k_y y - \omega t)} g_i(z). \tag{87}$$

The components $f_3(z)$ and $g_3(z)$ are given by a linear combination of other components, so that the Maxwell equations reduce to a system of differential equations of the form

$$F'(z) + (\alpha + \beta_1 e^{2ipz} + \beta_2 e^{-2ipz})F(z) = 0. \tag{88}$$

Here $F(z) = (f_1(z), f_2(z), g_1(z), g_2(z))^t$ is a 4-vector and α, β_i are 4×4 matrices, given by

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \frac{-i|\kappa|^2}{2\varepsilon_\perp\mu_{||}\omega} + i\omega \\ -1 & 0 & \frac{i|\kappa|^2}{2\varepsilon_\perp\mu_\perp\omega} - i\omega & 0 \\ 0 & \frac{i|\kappa|^2}{2\omega} - i\varepsilon_{||}\mu_\perp\omega & 0 & \frac{\mu_\perp}{\mu_{||}} \\ -\frac{i\mu_{||}|\kappa|^2}{2\mu_\perp\omega} + i\varepsilon_{||}\mu_{||}\omega & 0 & -\frac{\mu_{||}}{\mu_\perp} & 0 \end{pmatrix} \tag{89}$$

and

$$\beta_1 = -\overline{\beta_2} = \frac{\overline{\kappa}^2}{4\omega}\begin{pmatrix} 0 & 0 & \frac{-1}{\varepsilon_\perp\mu_\perp} & \frac{-i}{\varepsilon_\perp\mu_{||}} \\ 0 & 0 & \frac{-i}{\varepsilon_\perp\mu_\perp} & \frac{1}{\varepsilon_\perp\mu_{||}} \\ 1 & i & 0 & 0 \\ i\frac{\mu_{||}}{\mu_\perp} & -\frac{\mu_{||}}{\mu_\perp} & 0 & 0 \end{pmatrix}, \tag{90}$$

where we have introduced $\kappa = k_x + ik_y$. We see that the Euclidean symmetry of the original problem is still manifest since a rotation in the x–y plane sends $\kappa \to e^{i\theta}\kappa$, which is canceled by a suitable shift in the z coordinate. We may view (88) as a generalized Mathieu equation. Mathieu's equation itself may be written in this form with 2×2 matrices. By Floquet's theorem, the general solution of (88) will take the form

$$F(z) = e^{i\mu_1 z}h_1(z) + e^{i\mu_2 z}h_2(z) + e^{i\mu_3 z}h_3(z) + e^{i\mu_4 z}h_4(z), \tag{91}$$

where $h_i(z) = h_i(z + \pi/p)$ are 4-vectors. Making use of discrete symmetries of the equations, one may show that if μ is a Floquet exponent, then so is $-\mu$ and $\overline{\mu}$, implying relations among the μ_i. This equation may be studied using the infinite determinant techniques of Hill, an approach similar to that of [9] but that takes us beyond the scope of the current paper. We hope to address this issue in future work. Since we have retained the independence of the magnetic susceptibility and the permittivity, this analysis applies equally well to magnetic materials with helical phases [34].

V. CONCLUSION

We have shown that certain properties of the chiral phase of a nematic liquid crystal are intimately tied to the symmetries it possesses. We have shown that the Joets-Ribotta metric, which describes the propagation of extraordinary rays, is a left-invariant metric on $\tilde{E}(2)$ and we have shown how the underlying symmetry group can be practically used to understand properties of waves in such a medium.

We have separated the Hamilton-Jacobi equation and the wave equation for this metric. The wave equation can be reduced to Mathieu's equation, and the Hamilton-Jacobi equation to the quadrantal pendulum equation. We have also seen how Maxwell's equations for a general uniaxial material whose director field lies in a helical configuration can be reduced to coupled ordinary differential equations generalizing Mathieu's equation via a novel application of the theory of Lie groups. This new formalism is applicable to the macroscopic Maxwell equations whenever the medium has a continuous symmetry group. The approach taken generalizes transformation optics to permit non-impedance-matched media.

As we have seen even in this simple example, the extraordinary light rays propagating through a liquid crystal explore a much richer geometry than the usual flat geometry of light rays in the vacuum. This opens up the possibility of constructing analogues for the propagation of light in a gravitational field. In this case the light rays in the liquid crystal may be mapped onto light rays propagating in a Bianchi VII_0 cosmology [35] whose spatial sections have a fixed geometry, but one may imagine more ambitious possibilities.

APPENDIX: GENERALIZATION TO BIANCHI TYPE VII$_h$

It is interesting to ask whether the setup above generalizes to the Bianchi type VII$_h$ group. For this section we set $p = 1$, in order not to clutter up the formulas.

We now define left-invariant 1-forms and dual vector fields by

$$\lambda^3 = dz, \quad L_3 = \frac{\partial}{\partial z}, \tag{A1}$$

$$\lambda^1 = e^{hz}(\cos z dx + \sin z dy), \quad L_1 = e^{-hz}\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right), \tag{A2}$$

$$\lambda^2 = e^{hz}(\cos z dy - \sin z dx), \quad L_2 = e^{-hz}\left(\cos z\frac{\partial}{\partial y} - \sin z\right)\frac{\partial}{\partial x}. \tag{A3}$$

The right-invariant 1-forms and vectors fields are

$$\rho^3 = dz, \; R_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - h\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right), \tag{A4}$$

$$\rho^1 = dx + (1+h)ydz, \quad R_1 = \frac{\partial}{\partial x}, \tag{A5}$$

$$\rho^2 = dy - (1-h)xdz, \quad R_2 = \frac{\partial}{\partial y}. \tag{A6}$$

The metric[5]

$$n_e^2\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) = n_e^2(dz^2 + e^{2hz}(dx^2 + dy^2)) \tag{A7}$$

[5]In what follows n_e and n_o will be taken to be constant, that is, position independent.

is in fact that of hyperbolic three space. in the upper half space or Poincaré patch space model. Setting

$$e^{hz} = \frac{1}{Z}, \quad x = \frac{X}{h}, \quad y = \frac{Y}{h} \tag{A8}$$

it becomes

$$\frac{n_e^2}{h^2Z^2}(dZ^2 + dX^2 + dY^2), \tag{A9}$$

and we see that optically we can think of a vertically stratified isotropic medium with Cartesian coordinates (X,Y,Z) and refractive index

$$\frac{n_e}{hZ}. \tag{A10}$$

Rays are now circles orthogonal to the plane $Z = 0$.

The metric

$$ds_o^2 = n_e^2\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) + \left(n_o^2 - n_e^2\right)\lambda_1^2 \tag{A11}$$

may be thought of as describing a vertical stratified anisotropic medium with extraordinary and ordinary refractive indices varying with height Z in the same way, i.e., as

$$\frac{n_e}{hZ} \quad \text{and} \quad \frac{n_o}{hZ}, \quad \text{respectively.} \tag{A12}$$

Such a variation might be due to temperature variation within the material, for example. As before, the wave equation separates but $F(z)$ now satisfies

$$\frac{d^2F}{dz^2} + 2h\frac{dF}{dz} + \left(\omega^2 n_e^2 - \frac{e^{-2hz}}{2}\left(1 + \frac{n_e^2}{n_o^2}\right)\left(k_x^2 + k_y^2\right) + \frac{e^{-2hz}}{2}\left(k_x^2 + k_y^2\right)\left(1 - \frac{n_e^2}{n_o^2}\right)\cos(2z - \theta)\right)F = 0. \tag{A13}$$

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