Turbulent viscosity variability in self-preserving far wake with zero net momentum

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The profile of the self-preserving far wake with zero net momentum depends on the effective turbulent viscosity coefficient. The current model is based on the assumption of uniform viscosity in the wake cross section. It predicts the self-similar shape of the wake where the width W depends on the distance z from the body as $W \propto z^{1/5}$ for the axisymmetric case (or $z^{1/4}$ for the plane case). The observed wake width, however, demonstrates the dependence $W \propto z^{\alpha}$ (where $\alpha \leqslant 1/5$). We generalize the model of a self-preserving far wake for the case of the turbulent viscosity coefficient depending on the radius. Additional integrals of motion allow a new family of self-similar profiles with $\alpha \leqslant 1/5$.

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The existence of long-lived self-similar turbulent wakes behind moving (or streamlined) bodies at high Reynolds numbers $\gtrsim 10^5$ has been well known since the 1950s [1]. Such wakes are produced behind most artificial self-propelled bodies moving in the atmosphere and ocean (planes, ships, submarines). This feature has been used for, e.g., finding ships according to their wakes when another connection was lost. It was found that the wake width of self-propelled bodies has a power-law behavior [2], where the width of wake W depends on the distance z from the body as $W \propto z^{\alpha}$. This dependence has been studied in laboratory experiments [3,4], where the velocity profiles behind the body also was measured. Field observations by radar measurements and by measurements of surface tension [5–8] have also revealed the power-law self-similar broadening of the wake with the distance. If there is a nonzero net momentum flux across the wake, the conservation of momentum uniquely determines the power index $\alpha = 1/3$ (see, e.g., Ref. [9]). In the case of a self-propelled body the momentum flux is zero and the wake shape is determined by the properties of the turbulence, which are conveniently described in terms of the effective viscosity. So far the only model developed is the one in which the turbulent viscosity coefficient is uniform across the wake, which results in $\alpha = 1/5$ [9]. Yet a number of studies where accurate measurements of the wake spatial expansion rate have been performed have shown that typically $\alpha < 1/5$ [5,7,10]. Radar measurements [6,7] have revealed $\alpha = 1/5$ is an approximation that may be valid for a part of the pattern but does not describe properly the far ship wake. The uniform viscosity approximation implies quite specific profiles of the velocity behind the body. Laboratory studies of self-propelled bodies revealed velocity profiles that are inconsistent with this assumption [3,4]. Thus, it is desirable to analyze the wake behavior in a more general way, without invoking the uniform viscosity approximation.

In the present paper, we investigate the properties of a self-similar turbulent far wake behind a self-propelled body, like a ship. The wake is a mean velocity field that is produced as a result of the turbulence generated by the moving ship. The mean velocity field can be measured directly only inside the fluid, under the surface, which has been done so far only in laboratory studies [3,4]. Radar measurements provide information about the turbulent motions at the water surface and, thus, should be treated as indirect observations of the wake features [7,8]. Since the turbulent and the mean velocity distributions are closely interrelated, radar measurement do provide valuable information about the wake structure. In our approach we treat the turbulence features as given and derive the mean velocity profile as well as the wake expansion rate. Generation of the turbulence by the engine or surface waves [7,8] is not important for our analysis and is beyond the scope of the present paper.

In what follows we study the effect of the nonuniform viscosity and derive a new class of self-similar wake (profiles) solutions. We consider the axisymmetric wake with direct application to ships, although the results can be generalized on a plane wake (behind a long cylinder, for example).

The depth of the turbulence source is negligible relative to the width of the ship wake at noticeable distances where the wake can be expected to be self-similar. The mean flow is described by the velocities $U_z(z,r)$ and $U_r(z,r)$ for $0 \leqslant \varphi \leqslant \pi$. There is no dependence on φ within this region. Since $U_\varphi \equiv 0$ the normal component of the mean velocity at the water-air boundary vanishes as required. Let u_i be the components of the turbulent velocity and $q_{ij} = \langle u_i u_j \rangle$ be the ensemble averaged moments. The assumption of axisymmetry means that $q_{ij} = q_{ij}(z,r)$. It is worth noting that the presence of the water-air boundary does not impose additional constraints on the turbulent moments.

The equation for the mean flow takes the form

$$\left(U_r \frac{\partial}{\partial r} + U_z \frac{\partial}{\partial z}\right) U_z = -\frac{1}{r}\frac{\partial}{\partial r} r q_{rz} - \frac{\partial}{\partial z} q_{zz} - \frac{1}{\rho}\frac{\partial}{\partial z} P, \quad (1)$$

where $q_{rz} = \langle u_r u_z \rangle$, $q_{zz} = \langle u_z^2 \rangle$, and P is the mean pressure. Observations of the turbulent wakes show that the turbulent flow extends to large distances but only weakly expands. This means [9] that $|U_r| \ll U_z$. Let L be the typical spatial scale of inhomogeneity in the direction of the wake (z direction) while l be the typical spatial scale of inhomogeneity in the transverse direction (r), so that $\partial/\partial z \sim 1/L$ and $\partial/\partial r \sim 1/l$. As noted above, we are interested in the case where $l \ll L$. Let U_0 be the mean flow velocity in the z direction and $V = U_z - U_0$

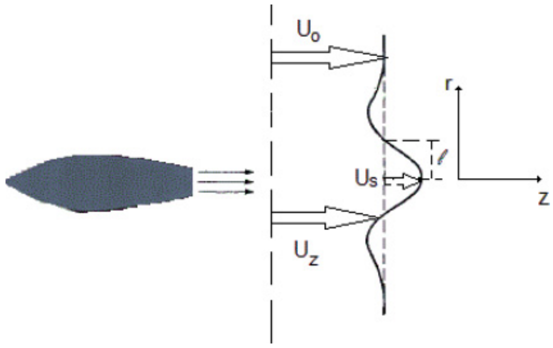

FIG. 1. The far wake scheme.

(see Fig. 1). We expect that $V \ll U_0$ far from the ship, so that in the far wake $U_z = O(U_0)$. With all this taken into account,

$$U_0 \frac{\partial U_z}{\partial z} = O(U_0 V/L). \tag{2}$$

For the turbulent velocities we cannot expect that any of the components dominate, so that q_{rr}, q_{zz}, and q_{rz} are expected to be of the same order.

The continuity for the slow varying velocity allows to estimate

$$\frac{U_r}{l} \sim \frac{\partial U_z}{\partial z} \sim \frac{V}{L}, \tag{3}$$

so that

$$U_r = O\left(\frac{Vl}{L}\right) \ll V \ll U_0, \tag{4}$$

in accordance with the initial assumption, and

$$U_r \frac{\partial}{\partial r} U_z = O(V^2/L). \tag{5}$$

The amplitude of the turbulent motion, on the other hand, can be comparable with the amplitude of the cross-stream variations of U_r, that is, all $q_{ij} = O(V^2)$. As a result, the pressure gradient in the r direction

$$\frac{1}{\rho}\frac{\partial}{\partial r} P = O(V^2/l), \tag{6}$$

while

$$\frac{1}{\rho}\frac{\partial}{\partial z} P = O(V^2/L) \tag{7}$$

Similarly,

$$\frac{\partial}{\partial z} q_{zz} = O(V^2/L). \tag{8}$$

With all of the above taken into account, the equation for the momentum transfer in the direction of the flow can be truncated to the following:

$$U_0 \frac{\partial}{\partial z} V = -\frac{1}{r}\frac{\partial}{\partial r}(r q_{rz}). \tag{9}$$

We shall seek solutions of the type

$$V = U_s f(\xi), \quad U_s = A z^{\beta}, \quad \xi = r/z^{\alpha}, \tag{10}$$

$$q_{rz} = -U_s^2 g(\xi). \tag{11}$$

Substitution in (1) gives $\beta = \alpha - 1$ and

$$s(\alpha - 1) f - s\alpha \xi f' = \frac{1}{\xi}\frac{d}{d\xi}(\xi g), \tag{12}$$

where $s = (U_0/A)$. Functions f and g are still to be found.

It is widely accepted that one express the turbulent stress in terms of the turbulent viscosity, which is defined by the relation

$$-q_{rz} = \nu_t \frac{\partial V}{\partial r}; \tag{13}$$

then the equation of motion takes the form

$$U_0 \frac{\partial V}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \nu_t \frac{\partial V}{\partial r}\right), \tag{14}$$

where $\nu_t = B z^{2\alpha - 1}(g/f')$, and the prime denotes ξ derivative.

The integral $I_0 = \int_0^\infty V r\, dr$ is proportional to the total momentum across the wake and is conserved. Indeed, $(d/dz)[\int_0^\infty U_0 V r\, dr] = -\int_0^\infty [\partial(r q_{rz})/\partial z]\, dr = 0$. If $I_0 \neq 0$, the conservation of momentum, together with the assumption that $V \propto z f(\xi)$, immediately gives $I_0 \propto z^{3\alpha - 1} \int_0^\infty f(\xi) \xi\, d\xi = \text{const}$, and therefore we obtain a well-known value $\alpha = 1/3$.

For the self-propelled body far wake, when $uL/U_0 l \sim 1$, one has $I_0 = 0$ so that the momentum conservation does not provide additional constraint, because the source and dissipation are in balance [11]. Let us consider a higher moment $I_m = \int_0^\infty U_0 V r^{m+1}\, dr$. Assuming convergence, one has

$$\begin{aligned} \frac{d}{dz} I_m &= -\int_0^\infty r^m \frac{\partial (r q_{rz})}{\partial r}\, dr \\ &= \int_0^\infty r^m \frac{\partial}{\partial r}\left(r \nu_t \frac{\partial V}{\partial r}\right) dr \\ &= m \int_0^\infty V \frac{\partial}{\partial r}\left(r^m \nu_t\right) dr, \end{aligned} \tag{15}$$

where we have taken into account that $V \to 0$, $\partial V/\partial r \to 0$ as $r \to \infty$, and q_{rz} is finite. If $[\partial(r^m \nu_t)/\partial r] \propto r$, then $\int_0^\infty V[\partial(r^m \nu_t)/\partial r]\, dr \propto I_0 = 0$ and I_m is conserved. The viscosity coefficient was assumed constant in Ref. [9]. However, it may depend on coordinates. In general, in the axisymmetric self-similar case it should have the form

$$\nu_t = K(z) \xi^{2-m}, \tag{16}$$

where $K(z)$ is some function on z. Such behavior of the turbulent viscosity was reported in Ref. [12]. Conservation of I_m gives for the self-similar solution $\alpha = 1/(m+3)$, $K(z) = B z^{2\alpha - 1}$, and $g = \xi^{2-m} f'$, where $B = \text{const}$. It should be noted that similar results can be obtained for the plane flow. In this case we obtain $\alpha = \frac{1}{m+2}$.

It is usually assumed that ν_t does not depend on ξ, which means $m = 2$ in (16) and $\alpha = 1/5$. However, this assumption is not well justified. For $m > 2$ the turbulent viscosity ν_t diverges as $r \to 0$. However, once $\partial V/\partial r$ vanishes sufficiently rapidly so that q_{rz} remain well defined, this divergence is formal only.

Substituting now the expression for g into (12) one arrives at the following equation:

$$f'' + \left[\frac{3-m}{\xi} + \frac{s}{m+3}\xi^{m-1}\right] f' + \frac{s(m+2)}{m+3}\xi^{m-2} f = 0 \tag{17}$$

with the constraint $\int_0^\infty f\xi\, d\xi = 0$. For arbitrary $m \geqslant 2$ and taking into account the self-similarity and boundary conditions, we obtain the general solution of Eq. (17):

$$f = C_1 F_1\left(1 + \frac{2}{m}, \frac{2}{m}, \frac{sm\xi^m}{3m^2 + m^3}\right), \tag{18}$$

where C is a constant and ${}_1F_1(a,b,z)$ is a confluent hypergeometric function. For integer values of m the solution of Eq. (17) takes the form

$$f = (a/s + \xi^m)\exp(-s\xi^m/b), \tag{19}$$

where a and b are some constants. For example, the observed $\alpha = 1/7$ would correspond to $m = 4$. In this case the equation takes the form

$$f'' + \left[-\frac{1}{\xi} + \frac{s}{7}\xi^3\right] f' + \frac{6s}{7}\xi^2 f = 0 \tag{20}$$

with the solution

$$f(\xi) = (C_1 + C_2 a\xi^4)\exp(-a\xi^4), \quad a = s/28. \tag{21}$$

The solution of Eq. (17) for certain values of m (and $s = 1$) is shown in Fig. 2.

As one can see, the velocity profile is significantly sensitive to the behavior of turbulent viscosity (Reynolds stresses). On the other hand, as follows from the definition (10) of parameter s, it specifies the initial amplitude and width (l) of the velocity profile; i.e., it is associated with the properties of the engine. The influence of the parameter s on the velocity profile is shown Fig. 3 for $m = 4$. As one can see, the overall shape of the curve (number of extrema and zeros) does not change, while the initial amplitude and width decrease with increasing s.

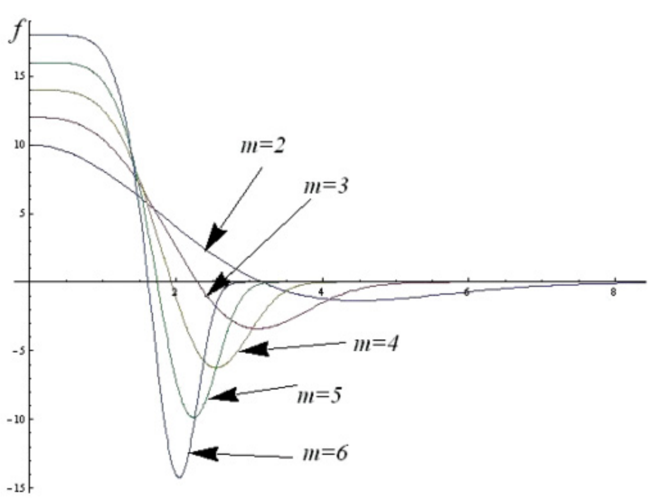

FIG. 2. Wake profiles for certain parameters m. $m = 2$ corresponds to $\nu_t = \text{const}$.

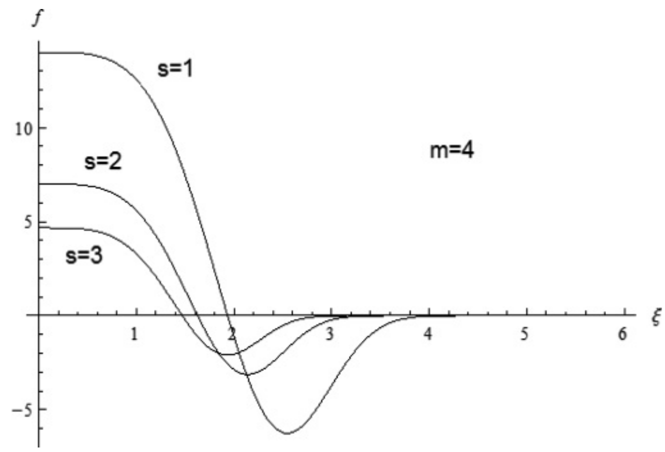

FIG. 3. Wake profiles for certain parameters s and $m = 4$.

Comparison of the obtained profiles with the experimental data [4] is shown in Fig. 4. The experimental data presented in the Fig. 4 were normalized on the amplitude of the profile at the distance $z/D = 46$. Only the data points corresponding to the far wake are shown. The experimentally measured profiles behind a self-propelled body are well fitted by the analytically derived profiles of the wake with $s = 2$ and $m = 3$ at the distances $z/D = 20 - 30$ and $m = 2$ at $z/D = 46$.

The width of the core of the flow (central part; see Fig. 2) decreases with increasing m. Simultaneously, the effective width of the wake also decreases. It is worth noting that the measured velocity profile is expanding slower (smaller α or larger m) near the body while starting to expand faster farther from it. In the far wake the profile approaches that of the uniform viscosity one, $m = 2$. On the other hand, the ship wake observations [5,7] show that at relatively short distances, the wake has a shape corresponding to uniform viscosity, while the far wake shape indicates that the viscosity depends on the radius ($\alpha < 1/5$). It is not clear what is the reason for the different behavior of the wakes in laboratory experiments and in the sea. The difference may arise from the different generation of turbulence and mixing in the wake. Under natural

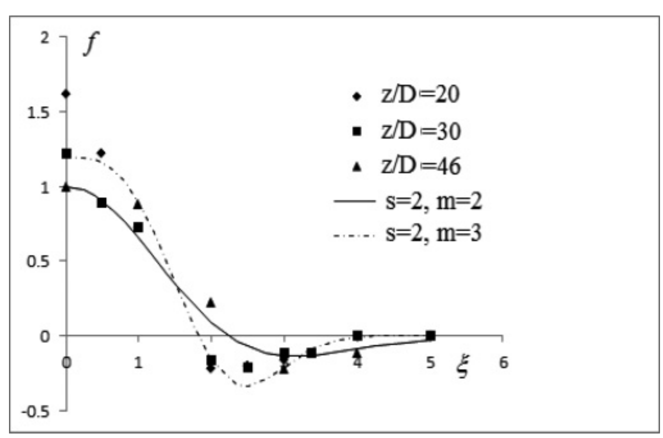

FIG. 4. Comparison of model and experimental data [4]. Experimental data normalized on the amplitude of the profile at distance $z/D = 46$.

conditions (e.g., in the sea) turbulence is created near the ship by its propeller. Mixing is efficient, and the turbulence becomes homogeneous at relatively small distances. In the laboratory experiments there is no propeller and a wake is formed in the flow behind bluff body (with or without rotation), which leads to the formation of an M-shaped wake (see, for example, Ref. [13] and references therein). Mixing is slower, and, respectively, turbulent becomes homogeneous father away from the body. Similar behavior would be observed behind a wind-propelled yacht if the velocity could be comparable to that of the laboratory experiments.

It should be noted that the condition $U_\varphi = 0$ can be omitted for the flooded wake, which does not affect the results obtained in this work.

To summarize, we have derived a family of self-similar solutions for the axisymmetric turbulent wake by relaxing the assumption of the uniform turbulent viscosity for a self-propelled body wake and considering a power-law dependence $\nu_T \propto r^{2-m}$. The latter allowed us to construct a new integral of motion, $I_m = \int_0^\infty r^m V r\, dr$, instead of the vanishing momentum flux $I_0 = \int_0^\infty V r\, dr = 0$. Each of these integrals corresponds to a flow profile width $W(z) \propto z^{1/(m+3)}$. The velocity profiles are derived for different m. These profiles are compared directly with existing measurements and may be further verified in future experiments. The profiles and exponents have certain implications for the statistical properties of turbulence and may be used for studies on drag reduction (see, for example, Refs. [14,15]).

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