

# Nonlinear ion modes in a dense plasma with strongly coupled ions and degenerate electron fluids

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The properties of solitary and shock structures associated with nonlinear ion modes in a dense plasma with strongly coupled nondegenerate ions and degenerate electron fluids are presented. For this purpose, we have used the viscoelastic fluid model for the ions, the inertialess electron momentum equation with weakly and ultrarelativistic pressure laws for the degenerate electron fluids, and Poisson's equation to derive the Burgers and Kortweg–de Vries equations. Possible stationary solutions of the latter are the shock and solitary structures, respectively. It is found that the speed, amplitude, and width of the shock and solitary waves critically depend on the strong coupling between ions and electron degeneracy effects. The relevance of our investigation to the role of localized excitations in dense astrophysical objects is briefly discussed.

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## I. INTRODUCTION

In the past, a number of authors have been concerned with the study of matter under extreme conditions [1–6], which occur in compact astrophysical objects and planetary systems. Examples of the latter are white and brown dwarf stars [7–9], as well as massive Jupiter [10] (serving as the benchmark for giant planets) and super-Earth terrestrial planets around other stars [11]. White dwarf stars have low luminosity and high surface emissivity, with masses typically less than  $1M_{\odot}$  and radii typically less than  $10^{-2}R_{\odot}$  (where  $M_{\odot}$  and  $R_{\odot}$  are the solar mass and radius, respectively). The average bulk density of a white dwarf star is typically  $\sim 10^{30} \text{ cm}^{-3}$ . Astrophysical aspects of high density have been recently discussed by Fortov [12].

In high-density plasmas (in which the particle number density is of the order mentioned above), electron fluids are degenerate, and nondegenerate ions are strongly coupled because the ion Coulomb coupling parameter  $\Gamma_i = Z_i e^2 / a_i k_B T_i$  is larger than unity. Here,  $Z_i$  is the ion charge state,  $e$  is the magnitude of the electron charge,  $a_i = (3/4\pi n_i)^{1/3}$  is the inter-ion spacing,  $n_i$  is the ion number density,  $k_B$  is the Boltzmann constant, and  $T_i$  is the ion temperature. Because electrons at high densities are degenerate, one must use Fermi-Dirac statistics to deduce the electron pressure. In his classical papers, Chandrasekhar [1,3] presented a general expression for the relativistic electron degeneracy pressure  $P_e = (\pi m_e c^5 / 3\hbar^3) [\alpha(2\alpha^2 - 3)(\alpha^2 + 1)^{1/2} + 3 \sinh^{-1} \alpha]$ , where  $m_e$  is the electron rest mass,  $c$  is the speed of light in vacuum,  $\hbar$  is the Planck constant divided by  $2\pi$ , and  $\alpha = p_e / m_e c$ , with

$p_e = (3\hbar^2 n_e / 8\pi)^{1/3}$  being the momentum of an electron on the Fermi surface. One can obtain explicitly expressions for  $P_e$  in the weakly and ultrarelativistic limits, which are characterized by  $\alpha \ll 1$  and  $\alpha \gg 1$ , respectively. We have

$$P_e = K n_e^\gamma, \quad (1)$$

where

$$\gamma = \frac{5}{3}, \quad K = \frac{3}{5} \left( \frac{\pi}{3} \right)^{\frac{1}{3}} \frac{\pi \hbar^2}{m_e} \simeq \frac{3}{5} L_c \hbar c \quad (2)$$

for the weakly relativistic degenerate electron fluids (where  $L_c = \pi \hbar / m_e c = 1.2 \times 10^{-10} \text{ cm}$ ) and

$$\gamma = \frac{4}{3}, \quad K = \frac{3}{4} \left( \frac{\pi^2}{9} \right)^{\frac{1}{3}} \hbar c \simeq \frac{3}{4} \hbar c \quad (3)$$

for the ultrarelativistic degenerate electron fluids. Here,  $n_e$  is the electron number density.

Recently, several authors [13–22] have used the pressure laws (1) and (2) to investigate the linear and nonlinear properties of electrostatic and electromagnetic waves, by using the nonrelativistic quantum hydrodynamic [13] and quantum magnetohydrodynamic [15] models and by assuming either immobile ions or nondegenerate uncorrelated mobile ions. It turns out that the presence of the latter and degenerate ultrarelativistic electrons with the pressure law (3) admits one-dimensional localized ion modes (IMs) supported by linear and nonlinear ion inertial forces and the pressure of degenerate electron fluids in a dense quantum plasma that is unmagnetized. Furthermore, modified Volkov solutions of the Dirac equation for electrostatic and electromagnetic waves in relativistic quantum plasmas have been discussed by Mendonça and Serbeto [23].

In the present paper, the properties of weakly nonlinear IMs in a dense quantum plasma composed of degenerate

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electron fluids and strongly coupled nondegenerate ion fluids are studied. To describe the dynamics of the IMs, the ion continuity and viscoelastic ion momentum equations (similar to those in Refs. [24–26]), as well as the inertialess electron momentum equation with the pressure laws (2) and (3) and Poisson’s equation are used. Strong ion coupling effects enter in the generalized ion momentum equation through the viscoelastic relaxation time for the decay of ion correlations and the bulk ion viscosity. By using the standard reductive perturbation technique [27,28], we derive the Burgers and Kortweg–de Vries (KDV) equations from the governing nonlinear equations for IMs. Stationary solutions of the Burgers and KDV equations in the form of shock and solitary waves are presented. The effects of ion correlations and electron degeneracy on the speed, width, and amplitude of both shock and solitary waves are examined.

## II. MODEL NONLINEAR EQUATIONS

We consider one-dimensional nonlinear propagation of electrostatic IMs associated with inertialess degenerate electron fluids and strongly coupled nondegenerate inertial ions in an unmagnetized dense plasma. The dynamics of nonlinear IMs in our plasma is governed by the momentum equation for inertialess degenerate electron fluids, given by

$$0 = en_e \frac{\partial \phi}{\partial x} - \frac{\partial P_e}{\partial x}, \quad (4)$$

and by the generalized viscoelastic ion hydrodynamic equations (similar to those in Refs. [24–26]), composed of the ion continuity and ion momentum equations

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \quad (5)$$

$$D_\tau \left[ m_i n_i D_t u_i + Z_i e n_i \frac{\partial \phi}{\partial x} + k_B T_{ef} \frac{\partial n_i}{\partial x} \right] = \eta_l \frac{\partial^2 u_i}{\partial x^2}. \quad (6)$$

The equations are closed by Poisson’s equation,

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - Z_i n_i), \quad (7)$$

where  $\phi$  is the electrostatic potential;  $u_i$  is the component of the ion fluid velocity along the  $x$  axis in a Cartesian coordinate system;  $t$  and  $x$  are time and space variables, respectively;  $m_i$  is the ion mass; and  $T_{ef} = (\mu_i T_i + T_\star)$  is the effective ion temperature. The latter consists of two parts, one ( $T_\star$ ) arising from the electrostatic interaction among strongly correlated positive ions and the other ( $\mu_i T_i$ ) arising from the ion thermal pressure.  $D_\tau = 1 + \tau_m(\partial/\partial t + u_i \partial/\partial x)$ ,  $D_t = \partial/\partial t + u_i \partial/\partial x$ ,  $\tau_m$  is the viscoelastic ion relaxation time,  $\mu_i$  is the ion compressibility, and  $\eta_l$  is the longitudinal ion viscosity coefficient. There are various approaches for calculating the ion transport coefficients, similar to those for one-component strongly coupled plasmas [25,26,29,30]. The parameter  $T_\star$  (which arises from the electrostatic interactions among strongly correlated positive ions), viscoelastic ion relaxation time  $\tau_m$ , and ion compressibility  $\mu_i$ , for our purposes, are written as [25,26,30]

$$T_\star = \frac{N_{nn}}{3} \frac{Z_i^2 e^2}{a_i k_B} (1 + \kappa) e^{-\kappa}, \quad (8)$$

$$\tau_m = \frac{\eta_l}{n_{i0} k_B T_i} \left[ 1 - \mu_i + \frac{4}{15} u(\Gamma) \right]^{-1}, \quad (9)$$

$$\mu_i = \frac{1}{k_B T_i} \frac{\partial P_i}{\partial n_i} = 1 + \frac{1}{3} u(\Gamma) + \frac{\Gamma_i}{9} \frac{\partial u(\Gamma)}{\partial \Gamma_i}, \quad (10)$$

where  $N_{nn}$  is determined by the ion structure and corresponds to the number of nearest neighbors (viz., in crystalline state,  $N_{nn} = 8$  for a body-centered cubic lattice,  $N_{nn} = 12$  for a face-centered cubic lattice, etc.);  $\kappa = a_i/\lambda_D$ , with  $\lambda_D$  being the Thomas-Fermi screening length; and  $u(\Gamma_i)$  is a measure of the excess internal energy of the system, calculated for weakly coupled plasmas ( $\Gamma_i < 1$ ) as [26]  $u(\Gamma_i) \simeq -(\sqrt{3}/2)\Gamma_i^{3/2}$ . To express  $u(\Gamma_i)$  in terms of  $\Gamma_i$  for a range of  $1 < \Gamma_i < 100$ , Slattery *et al.* [29] have analytically derived the relation

$$u(\Gamma_i) \simeq -0.89\Gamma_i + 0.95\Gamma_i^{1/4} + 0.19\Gamma_i^{-1/4} - 0.81, \quad (11)$$

where a small correction term due to a finite number of particles has been neglected. The dependence of the other transport coefficient,  $\eta_l$ , on  $\Gamma_i$  is somewhat more complex and cannot be expressed in such a closed analytical form. However, tabulated and graphical results of their functional behaviors, derived from molecular dynamic simulations, and a variety of statistical schemes are available in the literature [25].

It is obvious that Eqs. (5)–(7) are coupled with the electron number density  $n_e$ , which can be deduced from Eqs. (1) and (4) as

$$n_e = n_{e0} \left[ 1 + \frac{(\gamma - 1)e\phi}{\gamma K n_{e0}^{\gamma-1}} \right]^{\frac{1}{\gamma-1}} \simeq n_{e0} \left[ 1 + \frac{C_1}{1!} e\phi - \frac{C_2}{2!} (e\phi)^2 + \frac{C_3}{3!} (e\phi)^3 + \dots \right], \quad (12)$$

where  $n_{e0} = Z_i n_{i0}$ , and

$$C_1 = \frac{1}{\gamma K n_{e0}^{\gamma-1}},$$

$$C_2 = \frac{\gamma - 2}{\gamma^2 K^2 n_{e0}^{2(\gamma-1)}},$$

$$C_3 = \frac{(\gamma - 2)(2\gamma - 3)}{\gamma^3 K^3 n_{e0}^{3(\gamma-1)}}.$$

Hence,  $n_e$  for the weakly and ultrarelativistic electron fluids can be obtained by substituting Eqs. (2) and (3) into Eq. (12).

## III. IM SHOCK

To derive a dynamical equation for the shock waves from Eqs. (5)–(7) and (12), we use the reductive perturbation technique [27] and the stretched coordinates [28]

$$\xi = \epsilon(x - V_p t), \quad (13)$$

$$\tau = \epsilon^2 t,$$

where  $\epsilon$  is a smallness parameter measuring the weakness of the dispersion, and  $V_p$  is the phase speed of the electrostatic

IMs. We can expand the perturbed quantities  $n_i$ ,  $u_i$ , and  $\phi$  about their equilibrium values in a power series of  $\epsilon$  as

$$\begin{aligned} n_i &= n_{i0} + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \dots, \\ u_i &= \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots, \\ \phi &= \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \dots. \end{aligned} \quad (14)$$

Now use Eqs. (12)–(14) in Eqs. (5)–(7) and develop equations in various powers of  $\epsilon$ . To the lowest order in  $\epsilon$ , i.e., taking the coefficients of  $\epsilon^2$  from both sides of Eqs. (5) and (6) and the coefficients of  $\epsilon$  from both sides of Eqs. (7) and (12), one obtains the first-order continuity equation, the momentum equation, and Poisson's equation, which, in turn, give

$$u_i^{(1)} = \frac{Z_i e V_p \phi^{(1)}}{m_i V_p^2 - k_B T_{ef}}, \quad (15)$$

$$n_i^{(1)} = \frac{Z_i e n_{i0} \phi^{(1)}}{m_i V_p^2 - k_B T_{ef}}, \quad (16)$$

$$V_p = C_h \sqrt{1 + \beta_T}, \quad (17)$$

where  $\beta_T = k_B T_{ef} / Z_i \gamma K n_{e0}^{\gamma-1}$ , and  $C_h = (Z_i \gamma K n_{e0}^{\gamma-1} / m_i)^{1/2}$  is the speed of the IMs in our dense plasma. We note that  $\beta_T = a_e^2 k_B T_{ef} / Z_i L_c \hbar c$  and  $\beta_T = a_e k_B T_{ef} / Z_i \hbar c$  for the weakly and ultrarelativistic degenerate electron fluids, respectively. Equation (17) represents the linear dispersion relation for the IMs, in which the restoring force comes from the pressure of degenerate electrons and the ion mass provides the inertia. It is obvious from Eq. (17) that the phase speed  $V_p$  is increased by  $T_{ef}$  (i.e., by the effective ion temperature). It is also clear that, for  $\beta_T \ll 1$  (which is valid for both the weakly and ultrarelativistic limits), we have  $V_p \sim C_h$ , which, along with  $\gamma$ ,  $K$ , and  $\beta_h$ , is provided in Table I. The expression for  $C_h$ , given in Table I, dictates that, unlike the usual ion-acoustic speed in the usual nondegenerate electron-ion plasma,  $C_h$  (or  $V_p$  for  $\beta_T \ll 1$ ) is independent of the electron temperature, but it depends on the unperturbed electron number density, and it is directly proportional to  $n_{e0}^{1/3}$ , i.e., inversely proportional to interelectron distance [ $a_e = (3/4\pi n_{e0})^{1/3}$ ]. It is also directly proportional to  $\sqrt{Z_i}$  and inversely proportional to  $\sqrt{m_i}$ . To the next higher order in  $\epsilon$ , i.e., taking the coefficients of  $\epsilon^3$  from both sides of Eqs. (5) and (6) and the coefficients of  $\epsilon^2$  from both sides of Eqs. (7) and (12), one obtains another set of coupled equations for  $n_i^{(2)}$ ,  $u_i^{(2)}$ , and  $\phi^{(2)}$ , which, along with the first set of coupled linear equations for  $n_i^{(1)}$ ,  $u_i^{(1)}$ , and  $\phi^{(1)}$ , reduces to the nonlinear dynamical equation

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} = C \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}, \quad (18)$$

TABLE I. Expressions for  $K$ ,  $C_h = V_p (T_{ef} = 0)$ , and corresponding  $\Phi_m^{(1)} (T_{ef} = 0)$  for the weakly relativistic ( $\gamma = 5/3$ ) and ultrarelativistic ( $\gamma = 4/3$ ) limits.

$\gamma$	$K$	$\beta_T$	$C_h$	$\Phi_m^{(1)} (T_{ef} = 0)$
$\frac{5}{3}$	$\frac{5}{3} L_c \hbar c$	$\frac{a_e^2 T_{ef}}{Z_i L_c \hbar c}$	$\left( \frac{Z_i L_c \hbar c}{a_e^2 m_i} \right)^{\frac{1}{2}}$	$\frac{3m_i U_0 C_h^{ur}}{4Z_i e}$
$\frac{4}{3}$	$\frac{3}{4} \hbar c$	$\frac{a_e T_{ef}}{Z_i \hbar c}$	$\left( \frac{Z_i \hbar c}{a_e m_i} \right)^{\frac{1}{2}}$	$\frac{6m_i U_0 C_h^{ur}}{7Z_i e}$

where the nonlinear coefficient  $A$  and the dissipation coefficient  $C$  are, respectively,

$$A = \frac{Z_i e}{2m_i C_h \sqrt{1 + \beta_T}} [1 + \gamma + (\gamma - 2)\beta_T], \quad (19)$$

$$C = \frac{\eta_l}{2n_{i0} m_i}. \quad (20)$$

Equation (18) is the well-known Burgers equation describing the nonlinear propagation of the IMs in our dense plasma. It is obvious from Eqs. (18) and (20) that the dissipative term, i.e., the right-hand side of Eq. (18), is due to the strong correlation among positive ions.

We now look for a stationary shock wave solution of Eq. (18), by introducing  $\zeta = \xi - U_0 \tau'$  and  $\tau' = \tau$ , where  $U_0$  is the shock speed (in the reference frame). This leads us to write Eq. (18), under the steady-state condition ( $\partial/\partial \tau' = 0$ ), as

$$-U_0 \frac{\partial \phi^{(1)}}{\partial \zeta} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \zeta} = C \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2}. \quad (21)$$

It can be easily shown [31] that Eq. (21) describes the IM shock whose speed  $U_0$  is related to the extreme values  $\phi^{(1)}(-\infty)$  and  $\phi^{(1)}(\infty)$  by  $\phi^{(1)}(\infty) - \phi^{(1)}(-\infty) = 2U_0/A$ . Thus, under the condition that  $\phi^{(1)}$  is bounded at  $\zeta = \pm\infty$ , the IM shock solution of Eq. (21) is [31]

$$\phi^{(1)} = \frac{\Phi_m}{2} [1 - \tanh(\zeta/\Delta)], \quad (22)$$

where  $\Phi_m$  and  $\Delta$  are the height and width of the IM shock, respectively. We have

$$\Phi_m = \frac{2U_0}{A}, \quad (23)$$

$$\Delta = \frac{2C}{U_0}. \quad (24)$$

The IM shock arises due to the balance between nonlinearities associated with the generation of harmonics that result from IM couplings and strong correlations between the positive ions in our dense plasma. It is seen from Eqs. (19), (20), and (23) that the IM shock height is independent of the longitudinal viscosity coefficient  $\eta_l$ , but the shock thickness is directly proportional to the longitudinal viscosity coefficient  $\eta_l$  (i.e., the IM shock thickness increases with the increase of the longitudinal ion viscosity coefficient) and is inversely proportional to the ion mass density ( $\rho_{i0} = n_{i0} m_i$ ) and to  $U_0$ . It is also seen from Eqs. (19) and (22) that, for  $\gamma < 2$  [which is satisfied for both the weakly relativistic ( $\gamma = 5/3$ ) and ultrarelativistic ( $\gamma = 4/3$ ) limits], the nonlinear coefficient  $A$  decreases; i.e., the IM shock height increases (nonlinearly and slowly) with the increase of  $T_{ef}$ . On the other hand, for  $\beta_T \ll 1$ , the expressions for the shock height [ $\Phi_m^{(1)}$ ] for nonrelativistic ( $\gamma = 5/3$ ) and ultrarelativistic ( $\gamma = 4/3$ ) limits are provided in Table I.

Table I shows that, for  $\beta_T \ll 1$ , the IM shock height is (i) directly proportional to  $U_0$  and  $\sqrt{m_i}$  but inversely proportional to  $\sqrt{Z_i}$  in both the weakly and ultrarelativistic limits and (ii) directly proportional to  $n_{e0}^{1/3}$  in the weakly relativistic limit but to  $n_{e0}^{1/6}$  in the ultrarelativistic limit.

#### IV. SOLITARY IMS

To derive a dynamical equation for the solitary IMs from Eqs. (5)–(7) and (12), we again use the reductive perturbation technique [27], with another set of stretched coordinates:

$$\begin{aligned}\xi &= \epsilon^{1/2}(x - V_p t), \\ \tau &= \epsilon^{3/2}t.\end{aligned}\quad (25)$$

We now use Eqs. (12)–(14) and (25) in Eqs. (5)–(7) and develop equations in various powers of  $\epsilon$ . To the lowest order in  $\epsilon$ , i.e., taking the coefficients of  $\epsilon^{3/2}$  from both sides of Eqs. (5) and (6) and the coefficients of  $\epsilon^3$  from both sides of Eqs. (7) and (12), we obtain the first-order continuity equation, the momentum equation, and Poisson's equation, which, in turn, give a set of equations that is completely identical to the set of equations given by Eqs. (15)–(17).

To the next higher order in  $\epsilon$ , i.e., taking the coefficients of  $\epsilon^{5/2}$  from both sides of Eqs. (5) and (6) and the coefficients of  $\epsilon^2$  from both sides of Eqs. (7) and (12), we obtain another set of coupled equations for  $n_i^{(2)}$ ,  $u_i^{(2)}$ , and  $\phi^{(2)}$ , which, along with Eqs. (15)–(17), reduces to the nonlinear dynamical equation

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \quad (26)$$

where the nonlinear coefficient  $A$  is given by Eq. (19), and the dispersion coefficient  $B$  is

$$B = \frac{C_{\tilde{h}}^3 (1 + \beta_T)^{3/2}}{2\omega_{pi}^2} = \left( \frac{Z_i}{m_i} \right)^{1/2} \gamma^{3/2} K^{3/2} n_{e0}^{1/2(3\gamma-1)} \frac{1}{8\pi e^2}. \quad (27)$$

Equation (26) is the well-known KDV equation describing the nonlinear propagation of IMs in our dense plasma system.

The stationary solitary IM solution of the KDV equation (26) is obtained by transforming the independent variables to  $\zeta = \xi - U_0 \tau'$  and  $\tau' = \tau$ , where  $U_0$  is the speed of the solitary IM, and imposing the appropriate boundary conditions, viz.,  $\phi^{(1)} \rightarrow 0$ ,  $d\phi^{(1)}/d\zeta \rightarrow 0$ , and  $d^2\phi^{(1)}/d\zeta^2 \rightarrow 0$  at  $\zeta \rightarrow \pm\infty$ . Accordingly, the stationary solitary IM solution of the KDV equation (26) is

$$\phi^{(1)} = \phi_m \operatorname{sech}^2(\zeta/\delta), \quad (28)$$

where  $\phi_m$  and  $\delta$  are the amplitude and the width of the solitary IM, respectively. We have

$$\phi_m = \frac{3U_0}{A}, \quad (29)$$

$$\delta = \sqrt{\frac{4B}{U_0}}. \quad (30)$$

It is obvious from Eqs. (23) and (29) that  $\Phi_m = 2U_0/A$  for the IM shock and  $\phi_m = 3U_0/A$  for the solitary IM, i.e.,  $\phi_m/\Phi_m = 3/2$ . This means that the solitary IM amplitude differs by a factor of 3/2 from the IM shock height and that the variation of the solitary IM amplitude with  $Z_i$ ,  $m_i$ ,  $n_{e0}$ , and  $T_{ef}$  is exactly the same as that of the IM shock height.

It is easy to show from Eqs. (2), (3), (27), and (30) that

$$\delta^{nr} = \left[ \frac{Z_i L_c^3 \tilde{h}^3 c^3}{m_i U_0^2 \pi^2 e^4} (1 + \beta_T)^3 \right]^{1/4} \quad (31)$$

for the weakly relativistic limit ( $\gamma = 5/3$ ) and

$$\delta^{ur} = \left[ \frac{Z_i a_e^3 \tilde{h}^3 c^3}{m_i U_0^2 \pi^2 e^4} (1 + \beta_T)^3 \right]^{1/4} \quad (32)$$

for the ultrarelativistic limit ( $\gamma = 4/3$ ).

Equations (31) and (32) reveal that the width  $\delta$  of the solitary IMs increases with the effective ion temperature  $T_{ef}$ . Furthermore, we see from Eqs. (31) and (32) that, for  $\beta_T \ll 11$ , (i) the width  $\delta$  is directly proportional to  $Z_i^{1/4}$  but inversely proportional to  $m_i^{1/4}$  and  $\sqrt{U_0}$  and (ii) it is independent of the plasma number density in the weakly relativistic limit but inversely proportional to  $n_{e0}^{3/4}$  in the ultrarelativistic limit.

#### V. DISCUSSION AND CONCLUSION

In this paper, we have investigated the nonlinear propagation of electrostatic IMs in a dense plasma composed of inertialess degenerate electron fluids and strongly coupled nondegenerate strongly ion fluids. By using the appropriate electron density response and the generalized hydrodynamic equations for ions [25,26], together with Poisson's equation, we have derived the governing equations for nonlinear IMs. When the longitudinal ion viscosity, resulting from strong ion correlations, prevails over the dispersion arising from the charge separation, one finds that the dynamics of nonlinear IMs is governed by the Burgers equation. On the other hand, when the charge separation effects overwhelm the strong ion coupling effects, the nonlinear IM dynamics is governed by the KDV equation. Here, the electron degeneracy plays a crucial role. Our main results are summarized as follows:

(1) The phase speed of the nonlinear IMs,  $V_p$  (which is, in fact, the critical phase speed for which the solitary or shock IMs are formed), increases due to the effect of the effective ion temperature  $T_{ef}$ .

(2) Unlike the usual nonlinear ion-acoustic waves in a nondegenerate electron-ion plasma,  $C_h$  [which is, in fact, the phase speed for  $\beta_T \ll 1$ , i.e.,  $C_h = V_p(T_{ef} = 0)$ ] is independent of the electron temperature, but it depends on the electron number density and is directly proportional to  $n_{e0}^{1/3}$ , i.e., inversely proportional to the interelectron distance  $a_e$ . However, like the usual nonlinear ion-acoustic waves in a nondegenerate electron-ion plasma, it is also directly proportional to  $\sqrt{Z_i}$  and inversely proportional to  $\sqrt{m_i}$ .

(3) The IM shock, which is due to the balance between nonlinearities and dissipation, exists with a positive potential only. The strong correlation among positive ions is the source of dissipation and is responsible for the formation of the IM shock.

(4) The IM shock height is independent of the longitudinal viscosity coefficient  $\eta_l$ , but the IM shock thickness is directly proportional to the longitudinal viscosity coefficient  $\eta_l$  and is inversely proportional to the ion mass density ( $\rho_{i0} = n_{i0} m_i$ ).

(5) The IM shock height is directly proportional to  $\sqrt{m_i}$  but inversely proportional to  $\sqrt{Z_i}$  in both the weakly and ultrarelativistic limits. However, it is directly proportional to  $n_{e0}^{1/3}$

(i.e., inversely proportional to interelectron distance  $a_e$ ) in the weakly relativistic limit but directly proportional to  $n_{e0}^{1/6}$  (i.e., inversely proportional to the interelectron distance  $\sqrt{a_e}$ ) in the ultrarelativistic limit.

(6) The solitary IM, which is due to the balance between nonlinearities and dispersion, exists with a positive potential only. The solitary IM amplitude differs by a factor of 3/2 from the IM shock height, and the variation of the solitary IM amplitude with  $Z_i$ ,  $m_i$ ,  $n_{e0}$ , and  $T_{ef}$  is exactly the same as that of the IM shock height.

(7) The width  $\delta$  of the solitary IM increases with the increase of  $T_{ef}$ . On the other hand, for  $\beta_T \ll 1$ , the width  $\delta$  is directly proportional to  $Z_i^{1/4}$  but inversely proportional to  $m_i^{1/4}$  in both the weakly and ultrarelativistic limits. However, it is independent of the plasma number density in the weakly relativistic limit but is inversely proportional to  $n_{e0}^{3/4}$  in the ultrarelativistic limit.

It is important to add here that we have followed Chandrasekhar [1,2] by assuming that the white dwarf core is pure He. However, in a recent study, Koester [32] assumes that the core may be pure C or pure O. The important point to note here is that, whatever the ‘‘chemical’’ composition of the core is (pure He, pure C, or pure O) for a given mass, the total number of nucleons (protons plus neutrons) would be the same, and because the number of electrons per nucleon would also be the same, namely 1/2 (He being 4 nucleons and 2 electrons, C being 12 nucleons and 6 electrons, and O being 16 nucleons and 8 electrons), we would end up with the same value of  $n_0 = n_{e0} = Z_i n_{i0}$  for all compositions. This means that our nonlinear theory for localized IMs is also valid for the recent assumption of Koester [32]. Our results have shown how the presence of the ions of the heavier elements C or O (instead of He) can modify the basic features [viz., the speed, height, and thickness, which are expressed as functions of  $m_i$  by Eqs. (23), (24), (29), and (30)] of the IM shock and solitary waves that are formed in our dense plasma with degenerate inertialess electron fluids and strongly coupled nondegenerate inertial ion fluids.

It is also important to note here that when the dispersion (dissipation) effect is much more pronounced than the dissipation (dispersion) effect, and the dissipation (dispersion) effect is neglected, strongly coupled degenerate dense plasmas support solitary (shock) waves. To neglect the effect of the IM dispersion in comparison with that of the dissipation, or vice versa, one has to choose a suitable scaling (stretching of coordinates), such as the one that we used in our investigation of the IM shock and solitary waves. However, if it would be possible to keep both effects, i.e., to derive the KDV-Burgers equation [31] by choosing the appropriate stretched coordinates (which have not been found so far), one would have an oscillatory IM shock in which the first few oscillations at the wave front would be close to IM solitons [31].

The KDV-Burgers equation has been derived by a number of authors, e.g., Shukla and Mamun [33] and Pakzad and Javidan [34], by using the stretched coordinates defined by Eq. (25), along with an additional stretching of the viscosity coefficient ( $\eta_l = \epsilon^{1/2}\eta_0$ ), in order to study the dust-acoustic [35] solitary and shock waves in a strongly coupled dusty plasma. However,

the use of this additional stretching ( $\eta_l = \epsilon^{1/2}\eta_0$ ) is not correct, in general, at least from the mathematical point of view, since this additional stretching ( $\eta_l = \epsilon^{1/2}\eta_0$ ) leads to the dissipation coefficient to contain the expansion parameter  $\epsilon$ . So, to avoid this additional stretching ( $\eta_l = \epsilon^{1/2}\eta_0$ ), in our present work, we used stretched coordinates other than those used by Shukla and Mamun [33] or Pakzad and Javidan [34]. The Burgers equation has also been derived by a number of authors, e.g., Shukla [36] and Rahman *et al.* [37], by using the same additional stretching in order to study the dust-ion-acoustic [38] shock waves in a dusty plasma.

On the other hand, Pakzad [39] has derived the KDV equation by using the stretched coordinates defined by Eq. (25) and the modified KDV equation by using another set of stretched coordinates [viz.,  $\xi = \epsilon(x - V_p t)$ ,  $\tau = \epsilon^3 t$ ], and he has studied the ion-acoustic solitons [40] in a plasma containing warm ions (which are weakly relativistic and weakly coupled) and electrons (which follow the Cairns distribution [40]) and positrons (which follow the Boltzmann distribution). It is, therefore, obvious that the plasma models (viz., dusty plasma [33–38] and nonthermal plasma [39,40]), as well as the scale and time lengths of the waves considered in all of these earlier investigations [33–40], are not valid for any degenerate plasma system and are completely different from what we have considered in our present work, concerning waves in strongly coupled degenerate (quantum) plasma systems or compact objects like white dwarfs [1,2]. We refer to the seminal works of Rao *et al.* [35], Shukla and Silin [38], and Cairns *et al.* [40] for the details of the dust-acoustic waves, dust-ion-acoustic waves, and nonthermal plasmas, respectively, which make clear how our present work (its model, results, and possible applications) is completely different from the works of Pakzad and Javidan [34], Rahman *et al.* [37], and Pakzad [39], respectively.

To conclude, the results of the present investigation should be useful in understanding the salient features of localized IM excitation in dense (degenerate) plasmas, such as those in white dwarf stars. Specifically, from the phase speed of nonlinear IM structures, one may infer the composition and structural properties of massive stars in which the plasma particles are densely packed.

It may be noted here that the effects of nonplanar geometry and the external magnetic field on the IM shock and solitary structures and their multidimensional instability are also problems of great importance but are beyond the scope of our present work.

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