

Numerical evidence against a conjecture on the cover time of planar graphs

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We investigate a conjecture on the cover times of planar graphs by means of large Monte Carlo simulations. The conjecture states that the cover time $\tau(G_N)$ of a planar graph G_N of N vertices and maximal degree d is lower bounded by $\tau(G_N) \geq C_d N (\ln N)^2$ with $C_d = (d/4\pi) \tan(\pi/d)$, with equality holding for some geometries. We tested this conjecture on the regular honeycomb ($d = 3$), regular square ($d = 4$), regular elongated triangular ($d = 5$), and regular triangular ($d = 6$) lattices, as well as on the nonregular Union Jack lattice ($d_{\min} = 4$, $d_{\max} = 8$). Indeed, the Monte Carlo data suggest that the rigorous lower bound may hold as an equality for most of these lattices, with an interesting issue in the case of the Union Jack lattice. The data for the honeycomb lattice, however, violate the bound with the conjectured constant. The empirical probability distribution function of the cover time for the square lattice is also briefly presented, since very little is known about cover time probability distribution functions in general.

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I. INTRODUCTION

The cover time of a graph is a classic problem in theoretical computer science and graph theory with many practical implications, e.g., in the development of query processing and routing algorithms in computer networks and distributed systems, and has attracted the attention of computer scientists, mathematicians, and physicists for more than 30 years [1–22].

For a finite, connected graph $G_N = (V, E)$ of order N , the cover time $\tau(G_N)$ is the maximum expected time over the possible starting vertices $v \in V$ it takes for a random walker jumping through the edges of G_N with uniform probabilities to visit every vertex of G_N at least once. Exact expressions for the cover time are rare except for the simpler graphs, e.g., for the complete graph, for which the problem reduces to the well-known coupon collector's problem, and for the path, cycle, and star graphs, among a few others [10,15,20,21].

Most results on graph cover times come in the form of bounds on their expectation values, although little is known about the limit distributions of the related quantities. The existence of a lower bound follows from $\tau(G_N) \geq N$ —although it can be proved that $\mathbb{P}(\tau(G_N) \leq cN) \leq e^{-\alpha N}$, with $\alpha > 0$ depending only on c and $d_{\max}(G_N)$, the maximal degree of the graph [23]—while the existence of an upper bound follows from the recurrence of the associated Markov chains.

For planar graphs, Jonasson and Schramm showed that [17]

$$\liminf_{N \rightarrow \infty} \frac{\tau(G_N)}{N (\ln N)^2} \geq C_d, \quad (1)$$

where $C_d > 0$ are universal constants depending only on $d \geq d_{\max}(G_N)$. The functional form $\tau(G_N) \sim N (\ln N)^2$ for the cover time of the square lattice had been guessed earlier on the basis of Monte Carlo simulations and scaling analysis, where a multiplicative correction $(1 + c/\ln N)$ to this form was detected, with c (the magnitude of the leading scaling correction) a constant depending on the boundary conditions of the finite graphs [5,6,9]. Following a sophisticated probabilistic-geometric analysis of the cover time of the square

lattice, Dembo *et al.* [19] conjectured that for the honeycomb, square, and triangular lattices, corresponding respectively to $d = d_{\max} = 3, 4$, and 6, the constants C_d appearing in Eq. (1) are exactly given by

$$C_d = \frac{d}{4\pi} \tan\left(\frac{\pi}{d}\right), \quad (2)$$

further conjecturing that, with this C_d , inequality (1) may actually hold as an equality for these geometries.

In this article we investigate numerically the above mentioned conjecture on some planar graphs to check whether the bound (1) with the conjectured constant Eq. (2) holds as a lower bound or as an equality. As we shall see, the empirical data for the cover time of the honeycomb lattice indicate a violation of the lower bound with the constant given by Eq. (2), requiring a smaller constant, while for some other planar graphs the conjecture seems to hold valid.

II. MONTE CARLO DATA

We investigate the conjecture encoded in Eqs. (1)–(2) on the regular honeycomb ($d = 3$), regular square ($d = 4$), regular elongated triangular ($d = 5$), and regular triangular ($d = 6$) lattices, and on the nonregular Union Jack lattice ($d_{\min} = 4$, $d_{\max} = 8$). These lattices are depicted in Fig. 1.

We computed the cover times on graphs with $N = L \times L$ vertices under periodic boundary conditions, with $256 \leq L \leq 1536$, i.e., on graphs with up to 2.359×10^6 vertices. For each graph geometry and size, $\tau(G_N)$ is obtained as an average over 10^6 samples. Our Monte Carlo data together with the conjectured values $\tau^*(G_N) = C_d N (\ln N)^2$ obtained from Eqs. (1)–(2) appear in Fig. 2.

Notice that the toroidal graphs obtained under periodic boundary conditions are not planar, although they are locally very close to planar. The really important fact in a planar graph for the cover time problem, however, is that its edges do not cross, not that it can be embedded in a plane. Moreover, finite graphs with open boundary conditions cannot be regular, since the vertices at the boundaries are of a smaller degree. The asymptotics in the two cases (open and periodic boundary conditions) are expected to be the same, and most results

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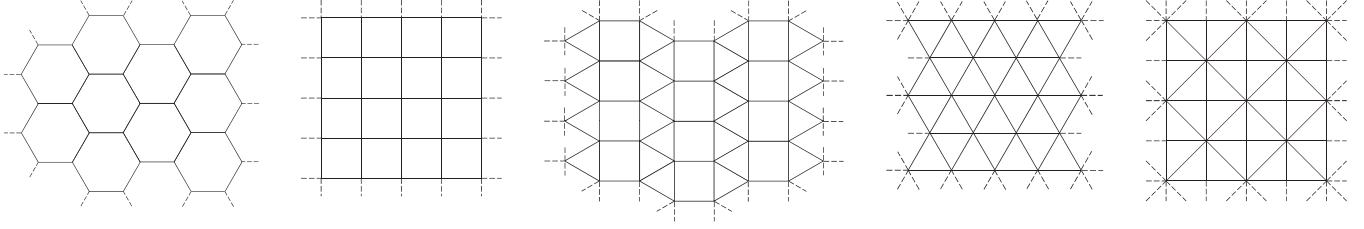


FIG. 1. Planar graphs investigated in this article. From left to right we have the honeycomb ($d = 3$), square ($d = 4$), elongated triangular ($d = 5$), triangular ($d = 6$), and Union Jack ($d_{\min} = 4$, $d_{\max} = 8$) lattices.

on graph cover times, including those to which we want to compare our own results, are obtained for graphs under periodic boundary conditions.

According to the conjecture, we must observe $\tau^*(G_N)/\tau(G_N) \leq 1$ for all planar graphs, a condition that our numerical data does not support for the honeycomb lattice. Other lattices observe the bound, with the square lattice being “borderline.” A naïve extrapolation of the ratios $\tau^*(G_N)/\tau(G_N)$ would give an extrapolated value greater than 1 in almost all cases depicted in Fig. 2. This indicates that possible corrections to $\tau^*(G_N)$ must go in the direction of decreasing its value. Since previous results in the literature suggest just the opposite, i.e., that, if anything, $\tau^*(G_N)$ may be missing a $(1 + c/\ln N)$ correction with $c > 0$ [5,6,9], we are led to believe that the constant C_d with $d = d_{\max}$ is overshooting. Supplementary evidence comes from the behavior of the Union Jack lattice with respect to d . Although the bound (1) requires $d \geq d_{\max}(G_N)$, our data suggest that this requirement is probably not optimal. We plot the ratios $\tau^*(G_N)/\tau(G_N)$ for the Union Jack lattice both with $d = d_{\max} = 8$ and with $d = \bar{d} = \frac{1}{2}(d_{\min} + d_{\max}) = 6$, the average degree of the lattice, and we found that the ratio with $d = \bar{d}$ provides a better lower bound than the ratio with $d = d_{\max}$; see Fig. 2. This makes us wonder if the average vertex degree

$$\bar{d}(G) = \frac{1}{|G|} \sum_{v \in V(G)} d(v), \quad (3)$$

where $d(v)$ is the degree of vertex v , is not a better constant to be used on purportedly universal formulas for planar graphs than the maximum degree d_{\max} . This could be tested on planar random graphs—e.g., on Delaunay triangulations of random points on the plane [24]—for which \bar{d} can assume different values, integer or not. Recently, planar random graphs have attracted the attention of physicists and mathematicians interested in their connectivity and percolation properties [25–30], but their cover times remain unexamined.

III. THE EMPIRICAL PROBABILITY DISTRIBUTION FUNCTION OF THE COVER TIMES OF THE SQUARE LATTICE

Very little is known about the probability distribution function (PDF) of $\tau(G_N)$. Actually, it is an open problem to prove that $\tau(G_N)$ has a nondegenerate limit law [15,19,20]. It seems that the only result on this regard available to date is a concentration result that states that, under mild conditions, the cover time is well approximated by its expected value as $N \rightarrow \infty$ [7]. It is thus of some interest to explore our empirical

data to characterize the PDF of the cover time, although we will not attempt to identify or infer this distribution here.

Figure 3 shows the histogram plot of 10^6 cover times sampled for a square lattice of $N = 1280 \times 1280$ vertices. For these data, we compute the sample mean $m = \langle \tau_i(G_N) \rangle$ and first few central moments $m_k = \langle (\tau_i(G_N) - m)^k \rangle$, from which we compute the sample standard deviation $s = \sqrt{m_2}$, skewness $g_1 = m_3/m_2^{3/2}$, and excess kurtosis $g_2 = m_4/m_2^2 - 3$ [31]. The values of these quantities are collected in Table I.

We also fitted the data to the beta PDF given by

$$P(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad (4)$$

rescaled to the interval $[a, b]$ with $a = \min\{\tau_i(G_N)\} = 73\,391\,036$ and $b = \max\{\tau_i(G_N)\} = 213\,039\,197$. We choose the beta distribution because it has finite support and can take several different shapes; moreover, beta-like PDFs for the cover times of some special graphs were found in previous investigations [5,6,21]. The empirical data together with the adjusted beta PDF appear in Fig. 3 [32].

The positive skewness g_1 indicates that the empirical PDF is right-tailed, with the bulk of the observed values lying to the left of the mean, although this feature is not very clear from the histogram (3) because g_1 is not very large. The moderately high value of the excess kurtosis g_2 , in turn, indicates that the empirical PDF is markedly non-normal, with a sizable proportion of the data in its right tail contributing to the variance observed. We notice that after 10^6 samplings, the ratio s/m has stationed at $\sim 11.5\%$.

IV. CONCLUSIONS

Our numerical data for the cover time of the honeycomb lattice provide evidence against the conjecture set forth in Ref. [19] regarding the constant C_d appearing on the lower bound (1) for planar graphs. Otherwise, for the other lattices investigated in this article the functional form given by Eqs. (1)–(2) seems to hold valid, possibly as an equality. In summary, for $d = 3$ our data seem to falsify the conjecture, for $d = 4$ it is “borderline,” and for $d \geq 5$ the conjecture holds easily. That there must be something special about the $d = 4$ case has been long recognized [15], and this might have showed in our finite-size simulations. Notice that the lower bound (1) remains valid for some C_d , just not with the C_d given by Eq. (2) with $d = d_{\max}$. Since the C_d given by Eq. (2) is monotone decreasing in d , it can be used in Eq. (1) to validate the bound, but then necessarily with some $d > d_{\max}$.

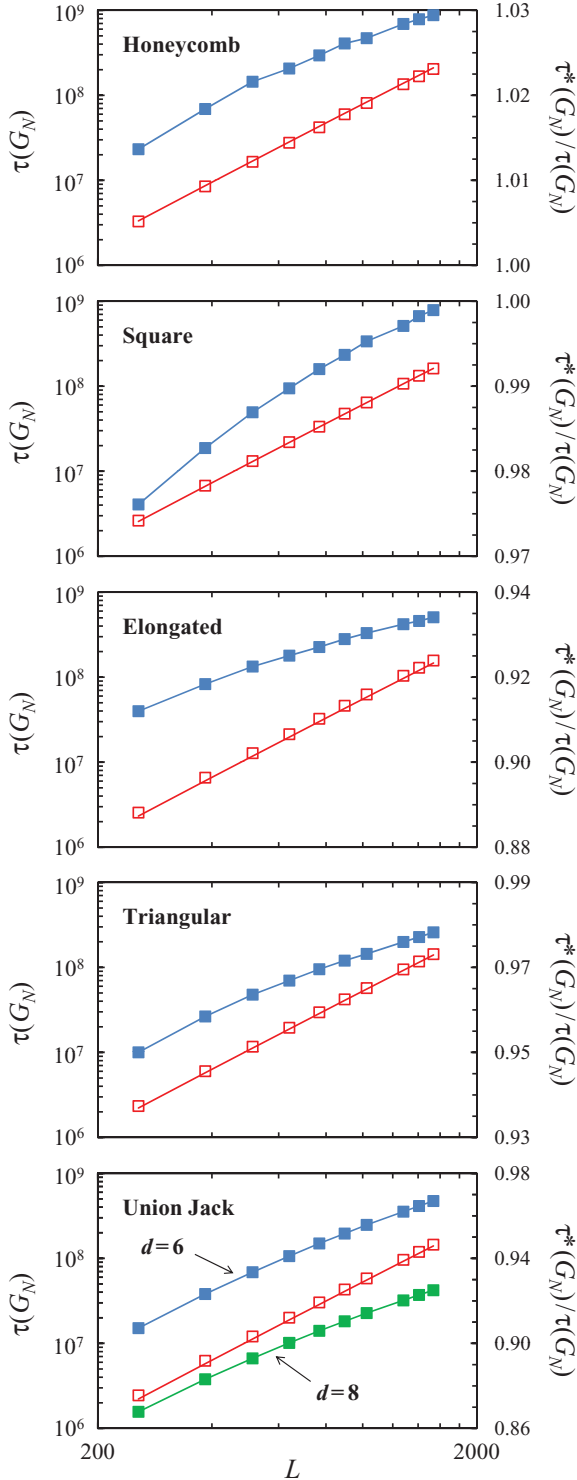


FIG. 2. (Color online) Cover times for the planar lattices depicted in Fig. 1. Each graph displays the empirical cover times $\tau(G_N)$ (open squares, left scale) together with the conjectured value $\tau^*(G_N) = C_d N(\ln N)^2$ (solid line, left scale) and the ratio $\tau^*(G_N)/\tau(G_N)$ (full squares, right scale). Each point of the $\tau(G_N)$ curves was obtained as an average over 10^6 samples. For the Union Jack lattice we display the ratio $\tau^*(G_N)/\tau(G_N)$ both for C_d with $d = d_{\max} = 8$ and $d = \bar{d} = \frac{1}{2}(d_{\min} + d_{\max}) = 6$; the plots of $\tau^*(G_N)$ for these two values of d are indistinguishable on the left scale of the graph and appear as a single line passing through the open squares.

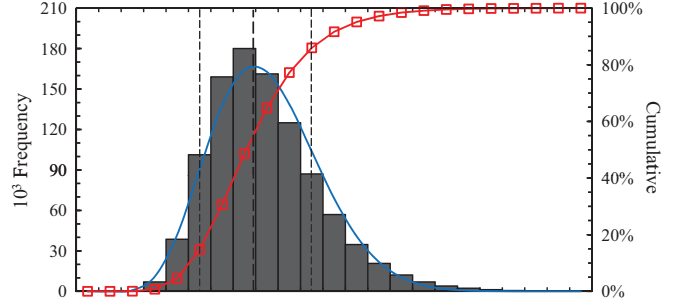


FIG. 3. (Color online) Histogram plot of 10^6 cover times sampled for a square lattice of $N = 1280 \times 1280$ vertices. There are 23 bins in the histogram, with the leftmost bin centered at 70×10^6 and the rightmost bin centered at 180×10^6 . The vertical dashed lines indicate the empirical mean m and $\pm s$ intervals. The continuous line corresponds to the adjusted beta PDF (4) with shape parameters $\alpha = 5.759 \pm 0.008$ and $\beta = 18.04 \pm 0.02$. The empirical cumulative distribution function is also shown (open squares, right scale).

It can be argued that our data were taken too short from the limit $N \rightarrow \infty$. However, if the conjecture is to be saved, that would mean a nonmonotone convergence of the ratios $\tau^*(G_N)/\tau(G_N)$, something that our data do not indicate.

Any case for the average degree \bar{d} given by Eq. (3) would be welcome. It may be that the situation here is similar to that of the determination of the critical percolation threshold p_c , for which good approximations and scaling relations were found based on the mean Euler characteristic of the critical percolation patterns [29]. It should be remarked, however, that for the critical percolation threshold problem one does not expect to find a “universal” formula in terms of the maximal vertex degree d alone, simply because of the empirical observation that several different lattices with the same d have disparate p_c . Notice that the constant C_d in Eq. (2) is closely related with the filling factor f introduced by Suding and Ziff in order to relate p_c to the number of sites per unit area of Archimedean lattices [33],

$$f = \pi \left[\sum_i a_i \cot \left(\frac{\pi}{n_i} \right) \right]^{-1}, \quad (5)$$

where $(n_1^{a_1}, n_2^{a_2}, \dots)$ is the Grünbaum-Shephard representation of the Archimedean lattice—e.g., the square lattice is denoted by (4^4) , while the elongated triangular lattice is denoted by $(3^3, 4^2)$ [34]. The investigation of different planar graphs of same degree (regular or average)—e.g., for $d = 4$, the square, kagomé, and Archimedean $(3, 4, 6, 4)$ lattices—may help to elucidate the above questions of geometric character.

Finally, our exploratory analysis of the PDF of the cover times of the square lattice in Sec. III is admittedly jejune; that

TABLE I. First moments of the empirical probability distribution function of the cover time of the square lattice depicted in Fig. 3.

	m	107 111 924
Mean	m	107 111 924
Variance	s	12 308 707
Skewness	g_1	0.91374
Excess kurtosis	g_2	1.51095

was not the focus of this work. We were nevertheless able to establish that the empirically observed PDF is not normal, with a leptokurtic shape ($g_2 > 0$). A proper investigation of the PDF of the cover times of planar graphs—e.g., by model selection among candidate left and right limited, univariate distributions—is still lacking and provides an interesting direction for further statistical work on cover times.

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