Fractal structure of a three-dimensional Brownian motion on an attractive plane

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Consider a Brownian particle in three dimensions which is attracted by a plane with a strength proportional to some dimensionless parameter α. We investigate the fractal spatial structure of the visited lattice sites in a cubic lattice by the particle around and on the attractive plane. We compute the fractal dimensions of the set of visited sites both in three dimensions and on the attractive plane, as a function of the strength of attraction α. We also investigate the scaling properties of the size distribution of the clusters of nearest-neighbor visited sites on the attractive plane and compute the corresponding scaling exponent τ as a function of α. The fractal dimension of the curves surrounding the clusters is also computed for different values of α, which, in the limit $\alpha \to \infty$, tends to that of the outer perimeter of planar Brownian motion, i.e., the self-avoiding random walk (SAW). We find that all measured exponents depend significantly on the strength of attraction.

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I. INTRODUCTION

The laws of Brownian motion, formulated first by Einstein more than a century ago [1], have now found many applications and generalizations in all quantitative sciences [2]. Many fractal structures in nature can be derived from the sample paths of Brownian motion characterized by appropriate fractal dimensions [3].

A d-dimensional Brownian motion is known to be recurrent, i.e., the particle returns to the origin, for $d \leqslant 2$ and escapes to infinity for $d > 2$. It is also known that the fractal (Hausdorff) dimension of the graph of a Brownian motion is equal to $\frac{3}{2}$ for $d = 1$ and 2 for $d \geqslant 2$.

The scaling limit of interfaces in various critical $2d$ lattice models are proven or conjectured to be described by the family of conformally invariant random curves, i.e., a Schramm-Loewner evolution (or SLE$_\kappa$) [4], which is driven by a $1d$ Brownian motion of diffusivity κ [5].

One of the most important invariance properties of planar Brownian motion is conformal invariance. Although the scaling limit of $2d$ random walk (i.e., $2d$ Brownian motion because of self-crossing) itself does not fall in the SLE category, variations of Brownian motion are described by SLE. Loop-erased random walk (LERW), where loops are removed along the way, is one of the examples that Schramm's studies have shown to be described by SLE$_2$. The external perimeter of $2d$ random walk is also a nonintersecting fractal curve that can be defined by SLE. Verifying an earlier conjecture by Mandelbrot [3], Lawler *et al.* used SLE techniques [6] to prove that the fractal dimension of the Brownian perimeter is $d_f = \frac{4}{3}$, i.e., the same as the fractal dimension of self-avoiding random walk (SAW) and the external perimeter of the percolation hull.

In this paper, we investigate the statistical and fractal properties of a $3d$ random walker attracted by a plane. We believe that this study can provide useful intuitive extensions for many related physical phenomena including the problems with a discrete time lattice walk [7,8], relaxation phenomena [9], exciton trapping [10], and diffusion-limited reactions [8,11].

II. THE MODEL

We consider a random walker moving along the bonds of a cubic lattice with the xy plane as an attractive plane. The "walker" source is considered to be the origin of the coordinate system. At each lattice point with $z \neq 0$, there are six possibilities for the random walker to select a link and move along. In our model, the random walker prefers walking on and near the attractive plane, and thus the probability that the random walker chooses the link that approximates it to the attractive plane is set to be αp, and for the remaining five links is considered to be p, such that $\alpha > 1$ (this will be called the *strength of attraction*) and $p = \frac{1}{\alpha+5}$. For each lattice point on the attractive plane with $z = 0$, the probability that each of the four links on the plane to be chosen is set to be $\alpha p'$ and for two other links perpendicular to the plane is considered to be p', where $p' = \frac{1}{4\alpha+2}$. The single parameter α in our model controls the strength of attraction. Note that in the limiting case $\alpha \to \infty$ our model reduces to the pure $2d$ random walk on the plane, and for $\alpha = 1$ the pure $3d$ random walk would be recovered.

Thus there are four possible probabilities: $\alpha p'$ for links that are in the attractive plane, p' for links from the attractive plane to either of the neighboring planes, p for links in all of the neighboring planes or leading from them into the bulk, and αp for links from all the neighboring planes to the attractive plane. By detailed balance, in equilibrium at inverse temperature β, the ratio $\alpha p/p'$ of the probabilities onto and off of the attractive plane defines an attraction energy $\beta\epsilon = \ln \frac{2\alpha(1+2\alpha)}{(\alpha+5)}$.

III. FRACTAL DIMENSION OF THE SET OF ALL VISITED SITES AND ITS LEVEL SET

In random walks, systems exhibit a *generic scale invariance*, meaning that the systems can exhibit self-similarity and

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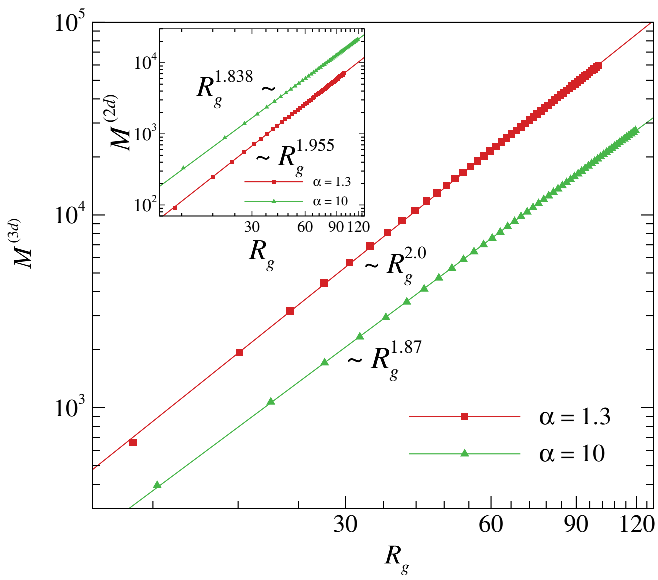

FIG. 1. (Color online) The average number of total lattice sites $M^{(3d)}$ visited at least once by the attracted random walker (ARW) (main panel), and those $M^{(2d)}$ on the attractive plane (inset), as function of their average radius of gyration for two different values of the strength of attraction $\alpha = 1.3$ (■) and $\alpha = 10$ (▲). The solid lines show the best fit to our data. The error bars are almost the same size as the symbols.

power laws without special tuning of parameters. This is why we already expect that our model would exhibit rich fractal properties for all values of α.

Let us first look at the fractal spatial structure of the $3d$ *attracted* random walk (ARW) and its intersection with the attractive plane. In order to estimate the fractal dimension d_f of the set of points visited at least once by the random walker, we examine the scaling relation between the average number of such points $M^{(3d)}$ and their corresponding radius of gyration R_g, i.e., $M^{(3d)} \sim R_g^{d_f}$. Each ensemble averaging for $M^{(3d)}$ (also for $M^{(2d)}$ in the following) and R_g was taken over 5×10^4 independent samples for a fixed number of random walk steps N. The measurements were done for $10^3 \leqslant N \leqslant 10^5$ with the number interval $\delta N = 2 \times 10^3$. We have also computed the fractal dimension of the total number of sites on the attractive plane (i.e., $M^{(2d)}$) visited by the random walker. (In this case the corresponding radius of gyration is computed for all sets of distinct visited sites only on the attractive plane. See Fig. 1)

We find that the fractal dimensions have a remarkably continuous dependence on the parameter α. The results of these fractal dimensions as function of the strength of attraction α are illustrated in Fig. 2. As can be seen from Fig. 2, for large values of α, since the problem reduces to the $2d$ random walk on the attractive plane, these two fractal dimensions converge to the same value close to the value $\sim$1.83. (This is comparable with the fractal dimension of the set of distinct sites visited by a $2d$ random walk (RW) on a square lattice, deduced from the results reported in [12].)

All error bars in this paper are estimated using standard least-squares analysis and are almost the same size as the symbols used in the figures.

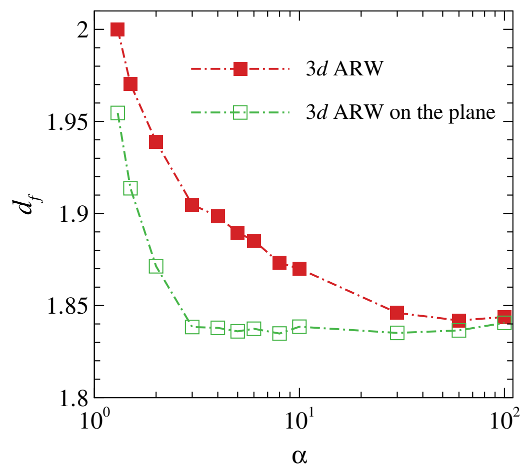

FIG. 2. (Color online) The fractal dimension of the set of all lattice points visited at least once by the attracted random walker (ARW) (■), and the set of all number of visited points on the attractive plane (□), as function of the strength of attraction α. The error bars are almost the same size as the symbols.

For an ideal linearly self-similar fractal of dimension d_f, one expects the fractal dimension of the intersection to be $d'_f = d_f - 1$ [3]. This is not apparently the case for $\alpha \neq 1$, since in our model the attractive plane has disturbed the homogeneity of the probability distribution in the z direction. Only for $\alpha = 1$ where $d_f = 2$[1] do we find $d'_f = 1 = d_f - 1$.

IV. CLUSTER SIZE DISTRIBUTION ON THE ATTRACTIVE PLANE

Henceforth, we investigate the fractal and scaling properties of the set of all distinct sites visited by the $3d$ ARW only on the attractive plane. Each of these sites is visited at least once by the $3d$ ARW and marked upon visiting (if not already visited and marked).

In this section, rather than analyzing the properties of the whole set after marking all visited sites on the plane, we identify with a specific color each cluster site as a set of all nearest-neighbor visited sites on the lattice. Two typical examples of such clustering are shown in Fig. 3 for two different values of the strength of attraction $\alpha = 2$ and $\alpha = 10$. As Fig. 3 shows, for lower values of α, there exist many isolated clusters of different scales that are accessed by the ARW only via the third dimension. By increasing the strength of the attraction, the number of isolated clusters decreases until $\alpha \to \infty$, for which there will be only one large cluster on the attractive plane.

To examine the possible scale invariance of cluster ensembles for small values of α, we compute the cluster size distribution and check whether it follows a power-law scaling. In critical statistical physics, the scaling properties of fractal

[1]The random walk on a simple cubic lattice is a *transient* process, since it has a finite escape probability of $\approx$0.66. Therefore, the number of distinct visited sites by the random walker is almost the same as the number of steps or (equivalently) the trajectory length, and thus it is expected for both to have the same fractal dimension 2.

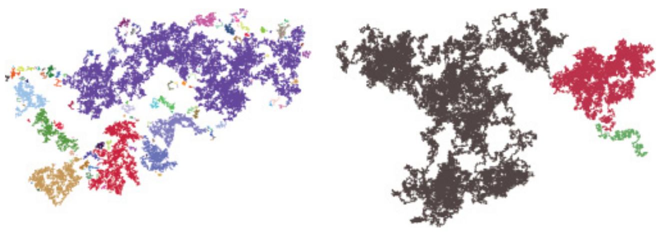

FIG. 3. (Color online) Typical samples of clusters of the visited sites on the attractive plane by a $3d$ ARW of $N = 10^6$, shown in different colors, for $\alpha = 2$ (left) and $\alpha = 10$ (right).

clusters can be described by the percolation theory [13], where the asymptotic behavior of cluster distribution $n_s(\lambda)$ near the critical point $\lambda \to \lambda_c$ has the general form

$$n_s(\lambda) = s^{-\tau} F[(\lambda - \lambda_c)s^{\sigma}], \quad (1)$$

where σ is a critical exponent and the scaling function $F(u)$ approaches a constant value for $|u| \ll 1$ and decays quickly for $|u| \gg 1$.

We undertook simulations for several values of α to measure the distribution of the cluster sizes of the visited lattice sites by the $3d$ ARW on the attractive plane (this is the probability that a visited lattice site on the attractive plane belongs to a cluster of size s). We gathered ensembles of a number (5×10^4 for smaller α and 1.5×10^6 for larger values of α) of independent samples of fractal patterns with marked-visited sites on the attractive plane. The number of random-walk steps was chosen to be $N = 4 \times 10^6$ in all simulations. The number density n_s of clusters of size s was then computed for each specific value of α by counting the number of clusters of size s divided by the total number of all clusters.

We find that for small- and intermediate-scale clusters, the distribution shows a power-law behavior compatible with the scaling relation in Eq. (1). As can be seen in the inset of Fig. 4, the curves for different values of α exhibit a sharp drop-off, indicating that they indeed contain only small clusters. By increasing α, the interval for the scaling region decreases and a peak appears, which signals the formation of large-scale clusters.

Our estimation of the cluster size distribution exponent τ in the scaling region as a function of α is also shown in Fig. 4. One observes that the exponent τ has a significant dependence on the strength of attraction α.

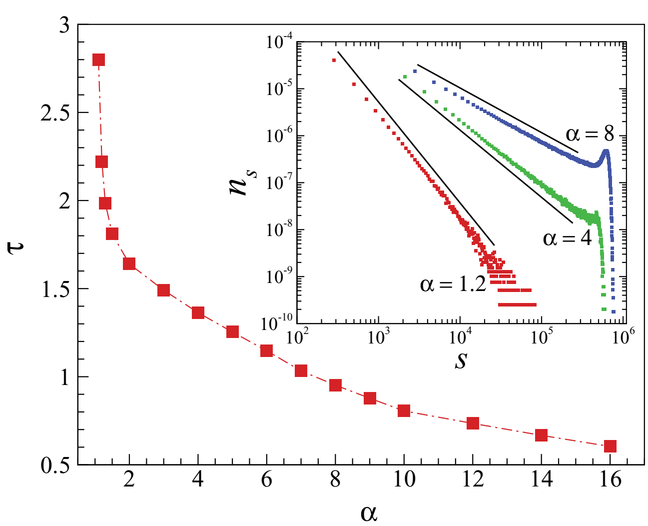

FIG. 4. (Color online) Cluster size distribution exponent τ defined in Eq. (1), as a function of the strength of attraction α. Inset: number density n_s of clusters of the visited lattice sites of size s on the attractive plane for three different values $\alpha = 1.2$, 4, and 8. The solid lines show the power-law behavior in the scaling region. The error bars are almost the same size as the symbols.

V. FRACTAL DIMENSION OF THE CLUSTER BOUNDARIES ON THE ATTRACTIVE PLANE

The remainder of this paper is dedicated to investigating the fractal properties of the boundaries of the visited-sites clusters on the attractive plane.

Given a configuration of visited sites by the $3d$ ARW on the attractive plane, the first step is to identify different clusters, as outlined before. After that, the boundary curve of each isolated cluster has to be identified. However, as the definition of interfaces and cluster boundaries on a square lattice can contain some ambiguities, there has been introduced a well-defined *tie-breaking* rule in [14] that generates nonintersecting cluster boundaries on a square lattice without any ambiguity.

To define the hull for each identified cluster according to the algorithm defined in [14], a walker (which, of course, has to be distinguished from the $3d$ ARW) moves clockwise along the edges of the dual lattice (which is also a square lattice) around the cluster starting from a given boundary edge on the cluster. The direction at each step is always chosen such that walking on the selected edge leaves a visited site on the

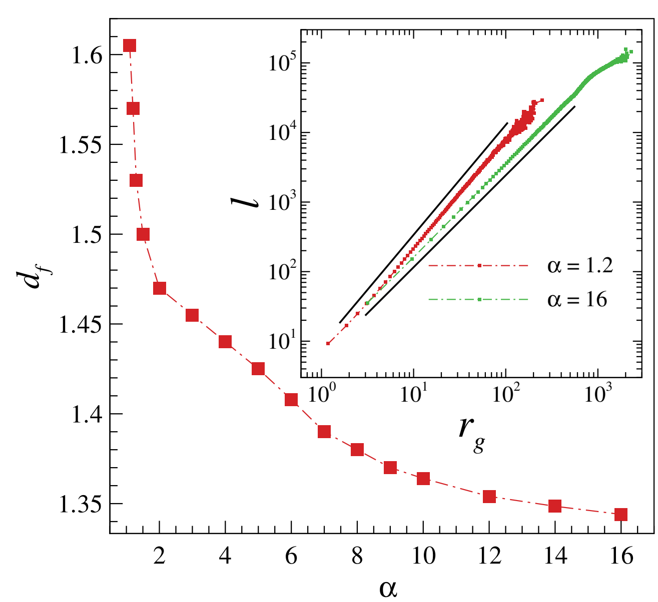

FIG. 5. (Color online) The fractal dimension of the perimeter of a cluster of visited sites on the attractive plane by $3d$ ARW as a function of the strength of attraction α. Inset: the average length of the perimeter l of a cluster versus its average radius of gyration r_g, for two different strengths of attraction, $\alpha = 1.2$ (upper graph) and $\alpha = 16$ (lower graph). The solid lines show the power-law behavior in the scaling region. The error bars are almost the same size as the symbols.

right and an empty plaquette on the left of the walker. If there are two possible ways of proceeding, the preferred direction is to the right of the walker. The directions *right* and *left* are defined locally according to the orientation of the walker.

According to this procedure, we have generated an ensemble of cluster-boundary loops for several different strengths of attraction in the range $1.1 \leqslant \alpha \leqslant 16$. Using the scaling relation $l \sim r_g^{d_f}$ between the average length of the perimeter of the loops l and their average radius of gyration r_g, we computed the fractal dimension d_f of the cluster boundaries as a function of α. The results are shown in Fig. 5.

The fractal dimension again shows a significant dependence on the strength of attraction α. In the limit $\alpha \to \infty$, d_f converges to the value $\frac{4}{3} = 1.3\bar{3}$, which is the fractal dimension of the SAW, i.e., the outer perimeter of the planar Brownian motion.

VI. CONCLUSIONS

In this paper, we have studied the scaling properties and the fractal structure of the lattice sites visited by a Brownian particle in $3d$ which is attracted by a plane with strength α. The fractal dimensions of the set of sites visited by the $3d$ random walker in both three dimensions and on the attractive plane are computed, which both converge to the same value of $\sim$1.83 for a large α. We also found that the size distribution of the cluster of visited sites by the particle on the attractive plane has a scaling form characterized by an exponent that depends significantly on the strength of attraction.

The fractal dimension of the surrounding loops of the clusters on the plane has been computed as a function of α. This also converges asymptotically to the expected value for SAW, i.e., the external perimeter of a planar Brownian motion.

These results, however, need some theoretical framework and mathematical proof. Another interesting feature for future investigation is the possible conformal invariance of the cluster boundaries on the attractive plane, which can be treated using SLE techniques (such study is already done only for the limiting case $\alpha \to \infty$ where the problem reduces to a $2d$ random walk in the attractive plane whose boundary is described by $\text{SLE}_{8/3}$). The fractal dimension of an SLE_κ curve is given by $d_f = 1 + \kappa/8$. In our model, when cluster boundaries on the attractive plane are conformally invariant, they would be defined by a diffusivity κ, which depends on the strength of attraction.

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