Control of solitons

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A method for adiabatic control of envelope solitons in the driven nonlinear Schrödinger equation is developed. The approach is based on the autoresonant effect, when the soliton is captured ("phase locked") by a two-phase resonant driving with slowly varying frequencies. Threshold conditions for amplitudes and variation rates of the driving required for the control of both the amplitude and the velocity of the soliton are found. Numerical simulations demonstrate that the method allows one to control solitons for a long time according to a given scenario, while the threshold conditions are fulfilled locally.

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I. INTRODUCTION

Envelope solitons are fundamental objects in physics of nonlinear waves. The envelope solitons are studied experimentally in different materials [1,2] and they have important applications in optical communications [3,4]. A principal problem in this context is how to excite solitons with predefined parameters and control soliton dynamics. Several approaches to this problem have been reviewed in Ref. [5]. In this paper we will use another approach, which is based on the effect of autoresonance [6,7].

The main idea of autoresonance dates back to Refs. [8,9] for the acceleration of relativistic particles. The adequate theory of the effect was proposed in Ref. [10] for the simplest model of a nonlinear pendulum. The autoresonance occurs in nonlinear systems under periodic driving with *slowly varying* parameters (usually, the driving frequencies). If the phase of oscillations of the system is captured by the drive in the vicinity of the resonance("phase locking"), and the amplitude of the driving exceeds some threshold value, the phase locking will be preserved for a long time allowing one to control the amplitude of oscillations by varying the driving frequency. A wide range of applications of the autoresonance associated with the nonlinear oscillator is discussed in Ref. [11] for plasmas, vortex, and planetary dynamics. At present, autoresonance has been applied in optical couplers [12], superconducting Josephson resonators [13], driven Bose-Einstein condensates [14], and excitation of antiproton plasmas [15]. Mathematical aspects of the problem have been reviewed in Ref. [16]. The idea of autoresonance was first applied to the control of nonlinear periodic waves [17] and solitons [18] in the Korteweg-de Vries equation. The main result of application of the autoresonance to nonlinear wave systems is excitation of waves to high amplitudes by a small resonant drive starting from a background noise. In the case of the nonlinear Schrördinger (NLS) equation, it was demonstrated in Refs. [6,7] for both one-phase and multiphase driving. A specific problem is the autoresonant control of already existing high amplitude solitons. For the one-phase driving, this problem has been studied for the sine-Gordon [19] and NLS equations [20].

In this paper we will study the autoresonant control of the solitons in the perturbed NLS equation

$$iu_t + \tfrac{1}{2}u_{xx} + |u|^2 u = \varepsilon f(x,t), \tag{1}$$

which is often used to describe dynamics of the envelope solitons in various areas of nonlinear wave physics. The perturbation in the right-hand side of Eq. (1) is supposed to be small ($0 < \varepsilon \ll 1$) and have a form of the periodic *two-phase* driving

$$f(x,t) = e^{i\psi(t)}\big(1 + g\, e^{ik[x-X(t)]}\big), \tag{2}$$

where ψ and X are given functions of time so that the frequency $\psi_t = \Omega(t)$ and the velocity $X_t = U(t)$ of the second phase are slowly varying functions: $\Omega_t = O(\alpha_1)$, $U_t = O(\alpha_2)$, $|\alpha_{1,2}| \ll 1$. We will assume that the small rates α_i are of the order of ε.

Starting from Ref. [21], a similar problem was actively studying for a damped and driven case with constant frequencies ($\alpha_1 = \alpha_2 = 0$) [22–25]. It was found that the soliton can be phase locked only if the dissipation coefficient was small enough in comparison with the amplitude of the driving ε. The phase locking without dissipation was studied in detail in Ref. [26] for solitons in asymmetric twin-core optical fibers. The problem is to preserve the phase locking while the frequency of the driving varies. If the phase locking is preserved, the frequency of the soliton (and, accordingly, its amplitude) should follow the frequency of the driving, which allows one to control the soliton amplitude by the drive. To find allowable variations of the driving frequency (i.e., threshold conditions on the rates α_1,α_2 in our case) when the phase locking occurs is the subject of the theory of autoresonance. In this paper we restrict ourselves with a dissipationless case because it is known that a small dissipation will preserve the main features of the autoresonance [27–29] (the linear dissipation coefficient should be less than $\varepsilon/2$ [16]).

An alternative method to excite NLS solitons was proposed in Ref. [30]. The approach did not associate with the phase locking and occurred when the varying driving frequency crossed the resonance frequency. A similar approach for a nonlinear pendulum was also discussed in Ref. [10] and was associated with the general nonlinear resonant phenomena [31], which allowed one to excite solitons up to amplitudes $O(\sqrt{\varepsilon})$ only. In contrast, autoresonance is a much more

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efficient method which allows one to excite solitons with high amplitudes $O(1)$ [6,16].

If $\varepsilon = 0$, the NLS equation has the soliton solution

$$u = \varphi_s(x,t) = \frac{A}{\cosh z} e^{i\Phi}, \quad \Phi = \frac{V}{A} z + \theta, \quad z = A(x - \xi), \tag{3}$$

where

$$\xi = Vt + \xi_0, \quad \theta = \omega t + \theta_0, \quad \omega = \frac{A^2 + V^2}{2}. \tag{4}$$

Here A and V are the amplitude and the velocity of the soliton. They are free parameters, which completely define the shape and the frequency ω of the soliton. The objective of the control is to vary the soliton parameters according to a given scenario by means of a driving. This problem was partially solved in Ref. [20], where it was shown that one of the soliton parameters, the frequency ω, could be controlled by a one-phase driving. In this paper we show that both soliton parameters A and V can be effectively controlled separately by a small two-phase driving of the form (2). Variations of $\Omega(t)$,$U(t)$ allow us to control the soliton parameters A and V via the autoresonant effect. The simplest dependence that will be used in the following is

$$\Omega = \Omega_0 + \alpha_1 t, \quad U = U_0 + \alpha_2 t. \tag{5}$$

The paper is organized as follows. In Sec. II we find equations for variations of the soliton parameters by the small two-phase driving of the form (2) with varying frequencies. In Sec. III we reduce the problem to a model of coupled pendulums and find the threshold conditions on parameters of the driving when the phase locking of the solitons occurs. Section IV deals with the numerical simulation of the autoresonance in the initial model (1). We show that the phase locking can be preserved for a long time, which allows us to considerably modify the soliton parameters by varying driving frequencies according to a given scenario.

II. EQUATIONS FOR SOLITON PARAMETERS

An external perturbation in Eq. (1) generates a background wave $\chi(x,t) \sim \varepsilon$. Since its amplitude is small, it may be described by the linear equation

$$i\chi_t + \tfrac{1}{2}\chi_{xx} = \varepsilon\, f(x,t). \tag{6}$$

Studying soliton dynamics, it is convenient to separate the localized and nonlocalized background parts of the solution:

$$u(x,t) = \varphi(x,t) + \chi(x,t), \tag{7}$$

where $\varphi(x,t) \to 0$ ($|x| \to \infty$). For a small perturbation, the localized component represents a slightly distorted soliton profile $\varphi(x,t) \sim \varphi_s(x,t) + O(\varepsilon)$ and satisfies the perturbed NLS equation [20]

$$i\varphi_t + \tfrac{1}{2}\varphi_{xx} + |\varphi|^2\varphi = -\chi^*\varphi^2 - 2\chi|\varphi|^2, \tag{8}$$

where the perturbation in the right-hand side is localized spatially on the soliton.

Equation (8) can be written in the variational form

$$\delta \iint \mathcal{L}(x,t)\,dx\,dt = 0, \tag{9}$$

with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}[i(\varphi\varphi_t^* - \varphi^*\varphi_t) + |\varphi_x|^2 - |\varphi|^4] - |\varphi|^2(\varphi\chi^* + \varphi^*\chi). \tag{10}$$

We will assume that the effect of the wave $\chi(x,t)$ on the soliton is reduced to a slow variation of the soliton parameters with time. This approximation is usually called adiabatic. In this approximation we can use the soliton profile φ_s, instead of φ, where A, V, ξ, and θ are now functions of time to be determined [32]. After the substitution, one obtains

$$\mathcal{L} = \frac{A^2}{\mathrm{ch}^2 z}\left[\theta_t - V\xi_t + \frac{V_t}{A}z - \frac{A^2}{\mathrm{ch}^2 z} + \frac{1}{2}(A^2 + V^2)\right] - \frac{A^3}{\mathrm{ch}^3 z}(\chi^* e^{i\Phi} + \chi e^{-i\Phi}). \tag{11}$$

Below, we will make sure that the above assumption is valid by comparing the theoretical conclusions with the results of our numerical simulation of Eq. (1).

With the Lagrangian (11), the integral over x in Eq. (9) can be calculated explicitly. One finds

$$\delta \int L\,dt = 0,$$

where

$$L = A\left(2\theta_t - 2V\xi_t + V^2 - \tfrac{1}{3}A^2\right) - A^3(Ie^{i\theta} + I^* e^{-i\theta}), \tag{12}$$

$$I = \int_{-\infty}^{\infty} \frac{e^{i(V/A)z}}{\mathrm{ch}^3 z}\chi^*(x,t)\,dx. \tag{13}$$

For the driving (2), the solution of Eq. (6) can be taken approximately [with an error of order $[O(\varepsilon\alpha_1 + \varepsilon\alpha_2)]$ as

$$\chi(x,t) = -\frac{\varepsilon}{\Omega} e^{i\psi}\,(1 + G\,e^{ik(x-X)}), \tag{14}$$

where

$$G = \frac{g\,\Omega}{\Omega - kU + k^2/2}. \tag{15}$$

Using the formula

$$F(\mu) = 2\int_{-\infty}^{\infty} \frac{e^{i\mu z}}{\cosh^3 z}\,dz = \frac{\pi(1+\mu^2)}{\cosh\frac{\pi\mu}{2}}, \tag{16}$$

we can calculate the integral (13):

$$I = -\frac{\varepsilon}{2A\Omega}e^{-i\psi}\left\{F\left(\frac{V}{A}\right) + Ge^{-ik(\xi - X)}F\left(\frac{V-k}{A}\right)\right\}.$$

Thus, the Lagrangian takes the final form

$$L = 2A\left(\delta_t + \Omega - \frac{V}{k}\phi_t - VU + V^2 - \frac{1}{6}A^2\right) + \frac{\varepsilon A^2}{\Omega}\left\{F\left(\frac{V}{A}\right)\cos\delta + G\,F\left(\frac{V-k}{A}\right)\cos(\delta - \phi)\right\}, \tag{17}$$

where we introduced differences of phases of the soliton and the driving

$$\delta = \theta - \psi, \quad \phi = k(\xi - X). \tag{18}$$

As a result, the Euler-Lagrange equations for variations of the soliton parameters take the form

$$\delta_t = \Delta\omega - \frac{\varepsilon A}{\Omega}\left\{\left[F_1 - \frac{V}{A}F_1'\right]\cos\delta + G\left[F_2 - \frac{V-\frac{k}{2}}{A}F_2'\right]\cos(\delta-\phi)\right\}, \tag{19}$$

$$\phi_t = k(V-U) + \frac{\varepsilon k}{2\Omega}\{F_1'\cos\delta + GF_2'\cos(\delta-\phi)\}, \tag{20}$$

$$A_t = -\frac{\varepsilon A^2}{2\Omega}\{F_1\sin\delta + GF_2\sin(\delta-\phi)\}, \tag{21}$$

$$V_t = \frac{\varepsilon A}{2\Omega}\{VF_1\sin\delta + G(V-k)F_2\sin(\delta-\phi)\}, \tag{22}$$

where we introduce

$$\Delta\omega = \omega - \Omega, \quad F_1 = F\left(\frac{V}{A}\right), \quad F_2 = F\left(\frac{V-k}{A}\right), \tag{23}$$

and the prime denotes the derivative of the function with respect to its argument. The system (19)–(22) describes the evolution of the soliton parameters under the action of the small perturbation (2).

III. A MODEL OF TWO COUPLED NONLINEAR PENDULUMS

It is convenient to rewrite the system (19)–(22) in the vector form

$$\mathbf{x}_t = \Delta\mathbf{\Omega} + \varepsilon\mathcal{F}, \tag{24}$$

where

$$\mathbf{x} = (\delta,\phi,A,V)^T, \quad \Delta\mathbf{\Omega} = [\Delta\omega,k(V-U),0,0]^T, \quad \mathcal{F}(\Omega,U,\mathbf{x}) = (\mathcal{F}^\delta,\mathcal{F}^\phi,\mathcal{F}^A,\mathcal{F}^V)^T. \tag{25}$$

In the vicinity of the resonance $\Delta\mathbf{\Omega} \sim O(\sqrt{\varepsilon})$ the soliton parameters should be slow functions of time. We introduce the "slow" time

$$\tau = \sqrt{\varepsilon}t$$

and consider the special asymptotic solution

$$\mathbf{x} = \mathbf{x}_0 + \sqrt{\varepsilon}\mathbf{x}_1 + \varepsilon\mathbf{x}_2 + \cdots, \quad \mathbf{x}_i = \mathbf{x}_i(\tau). \tag{26}$$

The slow parameters of the driving Ω and U will also be functions of τ. Supposing the linear dependence (5), one writes

$$\Omega = \Omega_0 + \sqrt{\varepsilon}\,\beta_1\tau, \tag{27}$$

$$U = U_0 + \sqrt{\varepsilon}\,\beta_2\tau, \tag{28}$$

where we introduced

$$\beta_1 = \alpha_1/\varepsilon, \quad \beta_2 = \alpha_2/\varepsilon. \tag{29}$$

Because the rates α_i were assumed to be small values of the order of ε, then $\beta_i \sim O(1)$.

The vector $\Delta\Omega$ will be the power series

$$\Delta\mathbf{\Omega} = \Delta\mathbf{\Omega}_0 + \sqrt{\varepsilon}\Delta\mathbf{\Omega}_1 + \varepsilon\Delta\mathbf{\Omega}_2 + \cdots, \tag{30}$$

where components of the coefficients can be found using Eqs. (26)–(28):

$$\Delta\Omega_0^\delta = \frac{A_0^2+V_0^2}{2} - \Omega_0, \quad \Delta\Omega_0^\phi = k(V_0-U_0), \tag{31}$$

$$\Delta\Omega_1^\delta = A_0A_1 + V_0V_1 - \beta_1\tau, \quad \Delta\Omega_1^\phi = k(V_1-\beta_2\tau), \tag{32}$$

$$\Delta\Omega_2^\delta = A_0A_2 + V_0V_2 + \frac{A_1^2+V_1^2}{2}, \quad \Delta\Omega_2^\phi = kV_2. \tag{33}$$

Here, the superscripts indicate the vector components in accordance with components of the vector $\mathbf{x}$, Eq. (25): $\Delta\mathbf{\Omega}_i = (\Delta\Omega_i^\delta,\Delta\Omega_i^\phi,\Delta\Omega_i^A,\Delta\Omega_i^V)^T$. One notes that the components $\Delta\Omega^A$ and $\Delta\Omega^V$ equal to zero by definition (25).

Substituting the series (26) and (30) into Eq. (24), one finds a set of vector equations for every power of $\sqrt{\varepsilon}$:

$$0 = \Delta\mathbf{\Omega}_0, \tag{34}$$

$$\mathbf{x}_{0,\tau} = \Delta\mathbf{\Omega}_1, \tag{35}$$

$$\mathbf{x}_{1,\tau} = \Delta\mathbf{\Omega}_2 + \mathcal{F}(\Omega_0,U_0,\mathbf{x}_0)\ \ldots. \tag{36}$$

Equation (34) gives

$$\frac{A_0^2+V_0^2}{2} = \Omega_0, \quad V_0 = U_0. \tag{37}$$

It means that, in the main order, the velocity and the amplitude of the soliton satisfy the resonant condition, when the frequency of the soliton coincides with the driving frequency and the soliton velocity equals the velocity of the driving.

To find equations for the phase differences $\delta_0(\tau)$ and $\phi_0(\tau)$ we differentiate Eq. (35) with respect to τ and eliminate $A_{1,\tau}$ and $V_{1,\tau}$ using Eq. (36). One finds

$$\delta_{0,\tau\tau} = A_0\mathcal{F}^A(\mathbf{x}_0) + V_0\mathcal{F}^V(\mathbf{x}_0) - \beta_1, \tag{38}$$

$$\phi_{0,\tau\tau} = k\mathcal{F}^V(\mathbf{x}_0) - k\beta_2. \tag{39}$$

Using the explicit form of $\mathcal{F}$, we rewrite the last equations in the final form:

$$\delta_{0,\tau\tau} = a\sin\delta_0 + b\sin(\delta_0-\phi_0) - \beta_1, \tag{40}$$

$$\phi_{0,\tau\tau} = c\sin\delta_0 + d\sin(\delta_0-\phi_0) - k\beta_2, \tag{41}$$

where

$$a = \frac{A_0}{2\Omega_0}F_1(V_0^2 - A_0^2), \tag{42}$$

$$b = \frac{A_0G}{2\Omega_0}F_2(V_0^2 - A_0^2 - kV_0), \tag{43}$$

$$c = \frac{kA_0V_0}{2\Omega_0}F_1, \tag{44}$$

$$d = \frac{kA_0G}{2\Omega_0}F_2(V_0-k). \tag{45}$$

Here the functions $F_{1,2}$ are taken at V_0, A_0 and G at Ω_0, U_0. A phase locking means that the phase differences δ_0 and ϕ_0 oscillate in a finite region around some mean values. It follows that the dynamical system (40), (41) should have stable stationary points (δ^*, ϕ^*).

At first, let us consider the special case. If $k = 0$ (and/or $g = 0$), the system (40),(41) reduces to the equation for the only phase δ_0:

$$\delta_{0,\tau\tau} = a(1+g)\sin\delta_0 - \beta_1, \tag{46}$$

It is a one-phase limit close to that which was studied in Ref. [20]. Equation (46) describes the dynamics of a "quasi-particle" in the potential $\mathbb{V}(\delta_0) = a(1+g)\cos\delta_0 + \beta_1\delta_0$. The stationary point δ^* is defined by the equation

$$\sin\delta^* = -\beta_1/a(1+g), \tag{47}$$

which has solutions only at

$$|\beta_1| \leqslant |a(1+g)|. \tag{48}$$

This condition is the crucial point in the theory of autoresonance. Returning to the definition (29) we conclude that the phase locking of a soliton by the slowly varying driving occurs if the rate of variation of the frequency $|\alpha_1|$ is less than some threshold value proportional to the amplitude of the driving:

$$|\alpha_1| \leqslant \alpha_{cr} = \varepsilon|a(1+g)|. \tag{49}$$

The condition differs from the case of the autoresonance of small amplitude waves [6,7,11] when $\alpha_{cr} \sim \varepsilon^{3/4}$. One notes also that studying the stability of the stationary points δ^* is trivial for the one-phase case. The stable points are located in the minima of the potential $\mathbb{V}(\delta_0)$, which always exist under the condition (48).

Let us return to the two-phase driving when $k \neq 0$ and $g \neq 0$. Equations for the stationary points follow from Eqs. (40) and (41):

$$\sin\delta^* = \frac{2\Omega_0}{kA_0^3F_1}\{\beta_1(V_0-k) - \beta_2(V_0^2 - A_0^2 - kV_0)\}, \tag{50}$$

$$\sin(\delta^*-\phi^*) = \frac{2\Omega_0}{kA_0^3F_2G}\{\beta_2(V_0^2-A_0^2) - \beta_1V_0\}. \tag{51}$$

It is obvious that they have solutions only if the absolute values of the right-hand sides of Eqs. (50) and (51) are less than a unit. For the given parameters of the soliton A_0, V_0 and the driving k, g, these conditions define a rhombus in the plane (β_1, β_2). The phase locking can occur only for the values (β_1, β_2) located inside the rhombus (see Fig. 1). These conditions extend the restriction (48) to the rates α_1, α_2 of the two-phase driving.

It is convenient to note another point of view to the threshold conditions. Let us fix β_1, β_2 and find the structure parameters of the driving k, g when Eqs. (50) and (51) have solutions, i.e., when the phase locking of a given soliton with A_0, V_0 can occur. The equation $|\sin\delta^*| = 1$ with respect to k has two roots,

$$k_{1,2} = \frac{\beta_1V_0 - \beta_2(V_0^2 - A_0^2)}{\pm\frac{A_0^3F_1}{2\Omega_0} + \beta_1 - \beta_2V_0}.$$

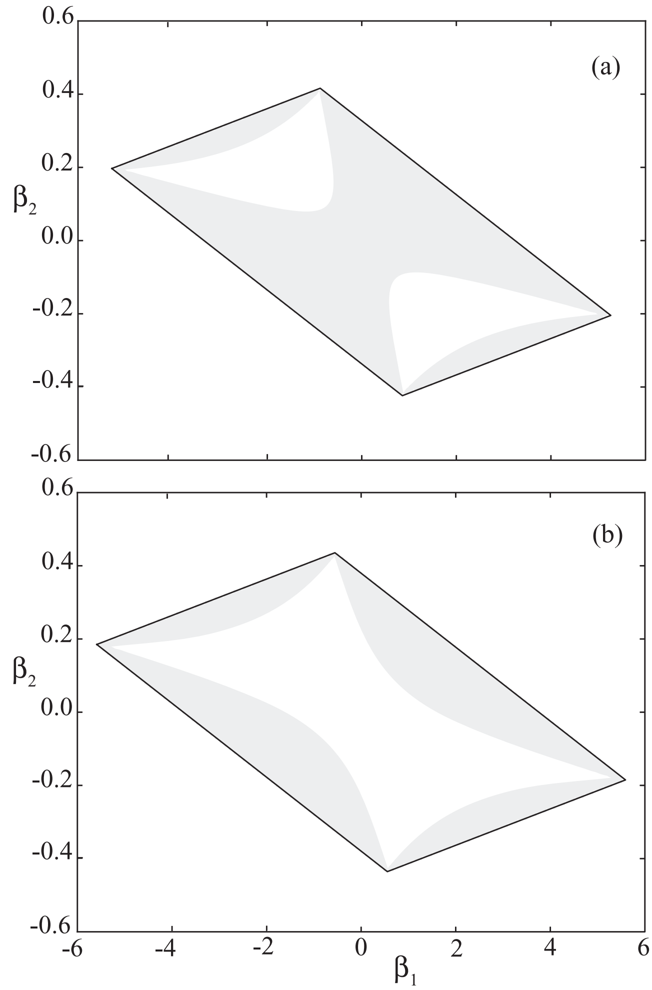

FIG. 1. The regions of the phase locking (gray domains): (a) – $g = 0.7$, (b) – $g = 0.8$; $g^{(1)} = 0.728$, $g^{(2)} = 1.325$; $A_0 = 1$, $V_0 = 0.1$, $k = 0.15$.

Sorting the roots so that $k_1 < k_2$, one finds that Eq. (50) has a solution for special values of the wave number k:

$$k_1 < k < k_2, \quad \text{if} \quad |2\Omega_0(\beta_2V_0 - \beta_1)/A_0^3F_1| > 1, \tag{52}$$

$$k < k_1, k > k_2, \quad \text{if} \quad |2\Omega_0(\beta_2V_0 - \beta_1)/A_0^3F_1| \leqslant 1. \tag{53}$$

The solvability of Eq. (51) gives

$$|g| \geqslant g_{cr} = \frac{2|\Omega_0 - kV_0 + \frac{k^2}{2}|}{kA_0^3F(\frac{V_0-k}{A_0})}|\beta_2(V_0^2 - A_0^2) - \beta_1V_0|. \tag{54}$$

Now we will study the stability of the stationary points. The standard linear analyses in the vicinity of the point (δ^*, ϕ^*) give the following characteristic equation:

$$\lambda^4 - p\lambda^2 + q = 0, \tag{55}$$

where

$$p = a\cos\delta^* + (b-d)\cos(\delta^*-\phi^*), \tag{56}$$

$$q = (bc - ad)\cos\delta^*\cos(\delta^*-\phi^*). \tag{57}$$

If all four roots are pure imaginary $\lambda = i\,\nu$, $\nu \in \mathbb{R}$, a stationary point will be stable. The conditions for that are

$$p < 0, \quad 0 < q < \frac{p^2}{4}. \tag{58}$$

These conditions restrict the region of the phase locking in the (β_1,β_2) plane inside the rhombus (gray regions in Fig. 1). One notes that the line $q = 0$ coincides with the boundaries of the rhombus because $\cos\delta^* \cos(\delta^* - \phi^*) = 0$ just in the boundaries. There are two qualitatively different types of behavior of the stability region depending on the parameter g [see Figs. 1(a) and 1(b)]. In the second case, which is always observed in a finite range $g^{(1)} < g < g^{(2)}$, the soliton cannot be phase locked by the steady drive with $\beta_1 = \beta_2 = 0$. This gap disappears when $|k| \ll |V_0|$. For any parameters, the phase locking always exists in the vicinity of the rhombus boundaries (see Fig. 1).

IV. NUMERICAL SIMULATION

In this section we present a detailed numerical simulation of the control of solitons in Eq. (1) by the autoresonant effect. The initial soliton was specified by its amplitude A_0 and velocity V_0. The driving at $t = 0$ was in resonance with the soliton according to Eq. (37). The phases θ_0, ξ_0 were consistent with initial phases of the driving $\psi(0)$, $X(0)$ so that the relevant phase differences (18) resided in the vicinity of the stationary point (δ^*,ϕ^*) defined in Eqs. (50) and (51). We have used the standard Fourier transform method (see, e.g., [33]) to solve Eq. (1) in an interval with the length much greater than the width of the initial soliton ($\sim 1/A_0$) to prevent the effects of boundaries on the dynamics of the soliton.

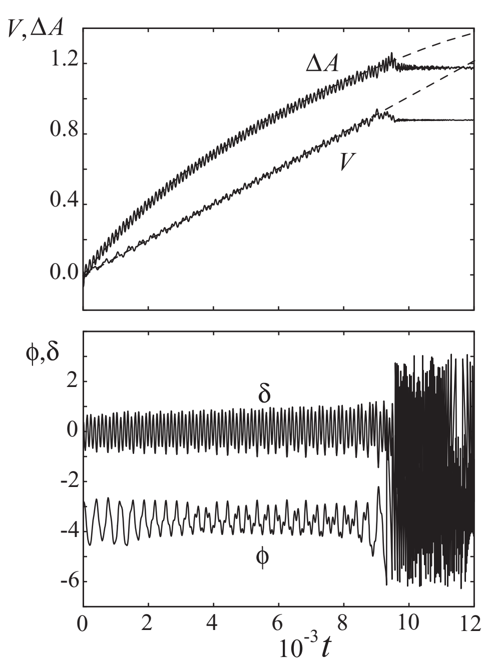

FIG. 2. Numerical simulation of NLS equation (1) for $g = 0.25 > g_{cr}$, where $g_{cr} = 0.08$. Other parameters: $\varepsilon = 10^{-3}$, $\beta_1 = 0.25, \beta_2 = 0.1$, $k = 1$, $\delta(0) = 0.5$ ($\delta^* = -0.05$), $\phi(0) = -2.8$ ($\phi^* = -2.77$), $A(0) = 1$, and $V(0) = 0$. The value ΔA is a deviation of the soliton amplitude from the initial value: $\Delta A = A - A(0)$.

Figure 2 shows a typical behavior of the soliton when driving parameters satisfy conditions of the phase locking obtained in Sec. III. It is clearly seen that the amplitude $A(t)$ and the velocity $V(t)$ of the soliton grow when parameters of the driving vary according to Eq. (5). At the same time, the phase differences $\delta(t)$,$\phi(t)$ oscillate in restricted ranges displaying phase locking of both phases.

One notes that the theory of Sec. III gives conditions for phase locking of the soliton in the initial time interval only ($t \ll \varepsilon^{-1}$). Numerical calculations demonstrate that, actually, the soliton being phase locked initially will preserve the phase locking during a long period and thus, sustain approximately the resonant conditions (37) locally in time. It allows one to considerably modify the soliton parameters according to the scenario $V \approx U(t)$ and $A \approx \sqrt{2\Omega(t) - U^2(t)}$, which is shown in Fig. 2 by dashed lines for the linear dependence of Eq. (5). The soliton amplitude and velocity grow until the threshold conditions for the phase locking are fulfilled. We observed destruction of the phase locking when the inequality (58) was broken at $t \approx 9.5 \times 10^3$. After the destruction, the soliton amplitude and velocity tend to steady levels forming a soliton which later does not depend on the small nonresonant driving.

We also observed no radiation from the soliton in the stage of phase locking if the amplitude of the soliton is large enough and the driving is far from the resonance with small amplitude background waves. The sufficient conditions follow from $|\chi| \ll A$. Using Eq. (14) we found

$$A \gg \varepsilon^{1/3}, \quad |\Omega - kU + k^2/2| \gtrsim g\varepsilon^{2/3}. \tag{59}$$

If these conditions are fulfilled, we observed that the background perturbations remain of the order of ε during the whole process, which confirmed that the adiabatic approximation was an appropriate supposition.

To verify the conditions of the phase locking in the initial model (1) we have performed simulations with different parameters of solitons and the driving. Varying the amplitude of the soliton A_0 and the driving parameter g we test the threshold condition (58). A comparison of the numerical and

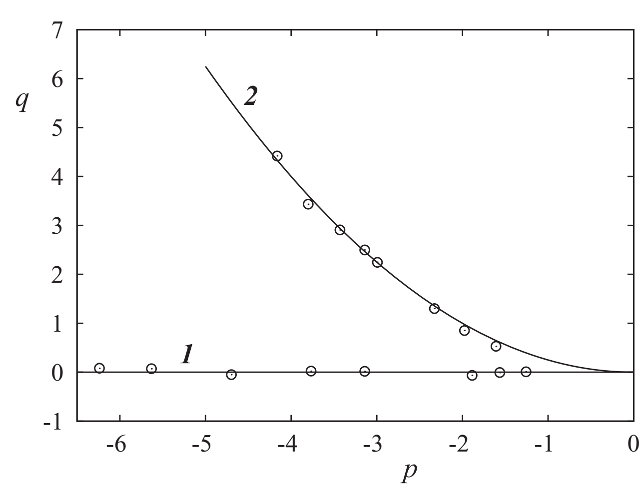

FIG. 3. The region of stability of the stationary points in the (p,q) plane is confined by lines 1 and 2. Circles mark the boundaries of the phase-locking region found numerically in NLS equation (1). $V_0 = 1$, $k = 0.1$, $\beta_1 = 1$, $\beta_2 = 0.2$, $\delta(0) = \delta^*$, and $\phi(0) = \phi^*$.

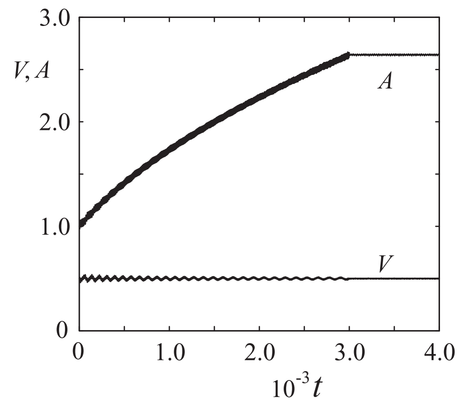

FIG. 4. Control of the amplitude A of the soliton, preserving its velocity V in NLS equation (1); $\beta_1 = 0.1, \beta_2 = 0, \varepsilon = 10^{-3}, g = 0.1, k = 1, \delta(0) = 0.5, \phi(0) = -2.82, A(0) = 1$, and $V(0) = 0$.

theoretical results is given in Fig. 3. The region of stability of the stationary points in the (p,q) plane is confined by lines 1 and 2 according to Eq. (58). It turned out that the region of phase locking of the solitons found numerically in NLS equation (1) was rather close to the theoretical values.

An important case of the control is shown in Fig. 4 when we amplify the soliton preserving its velocity. It occurs when the soliton is phase locked by the drive with $\alpha_2 = 0$. After switching off the driving at $t = 3 \times 10^3$ we observe a new steady soliton with increased amplitude and the same velocity. In contrast, the one-phase driving [20] does not allow one to control the amplitude and the velocity independently and thus, cannot preserve the velocity if $V(0) \neq 0$.

The autoresonance allows one to control the soliton when its parameters, the amplitude, and the velocity, can be varied as desired according to arbitrary variations of the driving parameters $\Omega(t),U(t)$, different from the linear dependence (5). It is needed for the control that the rates $\alpha_1 = d\Omega/dt$ and $\alpha_2 = dU/dt$ were sufficiently small at any stage of the process, i.e., conditions for the phase locking were satisfied locally in time. The control of the soliton for a complex dependence $\Omega(t),U(t)$ is demonstrated in Fig. 5. Starting from $A = 1$ and $V = 0$, the soliton parameters are varying along the circle in the (A,V) plane returning to the initial position at $t = 2 \times 10^4$. In our simulations we use periodic functions $\Omega(t),U(t)$ and, continuing the process, we can pass the circle several times. This behavior confirms stability of the control.

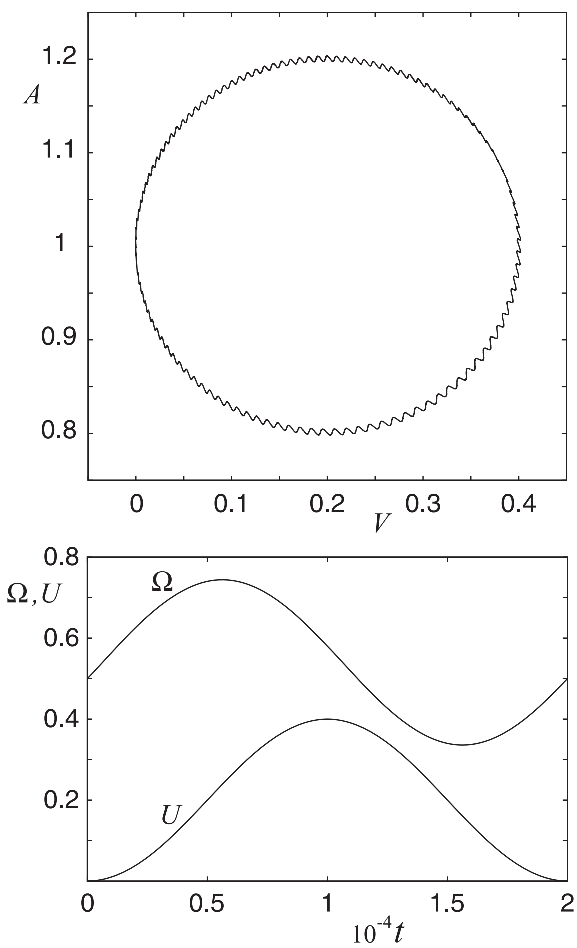

FIG. 5. A behavior of the velocity V and the amplitude A of the soliton in NLS equation (1) for the evolution of the driving parameters $\Omega(t),U(t)$ shown in the lower figure; $\varepsilon = 2 \times 10^{-4}, g = 0.1, k = 1, \delta(0) = -0.033, \phi(0) = -\pi - 0.033, A(0) = 1$, and $V(0) = 0$.

V. CONCLUSIONS

In this paper we have shown that the envelope solitons can be controlled by the resonant two-phase driving. We control both parameters of the soliton (3), the amplitude A and the velocity V, independently. Moreover, because of the phase locking of the soliton by the drive, the soliton phase θ and its coordinate ξ are adjusted to phases of the driving and thus, they are also controlled.

Let us rewrite the driving (2) in the form

$$f(x,t) = e^{i\psi(t)} + g\, e^{ikx+i\psi'(t)}, \tag{60}$$

where $\psi'(t) = \psi(t) - X(t)$. It is clearly seen that the drive is a superposition of two waves with different frequencies, $\Omega = \psi_t$, $\Omega' = \Omega - X_t$ and wave numbers. One notes that the wave number k cannot be zero if we need to phase lock both phases of the soliton. The control occurs when the frequencies of the drive slowly vary with time and the rates of these variations are limited by some thresholds defined in Sec. III. In this case the soliton will be phase locked by the drive and soliton parameters will follow after the variations of the driving frequencies to sustain the resonant conditions (37). This phenomenon is known as autoresonance. The phase locking was preserved for a long time while the threshold conditions were fulfilled, which allowed us to vary considerably the soliton parameters according to a given scenario defined by the driving parameters $\Omega(t)$ and $U(t)$.

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