# Phase-flip transition in relay-coupled nonlinear oscillators

Amit Sharma,[1] Manish Dev Shrimali,[1,*] Awadhesh Prasad,[2] Ram Ramaswamy,[3] and Ulrike Feudel[4]
[1]*The LNM Institute of Information Technology, Jaipur 302 031, India*
[2]*Department of Physics and Astrophysics, Delhi University, Delhi, 110 007, India*
[3]*School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India*
[4]*ICBM, University of Oldenburg, D-26111 Oldenburg, Germany*
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We study the dynamics of oscillators that are coupled in relay; namely, through an intermediary oscillator. From previous studies it is known that the oscillators show a transition from in-phase to out-of-phase oscillations or vice versa when the interactions involve a time delay. Here we show that, in the absence of time delay, relay coupling through conjugate variables has the same effect. However, this phase-flip transition does not occur abruptly at a certain critical value of the coupling parameter. Instead we find a parameter region around the phase-flip transition where bistability occurs. In this parameter interval in-phase and out-of-phase oscillations coexist with changing sizes of their basins of attraction. Further increase of the coupling strength leads to amplitude death and subsequently to the stabilization of a fixed point. These transitions are characterized through various quantities such as the average phase difference and crossings in the spectrum of Lyapunov exponents. Numerical results are presented for a specific case of coupled Rössler-like oscillators.

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## I. INTRODUCTION

Two themes that have separately been explored in recent studies of nonlinear dynamical systems are relay coupling [1,2] (i.e., coupling two systems via a third) and coupling systems through conjugate (or dissimilar) variables [3–5]. A primary interest in such studies has been on synchronization [6] in its many different forms, from both theoretical and experimental points of view. Depending on the nature and the strength of the coupling, different synchronization states are possible; these include complete or identical synchronization (CS) [7,8], in-phase (PS) [9], out-of-phase [10,11], lag synchronization (LS) [12], generalized synchronization (GS) [13], intermittent lag synchronization (ILS) [14], and mixed synchronization [15]. Several of these different synchronization phenomena have been explored in Rössler and Chua's oscillators [16,17], semiconductor lasers exhibiting chaotic emission on subnanosecond time scales [18], chemical reactions [19], and in biological systems such as neurons or ecological food webs [20].

When there is time delay in the coupling, a large class of nonlinear dynamical systems shows a phase-flip transition [21–24]. This transition is characterized by a change of the synchronized dynamics from being in-phase to out-of-phase or vice versa; the phase difference between the oscillators undergoes a jump of $\pi$ as a function of the coupling strength or the time delay. This phenomenon is of broad relevance as it has been observed in regimes of amplitude death, periodic, quasiperiodic, and chaotic dynamics. This particular transition (where the phase changes approximately to $\pi$) also occurs when the coupled systems are nonidentical (i.e., when there is a mismatch in the parameters) as well as when the oscillators are distinct. The examples range from coupled-laser systems to ecosystems [21].

In the present work, we study the effect of coupling via dissimilar (conjugate) variables [3,4] on the synchronization properties of indirectly coupled systems. Practical constraints often dictate the form and nature of the coupling. In particular, it is often not possible to couple systems via similar variables; for example, in coupled–semiconductor-laser experiments [25], in the study of electrical circuits [26], epidemiology [27], and many other natural systems. We show here that relay coupling involving conjugate variables leads to a rich variety of synchronization behavior.

The similarity of conjugate coupling to time-delay [3] has been explored in previous work and also underlies the process of attractor reconstruction [28] from a single time-series. By coupling systems in relay through conjugate variables, an effective time-delay is transmitted, causing different synchronization states [phase flip as well as amplitude-death (AD)]. The present studies are carried out for chaotic Rössler-like oscillators but we believe that the results are applicable more generally [29]. The transitions to different states of synchronization are studied by computing phase differences and Lyapunov exponents.

In the next section of this paper, the model of three coupled systems and the relay conjugate coupling scheme is described. This is followed by our main results on the synchronization properties, the phase-flip transition, and amplitude death in Sec. III. The paper concludes with a discussion and summary in Sec. IV.

## II. THE MODEL SYSTEM

Consider three systems coupled as in Fig. 1. The equations of motion in the conjugate coupling scheme are as follows:

$$\begin{aligned}\dot{\mathbf{X}}_1 &= f(\mathbf{X}_1) + \mathbf{K}g_1(\mathbf{X}_2',\mathbf{X}_1),\\ \dot{\mathbf{X}}_2 &= f(\mathbf{X}_2) + \mathbf{K}g_2(\mathbf{X}_1',\mathbf{X}_3',\mathbf{X}_2),\\ \dot{\mathbf{X}}_3 &= f(\mathbf{X}_3) + \mathbf{K}g_3(\mathbf{X}_2',\mathbf{X}_3),\end{aligned} \tag{1}$$

*Present address: Indian Institute of Technology Rajasthan, Jodhpur, 342 011, India.

 

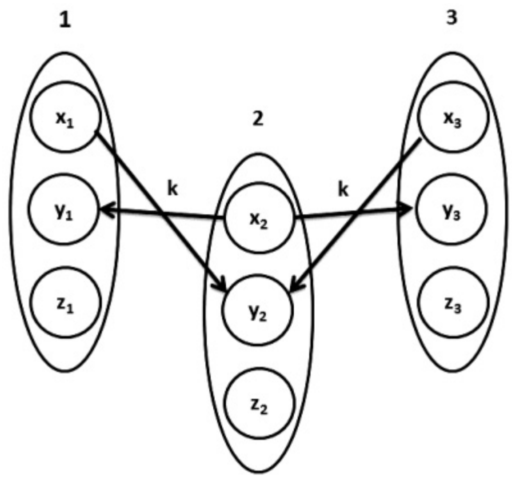


FIG. 1. Coupling scheme in the present model.

where $\mathbf{X}_i$ denotes the set of $m$ dynamical variables of the $i$th oscillator. The matrix $\mathbf{K}$ of dimension $m \times m$ contains information on the coupling topology and the $g_i$ are coupling functions. The superscript primes ($'$) on $\mathbf{X}$ denotes conjugate coupling; namely, that the arguments of the functions $g$ involve "dissimilar" variables. Note that the coupling is bipartite in that the first and third oscillator are coupled only to the second oscillator and are thus indirectly coupled to each other.

Each of the oscillators in Fig. 1 is a nonlinear dynamical system and, here, for definiteness, we take them to be Rössler-like [1]:

$$\begin{aligned} \dot{x_i} &= -\left[\omega_i + \epsilon\left(x_i^2 + y_i^2\right)\right] y_i - z_i, \\ \dot{y_i} &= \left[\omega_i + \epsilon\left(x_i^2 + y_i^2\right)\right] x_i + a y_i, \\ \dot{z_i} &= b + z_i(x_i - c). \end{aligned} \tag{2}$$

The modified Rössler system reduces to the usual Rössler model [30] when $\epsilon = 0$ but has a two-band chaotic attractor, shown in Fig. 2. The three oscillators are all taken to be identical, and the coupling is such that the only the equation for the $x_i$ variables is affected:

$$\dot{x_i} = -\left[\omega_i + \epsilon\left(x_i^2 + y_i^2\right)\right] y_i - z_i + \Sigma_{j=1}^{3} K_{ij}(y_j - x_i), \tag{3}$$

with $i = 1,2,3$. The matrix elements are $K_{ii} = K_{13} = 0$, $K_{12} = K_{23} = k/2$, and $K_{ij} = K_{ji}$. The coupling is diffusive; namely, the function $g$ is linear in its arguments. The coupling variables are dissimilar: $x_i$ is coupled to the conjugate variable $y_j$. The term $[\omega_i + \epsilon(x_i^2 + y_i^2)]$ in the modified Rössler equations is almost equal to the angular velocity of the $i$th oscillator and is perturbed by the chaotic amplitude $x_i^2 + y_i^2$ if $\epsilon \neq 0$. It has been shown in numerous studies that the dynamics of the usual Rössler oscillator ($\epsilon = 0$) is oscillatory over a range of the parameter $a$, $b$, and $c$ and can be also chaotic [30].

Lyapunov exponents for the coupled system are computed in the usual way [31]. To compute the oscillator phases, though, we use the concept of the analytical signal [32] to define the

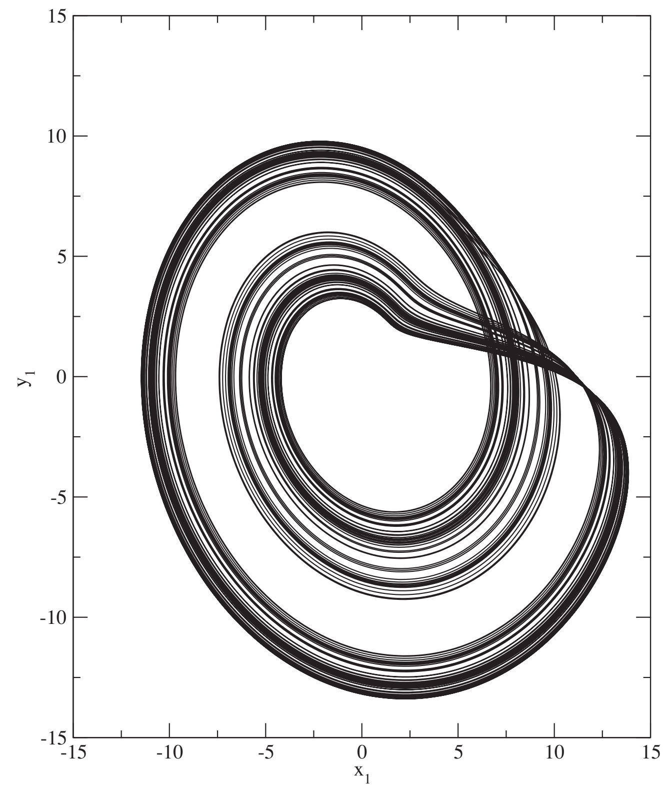


FIG. 2. Attractor of the modified Rössler model [Eq. (2)] with $\epsilon = 0.0026$ and $\omega_1 = 0.41$.

amplitude and the phase of an arbitrary variable $s(t)$. The analytic signal $\psi(t)$ is the complex function

$$\psi(t) = s(t) + \iota\tilde{s}(t) = R(t)e^{\iota\phi(t)}, \tag{4}$$

where the function $\tilde{s}(t)$ is the Hilbert transform of $s(t)$. The instantaneous amplitude $R_i(t)$ and the instantaneous phase $\phi_i(t)$ of the variable $s_i(t)$ of the $i$th oscillator can be defined as

$$\begin{aligned} R_i(t) &= \sqrt{s_i(t)^2 + \tilde{s}_i(t)^2}, \\ \phi_i &= \tan^{-1}\left[\tilde{s}_i(t)/s_i(t)\right], \end{aligned} \tag{5}$$

respectively. The phases of the individual oscillators are constructed from the variables $x_i(t)$, and the average phase difference $\Delta\phi_{ij}$ between two oscillators is

$$\Delta\phi_{ij} = \langle|\phi_i - \phi_j|\rangle \quad \text{for} \quad i,j = 1,2,3, \tag{6}$$

where $\langle\cdot\rangle$ denotes the time average.

## III. RESULTS

Our focus here is on the phase dynamics which we study in detail in the parameter space spanned by $\epsilon$, the magnitude of the perturbation, and the coupling strength $k$. We take all oscillators to be identical, $\omega_1 = \omega_2 = \omega_3 = 0.41$, and the other parameter values are fixed at $a = 0.15$, $b = 0.4$, $c = 8.5$. Before outlining the results for the three coupled systems let us briefly recall the dynamics for limiting cases.

For the uncoupled standard Rössler system $k = \epsilon = 0$, the dynamics are chaotic. Changing $k$ for $\epsilon = 0$ leads to phase synchronization and a phase-flip transition can be obtained at

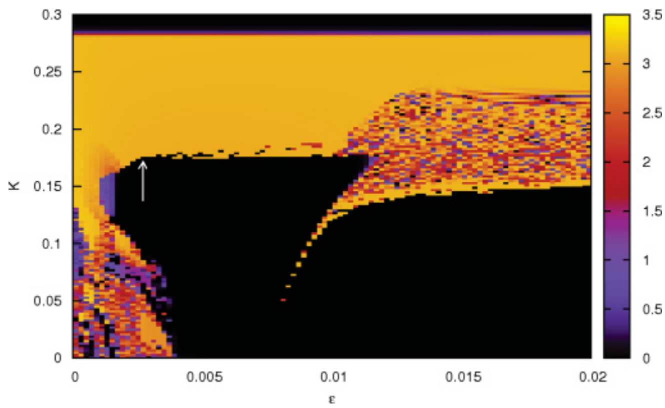


FIG. 3. (Color online) Phase difference between the first and third oscillators $\Delta\phi_{13}$ in parameter space $\epsilon$ and $k$. There is a sharp phase flip for $\epsilon \neq 0$ at a critical value of $k$ where systems 1 and 3 go from in-phase synchrony to antiphase synchrony, while the central oscillator undergoes amplitude death (marked by the arrow at $k_c = 0.177$ for $\epsilon = 0.0026$).

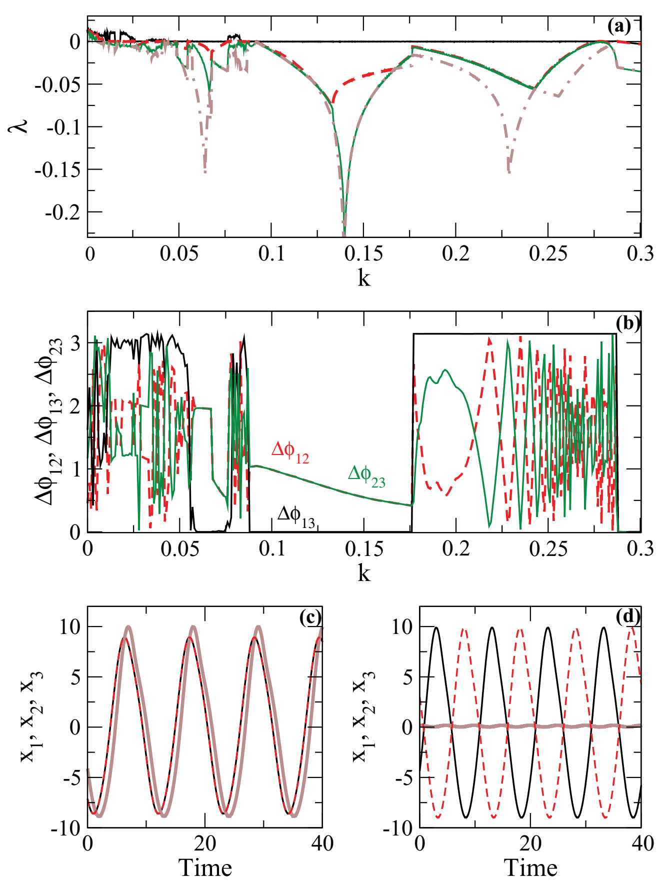


FIG. 4. (Color online) (a) Spectrum of Lyapunov exponents for three coupled modified Rössler systems. (b) The phase difference between any two oscillators ($\Delta\phi_{ij}$) as a function of the coupling strength $k$. At the phase flip $\Delta\phi_{13}$ (in black) jumps from 0 to $\pi$. In-phase and out-of-phase dynamics of first (black) and third (red dashed line) oscillators (c) before and (d) after the transition at $k = 0.16$ and 0.19, respectively. For (a) and (b), the averaging is done over $10^6$ time steps after removing transients of the same duration.

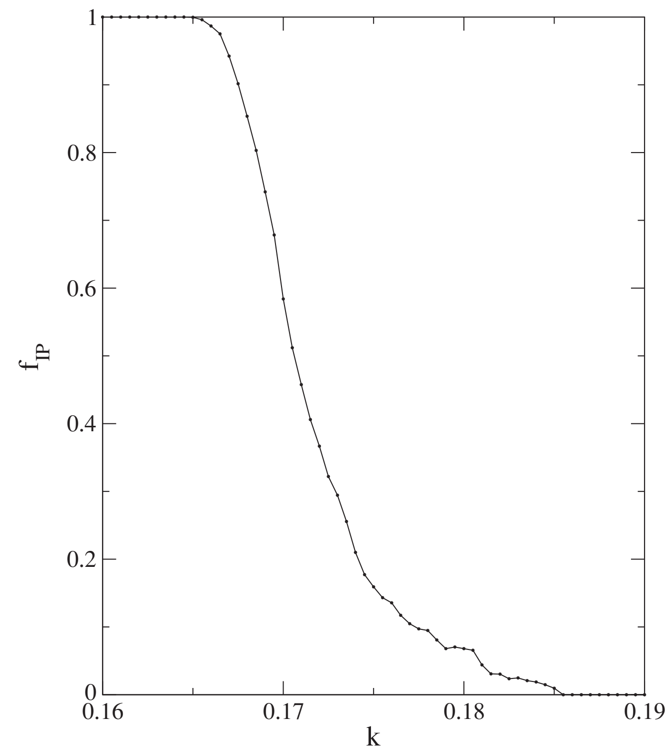


FIG. 5. Fraction of initial conditions (out of $10^4$ samples) going to in-phase state $f_{\mathrm{IP}}$ is plotted as a function of coupling strength $k$. When $0 < f_{\mathrm{IP}} < 1$, bistability occurs, where some of the initial conditions are going to the in-phase state while others converge to the out-of-phase state.

$k_c = 0.134$. By contrast, varying $\epsilon$ for $k = 0$ yields periodic oscillations at $\epsilon = 0.004$. The two limiting cases determine the dynamical behavior of the two largest regions in parameter space. Figure 3 shows the phase diagram for $\Delta\phi_{13}$ and the phase difference between the indirectly coupled oscillators labeled 1 and 3. Black dots indicate a zero-phase difference, while yellow dots correspond to a phase difference $\pi$. There is also a mixed regime where the phase difference is neither zero nor $\pi$. In order to examine the details of the different dynamical states in parameter space, the four largest Lyapunov exponents (LEs) have been calculated along a line in parameter space as a function of the coupling parameter $k$ where $\epsilon$ is fixed at 0.0026. These curves reveal various transitions from chaotic to periodic motion and vice versa. The Lyapunov exponents and the average phase difference between all pairs of oscillators in the coupled system ($\Delta\phi_{ij}$ for $i,j = 1,2,3$) are shown in Figs. 4(a) and 4(b), respectively.

As we increase the coupling strength from zero, positive LEs decrease while the zero LE becomes negative. At the moment, when one of the zero LE starts decreasing, phase synchronization sets in. This happens already for very small $k$: all oscillators show phase synchrony. There is a significant region of multistability when the coupling strength is small, $k < 0.09$, but upon increasing $k$ there are discontinuous changes in the phase differences accompanied by discontinuous changes in the nonzero LEs [Fig. 4(a)]. The two indirectly conjugate-coupled systems (1st and 3rd) show a transition to in-phase synchronization, where LEs shows a

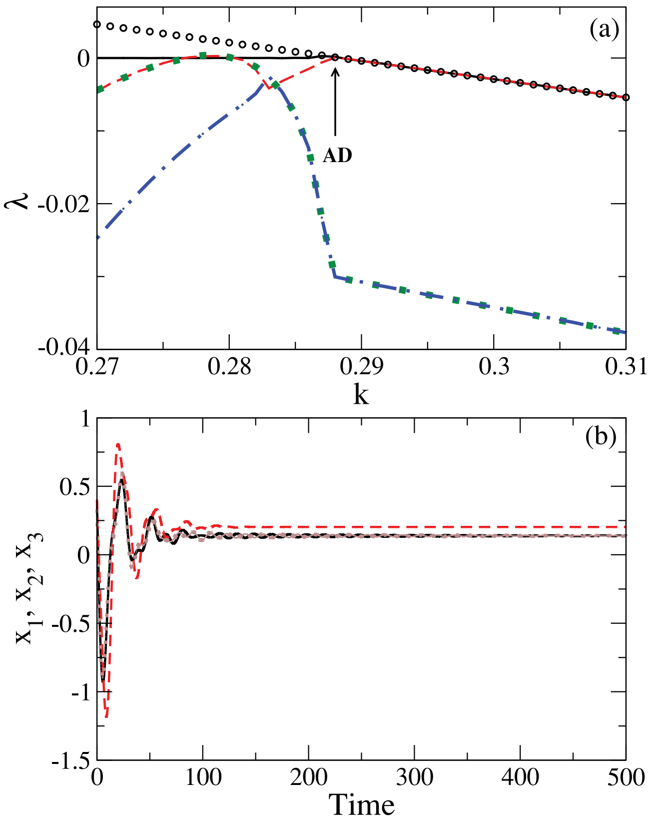


FIG. 6. (Color online) (a) Four largest Lyapunov exponents and real part of the largest eigenvalue (marked as open circles) as a function of coupling parameter $k$. Amplitude death occurs beyond the point marked AD. (b) The $x$ components of the three oscillators are shown for $k = 0.31$ in solid (black), dashed (red), and dotted lines (brown).

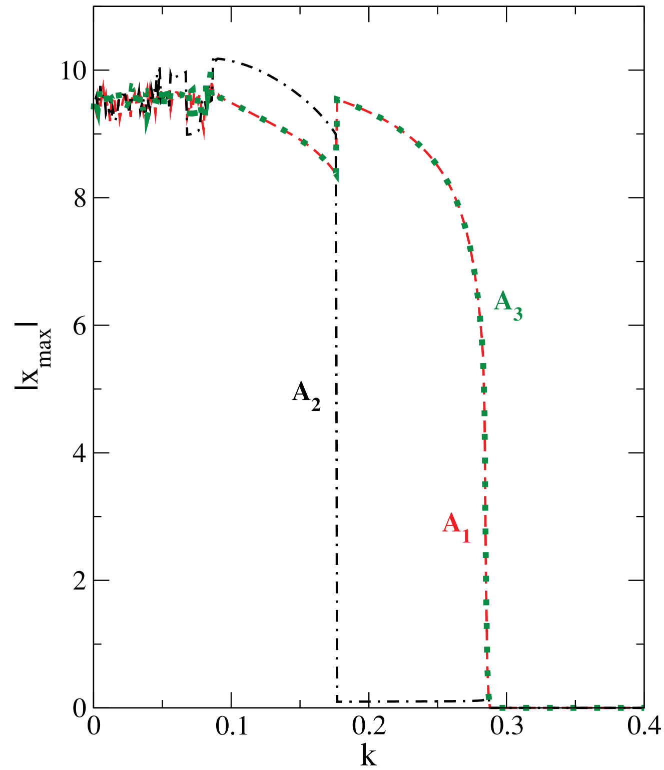


FIG. 7. (Color online) Transition to phase flip and the state of amplitude death for relay-coupled modified Rössler oscillators. Amplitudes of three oscillators are plotted with dashed (red) $A_1$, dotted-dash (black) $A_2$, and dotted lines (green) $A_3$.

discontinuous jump and the average phase difference between the two oscillators $\Delta\phi_{13} \to 0$, at $k = 0.057$. Further increase of the coupling strength gives rise to a regime of phase synchronization of all three oscillators for $0.068 \leqslant k \leqslant 0.076$. Oscillators 1 and 3 again go to an in-phase state at $k = 0.087$ with a discontinuous jump in LEs and $\Delta\phi_{13} = 0$. While the two indirectly coupled oscillators have a zero-phase difference, the phases of the directly coupled pairs $\Delta\phi_{12}$ and $\Delta\phi_{23}$ have the same nonzero phase difference. Inspection of the oscillator dynamics reveals that the first and third oscillators are completely synchronized while the middle oscillator has a different amplitude and a $k$-dependent phase shift compared to its neighbors.

There is an abrupt transition in the Lyapunov exponents and the phase difference between any two oscillators when the coupling strength passes through the critical value $k_c = 0.177$ corresponding to the phase-flip transition between the two indirectly coupled oscillators. Below the critical $k$, the mean phase difference $\Delta\phi_{13}$ computed over many cycles is nearly zero, indicating complete phase synchronization. When $k < k_c$, the two indirectly coupled systems become phase synchronized, and when $k > k_c$ they go into an antiphase synchronized state [Fig. 4(d) for $k = 0.19$]. It is important to note that, in contrast to the usual Rössler system ($\epsilon = 0$), the phase-flip transition for the modified Rössler system happens not in a chaotic regime but in a periodic dynamical regime. The two relay-coupled systems exhibit identical oscillations before the phase-flip transition and out-of-phase oscillations beyond the transition. In the spectrum of Lyapunov exponents, the phase-flip transition point can be identified by the sharp discontinuity in the slope of the Lyapunov exponents [Fig. 4(a)] and in the phase difference [Fig. 4(b)]. These results indicate that, even though the spatial systems are coupled instantaneously, there is an inherent delay in the communication among different indirectly coupled oscillators. Comparing the amplitude of the central oscillator with the outer ones in Figs. 4(c) and 4(d), we notice that it has decreased sharply with increasing coupling strength. This behavior is related to the approach of an amplitude death region as discussed below.

Near the phase flip there is bistability. This transition does not occur abruptly at a critical parameter value, but stretches over a parameter range ranging from $k \sim 0.165$ to $k \sim 0.19$. In this parameter interval we find the coexistence of in-phase and out-of-phase attractors. In Fig. 5 the fraction of initial conditions (out of $10^4$ samples) going to the in-phase synchronous state ($f_{IP}$) is shown as a function of the coupling strength $k$. Bistability has also been observed in two conjugate-coupled systems [3].

As mentioned, we also obtain the amplitude death phenomenon, which occurs when coupled oscillators drive each other to a stable fixed point and stop oscillating [22,24]. As shown in Fig. 6(a), amplitude death occurs when all Lyapunov exponents become negative at $k = 0.288$. The real part of the largest eigenvalue of the Jacobian matrix computed at the emerging fixed point coincides with the largest Lyapunov exponent in the AD region. The transient dynamics in the amplitude death regime is shown in Fig. 6(b). The coupling does not vanish on the synchronization manifold and the systems are still interacting through the dissimilar (conjugate) variable. This ongoing interaction is responsible for the suppression of oscillations [24].

The nature of the transitions to the phase flip and the state of amplitude death is further characterized by examining the amplitude of the individual oscillators in Fig. 7. Here, the transition from oscillatory state to phase flip between the first and third oscillators is apparent in the sudden drop in the amplitude of the second oscillator at a critical strength of coupling. Upon further increasing the coupling strength all three oscillators go into amplitude death.

## IV. DISCUSSION AND SUMMARY

We have shown that the phase-flip transition can occur in the absence of time-delay coupling when chaotic oscillators are indirectly coupled via conjugate variables. Using a relay coupling scheme, where a central system is coupled to two peripheral systems via conjugate variables, we observe a phase-flip transition of the indirectly coupled oscillators followed by an amplitude death transition for the central oscillator. Our results show that amplitude death and phase-flip phenomena as observed in delay-coupled systems can also be obtained in relay-coupled systems with indirect conjugate coupling. Indirect conjugate coupling gives rise to an effective delay in the two systems and thereby suppresses oscillations.

The phase-flip transition can be of considerable importance in switching oscillatory dynamics, and thus the present method has potential utility as a designing strategy to obtain either in-phase or out-of-phase states of chaotic and oscillatory systems. These results also indicate the possibility of the emergence of an inherent delay in spatially coupled systems, particularly when there is a large number of interacting oscillators that are indirectly coupled.

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