Solution of Fokker-Planck equation for a broad class of drift and diffusion coefficients

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A Langevin equation with variable drift and diffusion coefficients separable in time and space and its corresponding Fokker-Planck equation in the Stratonovich approach are considered. From this Fokker-Planck equation a class of exact solutions with the same spatial drift and diffusion coefficients is obtained. Furthermore, some details of this system are analyzed by using the spatial diffusion coefficient $D(x) = \sqrt{D}\,|x|^{-\theta/2}$.

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I. INTRODUCTION

In the past two decades, anomalous diffusion properties have been extensively investigated using several approaches in order to model different kinds of probability distributions such as long-range spatial or temporal correlations [1,2]. These approaches have been used to describe numerous systems in several contexts such as physics, hydrology, chemistry, and biology. The diffusion process is classified according to the mean square displacement (MSD)

$$\langle x^2(t)\rangle \sim t^{\alpha}. \tag{1}$$

In the case of normal diffusion, the MSD grows linearly with time ($\alpha = 1$). For $0 < \alpha < 1$ the process is called subdiffusive; for $\alpha > 1$ the process is called superdiffusive. The well-established property of the normal diffusion described by the Gaussian distribution can be obtained by the usual Fokker-Planck equation with a constant diffusion coefficient (without the drift term) [3,4] or by an integro-differential diffusion equation with the exponential function for the waiting time probability distribution [5]. Anomalous diffusion regimes can also be obtained by the usual Fokker-Planck equation; however, they arise from a variable diffusion coefficient that depends on time and/or space. In contrast, in terms of the Langevin approach, the regime is associated with a multiplicative noise term. In other approaches such as the generalized Fokker-Planck equation (nonlinear) and fractional equations, the equations can describe anomalous diffusion regimes with a constant diffusion coefficient.

The Langevin equation is a very important tool for describing systems out of equilibrium [3,4]. Moreover, this equation has been extensively investigated; many properties and analytical solutions of it have also been revealed. In this work solutions of a class of the Langevin equation with the deterministic drift and multiplicative noise terms in time and space are presented. To do so, the corresponding Fokker-Planck equation in terms of the Stratonovich definition is obtained as well as its solution for the probability distribution function (PDF).

II. LANGEVIN EQUATION AND CORRESPONDING FOKKER-PLANCK EQUATION

The following Langevin equation is considered in one-dimensional space with a multiplicative noise term:

$$\dot{\xi} = h(\xi,t) + g(\xi,t)\Gamma(t), \tag{2}$$

where ξ is a stochastic variable and $\Gamma(t)$ is the Langevin force. The averages $\langle\Gamma(t)\rangle = 0$ and $\langle\Gamma(t)\Gamma(\bar{t})\rangle = 2\delta(t-\bar{t})$ are assumed [3]; $h(\xi,t)$ is the deterministic drift. Physically, the additive noise [for $g(\xi,t)$ constant] may represent the heat bath acting on the particle of the system and the multiplicative noise term [for variable $g(\xi,t)$] may represent a fluctuating barrier. For $g = \sqrt{D}$ and $h(\xi,t) = 0$, Eq. (2) describes the Wiener process and the corresponding probability distribution is described by a Gaussian function. In the case of $g(\xi,t)$, some specific functions have been employed to study, for instance, turbulent flows ($g(x,t) \sim |x|^a t^b$) [6–8]. By applying the Stratonovich approach in a one-dimensional space [3], the following dynamic equation for the PDF is obtained:

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}[D_1(x,t)W(x,t)] + \frac{\partial^2}{\partial x^2}[D_2(x,t)W(x,t)], \tag{3}$$

where $D_1(x,t)$ and $D_2(x,t)$ are the drift and diffusion coefficients given by

$$D_1(x,t) = h(x,t) + \frac{\partial g(x,t)}{\partial x}g(x,t) \tag{4}$$

and

$$D_2(x,t) = g^2(x,t). \tag{5}$$

Note that Eq. (3) has a spurious drift due to the Stratonovich definition. Moreover, Eq. (3) can be written as

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}[h(x,t)W(x,t)] + \frac{\partial}{\partial x}\left(g(x,t)\frac{\partial g(x,t)W(x,t)}{\partial x}\right). \tag{6}$$

For the case of $h(x,t) = 0$ and $g(x,t) = T(t)D(x)$, the system has been considered in Ref. [9]; the solution for $W(x,t)$ is given by

$$W(x,t) = B(t)\frac{\exp\left(-\frac{\bar{x}(x)^2}{4\bar{t}(t)}\right)}{D(x)\sqrt{\bar{t}(t)}}, \tag{7}$$

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where

$$\frac{d\bar{t}}{dt} = T^2(t), \tag{8}$$

$$\frac{d\bar{x}}{dx} = \frac{1}{D(x)}, \tag{9}$$

and $B(t)$ is a normalization factor. Equation (7) can describe interesting properties such as a non-Gaussian distribution, a combination of behaviors such as Gaussian (for short distances) and exponential (for long distances), and a combination of behaviors such as Gaussian (for short distances) and power-law decay (for long distances). Further, it can describe many bimodal distributions for different forms of $g(x,t)$. For instance, if one considers $D(x) = \sqrt{D}\,|x|^{-\theta/2}$, then the probability distribution and MSD are given by

$$W(x,\bar{t}) = |x|^{\theta/2}\,\frac{\exp\left(-\frac{|x|^{2+\theta}}{D(2+\theta)^2\bar{t}}\right)}{\sqrt{4\pi D\bar{t}}} \tag{10}$$

and

$$\langle x^2(t)\rangle = \frac{[D^2(2+\theta)^4]^{1/(2+\theta)}\Gamma\left(\frac{6+\theta}{2(2+\theta)}\right)\bar{t}^{[2/(2+\theta)]}(t)}{\sqrt{\pi}}, \quad \theta > -2. \tag{11}$$

Moreover, the PDF can also be obtained for $\theta = -2$; in this case the PDF gives a log-normal distribution. One can see that the multiplicative noise term in space $D(x) = \sqrt{D}\,|x|^{-\theta/2}$ produces non-Gaussian shapes for the PDF Eq. (10); it presents a Gaussian shape only for $\theta = 0$. It can also reproduce the asymptotic behavior of the random-walk model and time fractional dynamic equation for $\bar{t} = t^{\beta(2+\theta)/2}$ [9], where $0 < \beta < 1$. There are two interesting processes that can be obtained from Eqs. (10) and (11) by taking $\theta > -2$. The first one considers a simple expression for $T(t)$ given by

$$T(t) = \frac{\sqrt{q}}{\sqrt{t}} \tag{12}$$

for $t \gg 1$. Using Eq. (8) this yields

$$\bar{t}(t) = q\ln t. \tag{13}$$

Equations (11) and (13) describe the ultraslow diffusion processes. This kind of diffusion has been found, for instance, in aperiodic environments [10].

The second process considers $T(t)$:

$$T(t) = \frac{\sqrt{\alpha t^{\alpha-1}}\sqrt{\sum_{j=0}^{n} c_j\lambda_j e^{-\lambda_j t^{\alpha}}}}{\sum_{i=0}^{n} c_i e^{-\lambda_i t^{\alpha}}}, \tag{14}$$

where c_j, λ_j, and α are constants. Using the function in Eq. (14), one can obtain anomalous diffusion processes with logarithmic oscillations. Note that the time behavior with a logarithmic oscillation is ubiquitous; examples have been observed, for instance, in epidemic spreading in fractal media [11], the financial stock market [12], and diffusion-limited aggregates [13]. In Fig. 1 the function $T(t)$ Eq. (14) is shown for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$, and $b = 0.3$; for these values the curves present logarithmic oscillations with different values of n and α. From Eq. (8) one obtains

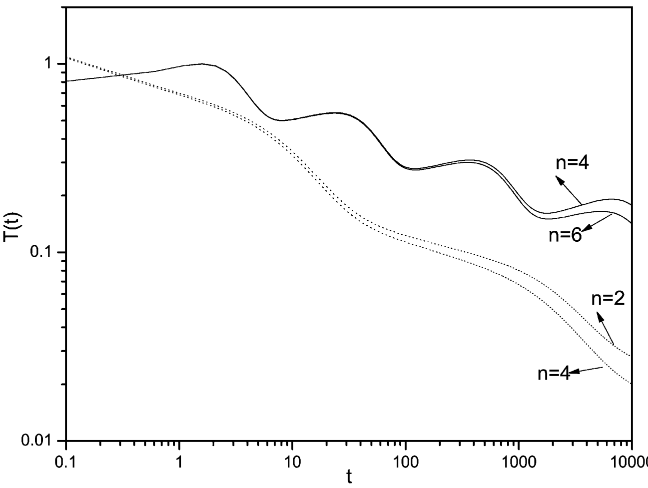

FIG. 1. Plots of the function $T(t)$ Eq. (14). The dotted lines correspond to $\alpha = 0.5$, whereas the solid lines correspond to $\alpha = 1$.

$$\bar{t}(t) = \frac{1}{\sum_{i=0}^{n} c_i e^{-\lambda_i t^{\alpha}}}. \tag{15}$$

Moreover, the PDF Eq. (10) presents unimodal states for $-2 < \theta \leqslant 0$ and bimodal states for $\theta > 0$ (see Fig. 2) with pronounced cusps. The numerical results show that the PDF remains practically unchanged for $n = 2$ and 6. Figure 3 shows the MSD Eq. (11) as a function of time t; it presents anomalous diffusion processes with logarithmic oscillations. It can be seen that the MSD tends to display power-law behaviors, which indicate subdiffusive regimes.

Consider now that the deterministic drift $h(x,t)$ and multiplicative noise term $g(x,t)$ are separable in time and space and are given by

$$h(x,t) = T_1(t)D(x) \tag{16}$$

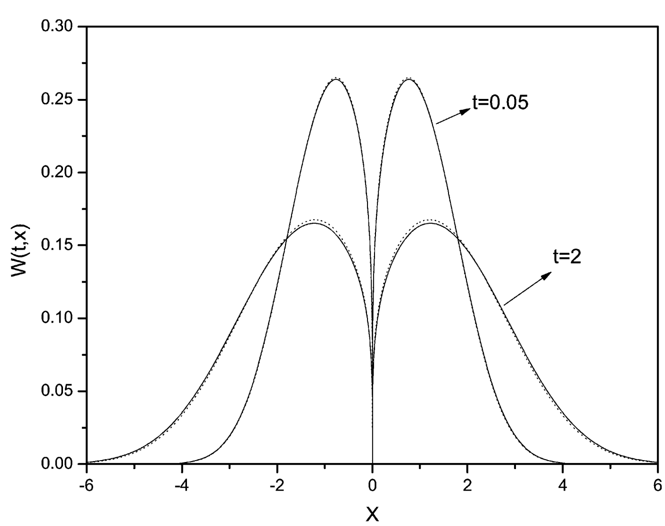

FIG. 2. Plots of the PDF Eq. (10) for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$, $b = 0.3$, $D = 1$, $\theta = 0.5$, and $\alpha = 1$. The solid lines correspond to $n = 2$, whereas the dotted lines correspond to $n = 6$.

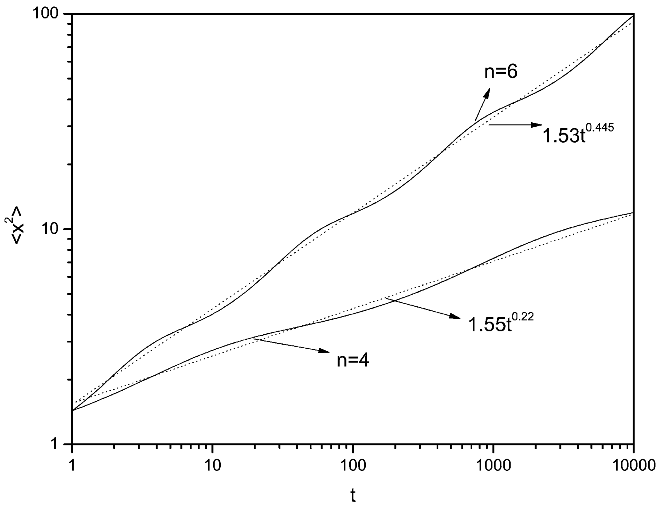

FIG. 3. Plots of the MSD Eq. (11) for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$, $b = 0.3$, $D = 0.3746$, and $\theta = 0.5$. The solid line with $n = 4$ corresponds to $\alpha = 0.5$, whereas the solid line with $n = 6$ corresponds to $\alpha = 1$. The dotted lines correspond to the power-law functions.

and

$$g(x,t) = T(t)D(x). \tag{17}$$

Thus Eq. (3) reduces to

$$\frac{\partial W(x,t)}{\partial t} = -T_1(t)\frac{\partial}{\partial x}[D(x)W(x,t)] + T^2(t)\frac{\partial}{\partial x}\left(D(x)\frac{\partial D(x)W(x,t)}{\partial x}\right). \tag{18}$$

Note that the coefficients $h(x,t)$ and $g(x,t)$ given by $h(x,t) = g(x,t) = D(x)$ have been used to study Brownian pumping in nonequilibrium transport processes [14]. By suitable transformations of variables it can be shown that Eq. (18) can be reduced to the constant-diffusion equation without the drift coefficient term. To do so, one takes the following transformations:

$$\rho(x,t) = D(x)W(x,t), \tag{19}$$

$$\frac{dt^*}{dt} = T^2(t), \tag{20}$$

and

$$x^* = \int \frac{dx}{D(x)} - \int dt\, T_1(t) + A, \tag{21}$$

where A is a constant, thus reducing Eq. (18) to

$$\frac{\partial \rho\,(t^*,x^*)}{\partial t^*} = \frac{\partial^2 \rho\,(t^*,x^*)}{\partial x^{*2}}. \tag{22}$$

Equations (20) and (21) give the time and space scaling factors that connect Eq. (18) to the ordinary diffusion equation (22). Equation (22) can be solved; the solution with a natural boundary condition is given by

$$\rho(t^*,x^*) = C\frac{\exp\left(-\frac{x^{*2}}{4t^*}\right)}{\sqrt{t^*}}, \tag{23}$$

where C is a normalization factor. Equations (19) and (23) show that the time-dependent coefficients $T(t)$ and $T_1(t)$ do not change the Gaussian form; however, the coefficient $D(x)$ can produce different forms for the distribution $W(x,t)$ [9]. Note that for $D(x) = \sqrt{D}$, $T(t) = 1$, and $T_1(t) = 0$ the Wiener process is recovered.

In order to investigate details of the solution of Eq. (23) one takes $D(x) = \sqrt{D}\,|x|^{-\theta/2}$. Using Eqs. (19) and (21), with $A = 0$, yields

$$W(x,t) = \begin{cases} \frac{(-x)^{\theta/2}}{\sqrt{4\pi D t^*(t)}} \exp\left(-\frac{\left((-x)^{(2+\theta)/2} + \frac{\sqrt{D}(2+\theta)}{2}H(t)\right)^2}{D(2+\theta)^2 t^*(t)}\right), \\ \quad x < 0 \\ W(x,t) = \frac{x^{\theta/2}}{\sqrt{4\pi D t^*(t)}} \exp\left(-\frac{\left(x^{(2+\theta)/2} - \frac{\sqrt{D}(2+\theta)}{2}H(t)\right)^2}{D(2+\theta)^2 t^*(t)}\right), \\ \quad x > 0, \end{cases} \tag{24}$$

where $H(t) = \int dt\, T_1(t)$ and $\theta > -2$. Equation (24) shows that the drift term produces an asymmetric PDF with respect to the coordinate x. For $T_1(t) = 0$ the PDF Eq. (24) reduces to the solution in Eq. (10) without the presence of the drift term and the symmetric PDF is recovered. In this case, the drift term $T_1(t)$ gives the duration of this asymmetry. Figure 4 shows the asymmetric PDF Eq. (24) for $t = 0.2$. The asymmetry of the PDF with $\theta = -0.1$ is more pronounced than the PDF with $\theta = -0.5$. From Eq. (24) one obtains

$$\langle x^2(t)\rangle = \frac{\Gamma\left(\frac{6+\theta}{2(2+\theta)}\right)}{\sqrt{\pi}}[D(2+\theta)^2 t^*(t)]^{2/(2+\theta)} \times e^{-H^2(t)/t^*(t)}\, {}_1F_1\left(\frac{6+\theta}{2\,(2+\theta)}, \frac{1}{2}, \frac{H^2(t)}{t^*(t)}\right), \tag{25}$$

where ${}_1F_1(a,b,z)$ is the Kummer confluent hypergeometric function [15]. For $H(t) = 0$, without the drift term, the result in Eq. (11) is recovered. Moreover, Eqs. (24) and (25) also present interesting results: For $H^2(t)/t^*(t)$ proportional to a constant they give results similar to Eqs. (10) and (11), without

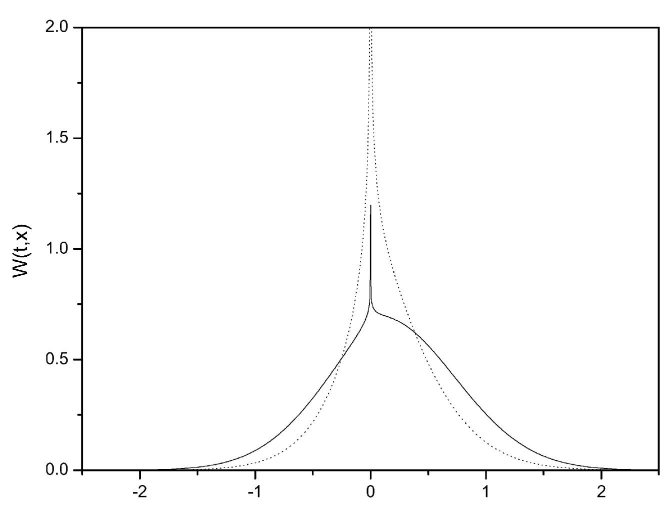

FIG. 4. Plots of the PDF Eq. (24) for $D = 1$, $t^*(t) = t$, and $H(t) = t$. The dotted line corresponds to $\theta = -0.5$, whereas the solid line corresponds to $\theta = -0.1$.

the drift term. In this case, the drift term only contributes an additional constant to the overall behavior of the system.

It should be noted that the solutions in Eqs. (19) and (23) can work adequately for positive $D(x)$. For negative $D(x)$ one should take $\rho(x,t) = -D(x)W(x,t)$. Consider, for example, $D(x) = x$. Taking

$$\rho(x,t) = -xW(x,t), \quad x < 0 \tag{26}$$

and

$$\rho(x,t) = xW(x,t), \quad x > 0, \tag{27}$$

from Eq. (21) one obtains

$$x^* = \ln|x| - H(t) - \ln|x_0| \tag{28}$$

and

$$W(x,t) = \frac{\exp\left(-\frac{[\ln|x|-H(t)-\ln|x_0|]^2}{4t^*(t)}\right)}{4\sqrt{\pi t^*(t)}\,|x|}. \tag{29}$$

This is the log-normal distribution, which is the same as the one given in Ref. [16] for $t^*(t) = t$, which has been obtained using the method of characteristics. It is worth mentioning that the coefficients $h(x,t)$ and $g(x,t)$, given by $h(x,t) \sim x$ and $g(x,t) \sim x$, might be used to investigate the barrier crossing problem in heavy-ion fusion reactions [17] as well as a limiting case of the Langevin equation for describing the tumor cell growth system [18].

III. CONCLUSION

When a multiplicative noise term is introduced into the simple Langevin equation (2), even separable in time and space, the system can exhibit complex behaviors and a rich variety of processes. A class of these processes has been presented analytically. It is hoped that they can be used to mimic a wide class of natural systems.

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