Lévy targeting and the principle of detailed balance

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We investigate confining mechanisms for Lévy flights under premises of the principle of detailed balance. In this case, the master equation of the jump-type process admits a transformation to the Lévy-Schrödinger semigroup dynamics akin to a mapping of the Fokker-Planck equation into the generalized diffusion equation. This sets a correspondence between above two stochastic dynamical systems, within which we address a (stochastic) targeting problem for an arbitrary stability index $\mu \in (0,2)$ of symmetric Lévy drivers. Namely, given a probability density function, specify the semigroup potential, and thence the jump-type dynamics for which this PDF is actually a long-time asymptotic (target) solution of the master equation. Here, an asymptotic behavior of different μ-motion scenarios ceases to depend on μ. That is exemplified by considering Gaussian and Cauchy family target PDFs. A complementary problem of the reverse engineering is analyzed: given *a priori* a semigroup potential, quantify how sensitive upon the choice of the μ driver is an asymptotic behavior of solutions of the associated master equation and thus an invariant PDF itself. This task is accomplished for so-called μ family of Lévy oscillators.

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I. INTRODUCTION

Many complex physical systems can be satisfactorily described in terms of the dynamics of a certain fictitious particle under the action of random forces, originating from its environment [1–4]. A microscopic locally defined impact of (conservative) external forces upon noise is typically quantified within the Langevin approach as an additive deterministic term. Whenever we can identify a Gaussian noise as an emergent property of the environment-particle coupling, after accounting for the presence of confining deterministic forces, the interrelated notions of (thermal) equilibrium, Boltzmann asymptotic probability density functions (PDFs), relaxation rate, and detailed balance generically follow. That is the case in the standard Brownian motion picture, based upon kinetic theory derivations.

For quite a large number of systems exhibiting stochastic features, experimental data show that the description based on the concept of the Gaussian noise is insufficient, since the involved fluctuations turn out to generate non-Gaussian heavy-tailed distributions. This happens in a broad range of systems of varied levels of complexity: physical, chemical, biological [1–3], geophysical, economic [4].

We restrict further considerations to so-called Lévy flights and specifically to a subclass of symmetric Lévy stable noises or drivers. The Langevin formalism (additive or multiplicative) is here a celebrated research standard and yields suitable fractional versions of Fokker-Planck equations [1–3] governing the dynamics of PDFs. Confining potentials and forces allow for the existence of stationary (invariant, "equilibrium") PDFs whose number of moments in existence may exceed 1 or 2, being finite but in principle arbitrarily large.

Contrary to the case of systems with Gaussian fluctuations, the notion of equilibrium, although natural under confining conditions, has no obvious *thermal* connotation in the context of Lévy processes. Within the Langevin modeling of Lévy flights one is faced with the nonexistence of the Gibbs-Boltzmann thermal equilibrium PDF, c.f. [5] for the pertinent no-go statement. That makes the Langevin approach to confined Lévy flights plainly incompatible with the conceptual framework of the Gibbs-Boltzmann thermodynamics, where the detailed balance is a generic property and a signature of thermal equilibrium.

Therefore, to address a thermalization of Lévy flights and the validity of the principle of detailed balance, one needs to resort to non-Langevin methods of quantifying an impact of external potentials upon Lévy flights. One of possible options is to invoke indirect (complex) implementations of the jump-type random motion. We note in passing that a description of subdiffusions, carried out [6,7] within the framework of a subordination of random processes and involving an additional fractional time-derivative in the transport equation, is known to admit an exponential relaxation to Boltzmann equilibria, c.f. also a discussion of the validity of the Nyquist theorem in [6]. In the present paper, we choose a direct option of dealing with Lévy flights and stay within the limitations of the standard theory of symmetric Lévy stable processes [1,2,8], where the only "fractional" input refers to spatial derivatives, and never to the time label.

It is clear that any conceivable "thermal equilibrium" concept for non-Gaussian jump-type processes needs to be addressed with care and should account for a number of precautions. In particular, an issue of physically motivated thermalization mechanisms for (confined) Lévy flights has received only a residual attention in the literature [9–13], mostly in connection with Tsallis statistics and kinetics issues. The main obstacle here may be that the source of Lévy noise is interpreted as extrinsic to the physical system under consideration, with no reliable kinetic theory background, i.e., with no obvious microscopic channels of an energy exchange with the environment.

Lévy flights are pure jump (jump-type) processes. Therefore, it seems useful to recall that various model realizations of standard jump processes (with jump size being bounded from below and above) can be consistently thermalized, by means of a locally defined scenario (through potentials affecting the jump size and direction) of an energy exchange with the thermostat, see [14–17] and references therein. That involves a suitable redefinition of, otherwise symmetric, transition rates for the jump process. The validity of the *principle of detailed balance* comes as a straightforward consequence. Inversely, if we demand the principle of detailed balance to hold true, this redefinition is a sufficient condition for a stationary PDF to appear in the Gibbs-Boltzmann form. We note that there is no known (additive or multiplicative) Langevin representation for Lévy processes respecting the canonical form of detailed balance of Refs. [14–17].

Interestingly, an analogous idea has been followed in Refs. [18–20], for Lévy flights in the systems with topological complexity like polymers. A different version of the fractional Fokker-Planck equation, suitable for systems in thermal equilibrium and with the symmetric Lévy-stable driver in action, has been derived and found to admit a transformation into the fractional version of the generalized diffusion equation. There, the above equation has been named as a fractional or generalized Schrödinger equation

$$\partial_t \psi(x,t) = -\hat{H}\psi(x,t), \tag{1}$$

where there is no imaginary unit i before the time derivative.

The latter transformation relates the fractional Fokker-Planck dynamics with the Lévy semigroup and is specific to non-Langevin systems with a symmetric stable-noise. Asymmetric (e.g., one-sided) stable processes cannot be transformed into a semigroup dynamics. The corresponding transport equation [21] is different from standard fractional Fokker-Planck equations known in the literature [1–3] and as well from those considered in Refs. [18–20], see also [9,13,22].

The main subject of the present paper is an exploration of the direct relation of the master equation governed dynamics (with the built-in detailed balance principle) and its Lévy semigroup transcription. The class of confined Lévy-stable driven systems, that is compatible with that correspondence, comprises those jump-type processes which are equilibrated (eventually, to a Gibbs-Boltzmann thermal equilibrium state) by a mild spatial disorder of the physical environment in which those jumps take place [13,19,20]. The environmental inhomogeneities may be modeled by microscopic potentials (various periodic ones were considered in Ref. [19]). We aim at a significant generalization of our recent findings [9,13,22] and, with a focus on a clarity of presentation and availability of explicit analytical and computational outcomes, we consider slightly simpler confining potentials.

While passing to the semigroup transcription of the master equation dynamics, an effective semigroup potential appears instead of an explicit input of Boltzmann (microscopic) potentials. The relation between these two types of potentials is indirect and will be clarified below. The resultant Kac (or Feynman-Kac) exponential weight, with that semigroup potential in the exponent, is responsible for a large-scale statistical (spatial) redistribution of random paths executed in the course of the pertinent Lévy process. A thorough study of that issue (statistics of random paths) in a simpler context of standard jump processes can be found in Refs. [23,24].

An impact of environmental potentials can thus be quantified as follows: (a) on the level of the master equation, by means of a locally defined microscopic potential $U(x)$ (entering a multiplicative modification of transition rates) which, if confining, explicitly appears in the exponent of the Boltzmann stationary PDF $\rho_*(x) \sim \exp(-U)$, (b) on the level of the associated semigroup dynamics, by means of an effective semigroup potential $\mathcal{V}(x)$ in the fractional Schrödinger-type equation; it is the positive ground state $\rho_*^{1/2}(x) \simeq \exp[-U(x)/2]$ of the latter which actually determines $\rho_*(x)$ of (a).

Contrary to the approach of Refs. [18–20], where model-study transition rates (and the involved microscopic potentials) were heuristically introduced, we put forward, as a primary motion picture, a semigroup version of the jump-type dynamics with symmetric Lévy-stable drivers. It is a functional form of the semigroup potential $\mathcal{V}(x)$, which we consider to be an arbitrary continuous and bounded from below function, that ultimately ensures the existence of an asymptotic invariant PDF for the associated master equation and gives rise to Markovian realizations of the pertinent jump-type dynamics. Here we encounter the following large time behavior: $-\ln \psi(x,t \to \infty) \to -\ln[\rho_*^{1/2}(x)] \sim U(x)/2$, where $U(x)$ is the pertinent Boltzmann potential and $\psi(x,t)$ is a solution of Eq. (1).

The structure of the paper is as follows. First we discuss an issue of detailed balance for standard jump processes and next define its immediate generalization to Lévy flights [μ-family of Lévy-stable laws with $\mu \in (0,2)$] in Secs. II and III. A mapping of the resultant master equation to a fractional version of the generalized diffusion equation follows in Sec. IV. For clarity of presentation, we make a Brownian detour in Sec. V to indicate how the semigroup framework is related to the standard Fokker-Planck dynamics of diffusion-type processes. Some material presented in Secs. II–V have already been presented in our previous works. The purpose of this inclusion is to make the paper self-contained.

Essentially new results are presented in Secs. VI–VIII; Eqs. (18), (25), and (33) comprise the main novel results of the paper. In Sec. VI we describe the Lévy μ targeting under an assumption that target PDFs are selected from so-called Cauchy α family of PDFs. For a computationally advantageous example of $\alpha = 2$ and arbitrary $\mu \in (0,2)$ we provide analytic formulas for the associated semigroup potentials (they define the semigroup dynamics which makes the considered PDFs to be genuine asymptotic targets of the jump-type process). In Sec. VII the Lévy targeting is considered for Gaussian target PDFs. Section VIII presents a complete solution of the reverse engineering problem for the μ family of Lévy oscillators, corresponding to quadratic semigroup potential. The obtained analytic formulas for asymptotic PDFs are depicted in Figs. 4 and 5.

Not to overburden the paper with formal arguments, a general solution of the reverse engineering problem for arbitrary semigroup potential has been moved to another publication.

II. JUMP PROCESSES AND DETAILED BALANCE

Let K be a finite state space, with $x,y \in K$. We consider Markovian stochastic dynamics for a finite random system, with transition rates $k(x|y) \equiv k(y \to x)$. Given an initial probability distribution $\rho_0(x)$, its time evolution for times $t \geqslant 0$ is governed by the master equation:

$$\frac{d}{dt}\rho_t(x) = \sum_{y\in K}[k(x|y)\rho_t(y) - k(y|x)\rho_t(x)]\,. \tag{2}$$

Given a stationary solution $\rho_{\rm eq}(x)$ of the master equation, $\dot{\rho}_{\rm eq}(x) = 0$. If we have

$$k(x|y)\rho_{\rm eq}(y) = k(y|x)\rho_{\rm eq}(x) \tag{3}$$

one says that the *condition of detailed balance* is fulfilled.

Let $\rho_{\rm eq}(x) \propto \exp[-U(x)]$, where U is a suitable function on K. (The inverse temperature β can be safely absorbed in the definition of U. As well, for clarity of discussion, we can set $\beta = 1$.) Accordingly

$$k(x|y) = k(y|x)\exp[U(y) - U(x)]\,. \tag{4}$$

We note that $k_0(x|y) = k_0(y|x)$, in a finite state space, yields a uniform distribution $\rho_{\rm eq}(x) = \text{const}$ for all $x \in K$. Let us consider a simple multiplicative modification of a symmetric transition intensity $k_0(x|y)$:

$$k_0(x|y) \Longrightarrow k_U(x|y) = k_0(x|y)\exp\left[\frac{U(y)-U(x)}{2}\right]. \tag{5}$$

By inspection [simply replace $k(x|y)$ by $k_U(x|y)$ in Eqs. (2)–(4)] one verifies the validity of the detailed balance condition, with $\rho_{\rm eq}(x) \propto \exp[-U(x)]$ as the corresponding stationary distribution.

We assume that an equilibrium density $\rho_{\rm eq}(x) > 0$ is unique and presume the detailed balance conditions (3) and (4) to be respected. Then, the relative entropy (negative of the Kullback-Leibler entropy) becomes

$$\mathcal{S}(\rho_t|\rho_{\rm eq}) = \sum_{x\in K}\rho_t(x)\ln\frac{\rho_t(x)}{\rho_{\rm eq}(x)} = \mathcal{F}(\rho_t) - \mathcal{F}(\rho_{\rm eq}) \geqslant 0. \tag{6}$$

Here an obvious analog of the familiar Helmholtz free energy $\mathcal{F}(\rho_t) = \sum_{x\in K}U(x)\rho_t(x) - \mathcal{S}(\rho_t)$ has been introduced, with $\mathcal{S}(\rho_t) = -\sum_{x\in K}\rho_t(x)\ln\rho_t(x)$ being the Shannon entropy of the probability distribution $\rho_t(x)$. We have $\mathcal{F}(\rho_t) \geqslant \mathcal{F}(\rho_{\rm eq}) = -\ln\sum_{x\in K}\exp[-U(x)]$. The relative entropy is monotonous in time and converges to zero, which is accompanied by a decrease of the free energy $\mathcal{F}(\rho_t)$ to its minimal value $\mathcal{F}(\rho_{\rm eq})$.

It is useful to mention an interesting inverse stationary problem of Refs. [15,16]. Namely, for an arbitrary positive probability distribution $\rho_{\rm eq}(x) > 0$ on K there exists a function $U(x)$ such that $\rho_{\rm eq}(x)$ is invariant under the jump dynamics with the transition rate $k_U(x,y)$ of the form (5). In the original formulation of Ref. [16], the reference transition rate $k_0(x,y)$ needs not to be symmetric.

III. DETAILED BALANCE FOR LÉVY FLIGHTS

The above reasoning gives an immediate justification to the strategy adopted before in the context of Lévy-stable processes, albeit with no explicit reference to the detailed balance principle, in a number of papers [18–20]. We also note Refs. [9,13,22], where "stochastic targeting" and related "inverse engineering" (terms, originally coined in Ref. [5]) have been exploited to this end.

To proceed further, we recall that a characteristic function of a random variable X completely determines a probability distribution of that variable. If this distribution admits a PDF $\rho(x)$, we can write $\langle\exp(ipX)\rangle = \int_R \rho(x)\exp(ipx)dx$. A classification of infinitely divisible probability laws is provided by the Lévy-Khintchine formula for the exponent $-F(p)$ of $\langle\exp(ipX)\rangle = \exp[-F(p)]$.

We restrict subsequent considerations to a subclass of stable probability distributions with $F(p) = |p|^\mu$, with $0 < \mu \leqslant 2$. The induced jump-type dynamics $\langle\exp(ipX_t)\rangle = \exp[-tF(p)]$ is conventionally interpreted in terms of Lévy flights and quantified by means of a pseudodifferential (fractional) analog of the heat equation for corresponding PDF

$$\partial_t\rho = -|\Delta|^{\mu/2}\rho = \int[w_\mu(x|y)\rho(y) - w_\mu(y|x)\rho(x)]dy, \tag{7}$$

which has been rewritten as a master equation for a random system on real axis, with a pure jump dynamics. The jump rate $w_\mu(x|y) \propto 1/|x-y|^{1+\mu}$ is a symmetric function, $w_\mu(x|y) = w_\mu(y|x)$ akin to $k_0(x|y)$ of the previous subsection. We recall that the action of a fractional operator $|\Delta|^{\mu/2}$ on a function from its domain is defined by means of the Cauchy principal value of an involved integral, which is singular at $x = z$:

$$-(|\Delta|^{\mu/2}f)(x) = \frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi}\int_{-\infty}^{\infty}\frac{f(z)-f(x)}{|z-x|^{1+\mu}}dz. \tag{8}$$

The above singularity can be consistently handled. To this end, we change the variables in Eq. (8) to obtain

$$-|\Delta|^{\mu/2}f(x) = \frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi}\int_{-\infty}^{\infty}dy\,\frac{f(x+y)-f(x)}{|y|^{1+\mu}} \tag{9}$$

and pay attention to the fact that the integral in (9) should be taken as its Cauchy principal value. The existence of the principal value for arbitrary μ can be proven by the expansion of $f(x+y)$ in (9) in the Taylor series at small y: $f(x+y) - f(x) \approx yf'(x)$. Substituting the above Taylor expansion into the integral (9), we find that its behavior in the vicinity of $y = 0$ is dictated by the integral

$$\int_{-\varepsilon}^{\varepsilon}\frac{y}{|y|^{1+\mu}}dy = \int_{-\varepsilon}^{\varepsilon}|y|^{-\mu}\,\text{sign}\,y\,dy \equiv 0 \tag{10}$$

as the integrand is odd at any $0 < \mu < 2$. The expression (10) shows that the Cauchy principal value of the integral (8) exists for any function $f(z)$, the only regularity condition is that this function should decay at infinities faster then $1/z^2$, which yields $f(z) \sim |z|^{\mu-1-\delta}$, $\delta > 0$ as $|z| \to \infty$.

Mimicking the previous step (5), we open a possibility of a locally controlled energy exchange with an environment, by modifying the jump rate $w_\mu(x|y)$ of the free (neither external forces nor potentials) fractional dynamics to the nonsymmetric form $w_\mu^U(x|y) \neq w_\mu^U(y|x)$: $w_\mu^U(x|y) = w_\mu(x|y)\exp([U(y) - U(x)]/2)$. With $w_\mu^U(x|y)$ replacing $w_\mu(x|y)$, the master equation (7) ultimately takes a slightly discouraging form, known from a number of previous publications:

$$\begin{aligned}\partial_t \rho &= -|\Delta|_U^{\mu/2} \rho = \int \left[w_\mu^U(x|y)\rho(y) - w_\mu^U(y|x)\rho(x)\right] dy \\ &= -[\exp(-U/2)]\,|\Delta|^{\mu/2}[\exp(U/2)\rho] \\ &\quad + \rho \exp(U/2)|\Delta|^{\mu/2}\exp(-U/2). \end{aligned} \tag{11}$$

The above transport equation cannot be transformed to any known form of the fractional Fokker-Planck dynamics, based on the standard (Lévy-stable) Langevin modeling (c.f. [26–30] for literature sample). These two dynamical patterns of behavior are inequivalent [9,22].

For a suitable (to secure normalization) choice of $U(x)$, $\rho_{\rm eq}(x) \propto \exp[-U(x)]$ is a stationary solution of Eq. (11). The detailed balance principle of the form (3) and (4) holds true.

For the record, let us mention that the free fractional Fokker-Plack equation (7) has no stationary solutions. Thus, the jump-type dynamics with properly modified jump rates clearly may give rise to confined Lévy flights. Their asymptotic PDFs in principle may have an arbitrary, not necessarily finite and/or small, number of moments. The reference stable laws generically have no moments of order higher than one.

IV. LÉVY SEMIGROUP MODELING

The master equation (11) cannot be derived within the standard Langevin modeling of confined Lévy flights [13,22,25]. The latter motion scenario (with an ample coverage in the literature [26–28]) is incompatible with that based on the detailed balance principle (3), (4) and the resultant Eq. (11), c.f. [9,22].

The form of Eq. (11) is not handy. However, there exists an equivalent description of the pertinent dynamics in terms of a Lévy-stable semigroup or a fractional (Lévy-)Schrödinger-type equation [13,19,20,25]. The difference with the standard time-dependent Schrödinger equation is the absence of an imaginary unit i before time derivative [e.g., in Eq. (11)].

To this end let us consider the Lévy-Schrödinger Hamiltonian operator with an external potential

$$\hat{H}_\mu \equiv |\Delta|^{\mu/2} + \mathcal{V}(x). \tag{12}$$

Suitable properties of $\mathcal{V}$ need to be assumed, so that $-\hat{H}_\mu$ is a legitimate generator of a dynamical semigroup $\exp(-t\hat{H}_\mu)$ and $\partial_t \Psi = -\hat{H}_\mu \Psi$ holds true for real functions $\Psi(x,0) \to \Psi(x,t)$.

Let us *a priori* select an invariant probability density $\rho_{\rm eq}(x) \doteq \rho_*(x) \propto \exp[-U(x)]$ of Eq. (11). To make it an asymptotic PDF of a well-defined jump-type process we address an issue of the existence of a suitable semigroup dynamics.

Looking for stationary solutions of the affiliated semigroup equation $\partial_t \Psi = -\hat{H}_\mu \Psi$, we realize that if a square root of a positive invariant PDF $\rho_*(x)$ is asymptotically to come out via the semigroup dynamics $\Psi \to \rho_*^{1/2}$, then the resulting fractional Sturm-Liouville equation $\hat{H}_\mu \rho_*^{1/2} = 0$ imposes a *compatibility condition* upon the functional form of $\mathcal{V}(x)$, that needs to be respected. Namely, the potential function and invariant PDF $\rho_*^{1/2}$ should be related as

$$\mathcal{V} = -\frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}. \tag{13}$$

The resulting semigroup dynamics provides a solution for the Lévy stable *targeting problem*, with a predefined invariant PDF.

Inversely, if we predefine a concrete potential function $\mathcal{V}(x)$, then the functional form of an asymptotic invariant PDF $\rho_*(x)$ [actually $\rho_*^{1/2}(x)$] comes out from the above compatibility condition. We call the problem of derivation of ρ_* from a predefined semigroup potential $\mathcal{V}(x)$ as *reverse engineering problem*, see Ref. [5] where this idea had been put forward.

For $\mathcal{V} = \mathcal{V}(x)$ bounded from below, the integral kernel $k(y,s,x,t) = \{\exp[-(t-s)\hat{H}]\}(y,x)$, $s < t$, of the dynamical semigroup $\exp(-t\hat{H})$ is positive. The semigroup dynamics reads $\Psi(x,t) = \int \Psi(y,s)\,k(y,s,x,t)\,dy$, so that for all $0 \leqslant s < t$ we can reproduce the dynamical pattern of behavior, actually set by Eq. (11), but now in terms of Markovian PDFs $p(x,s,y,t)$:

$$\rho(x,t) = \rho_*^{1/2}(x)\Psi(x,t) = \int p(y,s,x,t)\rho(y,s)\,dy, \tag{14}$$

where

$$p(y,s,x,t) = k(y,s,x,t)\frac{\rho_*^{1/2}(x)}{\rho_*^{1/2}(y)}.$$

An asymptotic behavior of $\Psi(x,t) \to \rho_*^{1/2}(x)$ implies $\rho(x,t) \to \rho_*(x)$.

A remark is in place here. The spectral theory of fractional operators of the form (12) has received a broad coverage in the mathematical [8,31–34] and mathematical physics literature [35,36]. An explicit functional form of asymptotic invariant PDFs of confined Lévy flights ρ_* ($\rho_*^{1/2}$ in the semigroup notations) is seldom accessible, with a notable exception of those for Cauchy flights [22,28]. Therefore it is wise to rely on accumulated data that are available, about the near-equilibrium behavior and the decay of PDFs as $|x| \to \infty$, under very general circumstances. Various rigorous estimates pertaining to the decay at infinities of the eigenfunctions, quantify the number of moments of the associated PDFs for different classes of potential functions $\mathcal{V}(x)$. As well, fractional versions of Feynman-Kac formula determining an integral kernel of the semigroup operator, and thence the transition probability which generates [by virtue of Eq. (14)] the PDF $\rho(x,t)$ dynamics consistent with Eq. (11), have an ample coverage therein.

V. BROWNIAN DETOUR

The aim of this section is to describe the relation between above Lévy-Schrödinger semigroup framework and standard Fokker-Planck dynamics of diffusion-type processes. To make this description clear, here we put explicit relations, translating things from the language of partial differential equations (like the Fokker-Planck one) and dealing explicitly with PDFs into the operator language, inherent in (both normal and fractional) quantum mechanics and ultimately in Lévy-Schrödinger semigroup.

In the theory of standard Brownian motion, the Langevin equation or the like (stochastic differential equation with the Wiener noise input) allows to infer a corresponding Fokker-Planck equation. This in turn can be transformed into a Hermitian (strictly speaking, self-adjoint) spectral problem [29]. Contrary to the Lévy-stable case, for diffusion-type processes both these descriptions (e.g., semigroup and Langevin-based Fokker-Planck approaches) are similar descriptions of the dynamics of $\rho(x,t)$.

Given the spectral solution for the operator $\hat{H} = -\Delta + \mathcal{V}$, the integral kernel of $\exp(-t\hat{H})$ reads $k(y,x,t) = \sum_j \exp(-\epsilon_j t)\,\Phi_j(y)\Phi_j^*(x)$. Here, the sum may be replaced by an integral in case of a continuous spectrum and (generalized) eigenfunctions may be complex-valued.

If we set $\mathcal{V}(x) = 0$ identically, a purely continuous spectral problem arises. Then, one arrives at the familiar heat kernel

$$\begin{aligned} k(y,x,t) &= [\exp(t\Delta)](y,x) \\ &= (2\pi)^{-1/2}\int \exp(-p^2 t)\,\exp(ip(y-x))\,dp \\ &= (4\pi t)^{-1/2}\,\exp\left[-\frac{(y-x)^2}{4t}\right], \end{aligned}$$

which is a well-known transition probability density of the Wiener process [actually, upon setting $t \to (t-s)$].

When confining potentials are present, either entire spectrum or its part turns out to be discrete, the corresponding eigenfunctions being real-valued. A standard example is the harmonic oscillator, i.e., the Ornstein-Uhlenbeck process in its original stochastic version. Consider

$$\hat{H} = (1/2)(-\Delta + x^2 - 1).$$

The integral kernel of $\exp(-t\hat{H})$ is given by the classic Mehler formula [37]:

$$\begin{aligned} k(y,x,t) &= k(x,y,t) = \exp(-t\hat{H})(y,x) \\ &= \frac{1}{\pi\sqrt{1-e^{-2t}}}\exp\left[-\frac{x^2-y^2}{2} - \frac{(xe^{-t}-y)^2}{1-e^{-2t}}\right]. \end{aligned}$$

The normalization condition

$$\int k(y,x,t)\exp[(y^2-x^2)/2]\,dy = 1$$

directly employs [and defines upon setting $t \to (t-s)$] the transition probability density of the Ornstein-Uhlenbeck process,

$$p(y,x,t) = k(y,x,t)\frac{\rho_*^{1/2}(x)}{\rho_*^{1/2}(y)}$$

with $\rho_*(x) = \pi^{-1/2}\exp(-x^2)$ being its (Gaussian) invariant PDF.

VI. CAUCHY FAMILY OF PDFS AND LÉVY μ TARGETING

Here we describe in some detail the Lévy stable (with stability index μ) targeting strategy with the predetermined one-parameter family of Cauchy target PDFs:

$$\rho_*(x) \equiv \rho_\alpha(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma(\alpha-1/2)}\frac{1}{(1+x^2)^\alpha}, \quad \alpha > 1/2. \tag{15}$$

We consider functions (15) as asymptotic invariant PDFs for the stochastic jump-type process of Eq. (11). We wish to demonstrate that any μ-stable driver can be employed to this end.

Instead of addressing directly Eq. (11), we use the semigroup dynamics $\exp(-t\hat{H}_\mu)$ generated by the fractional operator (12), i.e., the integrodifferential equation

$$\partial_t \Psi = -|\Delta|^{\mu/2}\Psi - \mathcal{V}_\mu \Psi, \tag{16}$$

where $\Psi(x,t) \equiv \rho(x,t)/\rho_*^{1/2}(x)$ and $\mathcal{V}_\mu(x) = -(|\Delta|^{\mu/2}\rho_*^{1/2})/\rho_*^{1/2}$, $0 < \mu \leqslant 2$.

We note that the Cauchy family (15) has been chosen for computational convenience only. In principle, there is no restriction on the choice of any other target PDF $\rho_*(x)$. The qualitative outcome will be the same as that provided in terms of family (15). Hereafter we call such general procedure "μ targeting."

Let us add, as a side comment, that the Cauchy family of PDFs has played an important role in the previously mentioned search for "thermodynamic equilibria," that may possibly be associated with confined Lévy flights [10–12]. It is known [9,10] that an exponent α in principle can be directly related to the thermal equilibrium label $\alpha \propto 1/k_B T$. An analogous observation has been reported in Refs. [9,13], after transforming PDFs (15) into an "exponential form," which resembles Boltzmann one $\rho_* \propto \exp(-U)$, with $U(x) = \alpha\ln(1+x^2)$.

To pass over to the semigroup description we need to infer $\mathcal{V}_\mu(x)$, given ρ_*. This can be done analytically by means of the Fourier transform, specifically because Fourier images of functions (15) for arbitrary $\alpha > 0.5$ exist in a closed analytical form of MacDonald functions K_ν [38].

The Fourier image $g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{ikx}dx$ of a function $g(x)$, when adopted to $g(x) = |\Delta|^{\mu/2}f(x)$ reads $|k|^\mu f(k)$. Fourier images of the square roots of PDFs (15) read

$$\rho_\alpha^{1/2}(k) = \sqrt{\frac{2\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi\,\Gamma(\alpha-1/2)\,\Gamma(\alpha/2)}}\,|k|^{\frac{\alpha-1}{2}}K_{\frac{\alpha-1}{2}}(|k|). \tag{17}$$

An explicit expression for the α family of "μ potentials" $\mathcal{V}_{\mu,\alpha}(x) = -(|\Delta|^{\mu/2}\rho_\alpha^{1/2})/\rho_\alpha^{1/2}$ readily follows:

$$\begin{aligned} \mathcal{V}_{\mu,\alpha}(x) = &-\frac{2^\mu}{\sqrt{\pi}}(1+x^2)^{\alpha/2}\frac{\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{\alpha+\mu}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \\ &\times {}_2F_1\left(\frac{1+\mu}{2},\frac{\alpha+\mu}{2},\frac{1}{2},-x^2\right), \end{aligned} \tag{18}$$

where ${}_2F_1(a,b;c,x)$ is a hypergeometric function [38].

The expression (18) gives the general form of the semigroup potentials $\mathcal{V}_{\mu,\alpha}(x)$ for arbitrary α and μ. To have a better feeling about the properties of the function (18), we should explore

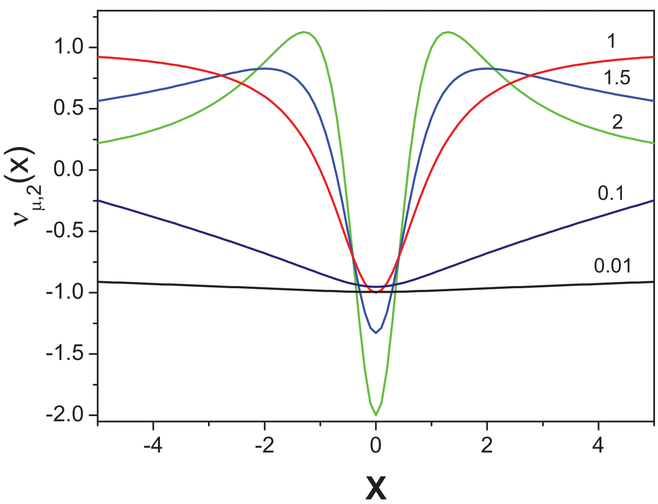

FIG. 1. (Color online) Dependence $\mathcal{V}_{\mu,2}(x)$ for ρ_2 terminating PDF. Figures near curves correspond to μ values. The potentials for $\mu = 1$ and 2 are given by Eqs. (21) and (22), respectively.

this expression for some specific values of parameter α. Further discussion is limited to the case of $\alpha = 2$, i.e.,

$$\rho_2^{1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{1+x^2} \rightarrow \rho_2^{1/2}(k) = e^{-|k|}. \tag{19}$$

We note that the Fourier image $\rho_2^{1/2}(k)$ directly comes from the general expression (17), if we use $K_{1/2}(x) = (\pi/2x)^{1/2}\, e^{-x}$. Then for all $0 < \mu < 2$ we have

$$\mathcal{V}_{\mu,2}(x) = -(1+x^2)^{\frac{1-\mu}{2}}\Gamma(1+\mu)\cos[(1+\mu)\arctan x]. \tag{20}$$

For $\mu = 1$ from (20) we recover our elder result, originally obtained in the context of Cauchy flights [22]:

$$\mathcal{V}_{1,2}(x) = \frac{x^2-1}{1+x^2}. \tag{21}$$

The plots of the μ dependence of (20) are reported in Fig. 1.

The stability index μ is constrained to stay within an interval $0 < \mu \leqslant 2$. The boundary value $\mu = 2$ takes us beyond the jump-type "territory" to continuous (Wiener noise) stochastic processes. It is interesting to observe that on the level of "μ potentials," the transition from $\mu < 2$ to $\mu = 2$ is actually smooth.

Analytically, recalling the fractional derivative transcription $(-\Delta)^{\mu/2} \equiv -\partial^\mu/\partial|x|^\mu$ and then setting "blindly" $\mu = 2$ in (13), we arrive at the semigroup potential for the operator $\hat{H} = -\Delta + \mathcal{V}_{2,2}$:

$$\mathcal{V}_{2,2}(x) = \mathcal{V}_{FP}(x) = \frac{\frac{d^2}{dx^2}\rho_2^{1/2}(x)}{\rho_2^{1/2}(x)} = \frac{2(3x^2-1)}{(1+x^2)^2}. \tag{22}$$

The notation $\mathcal{V}_{FP}(x)$ refers to the fact that this potential appears in the semigroup (self-adjoint) version (c.f. Ref. [29]) of the standard Fokker-Planck equation for a diffusion-type process. The same result (22) can be obtained from Eq. (20) at $\mu = 2$.

The expression (20) permits us to expand the potential $\mathcal{V}_{\mu,2}(x)$ near $\mu = 2$ to obtain

$$\mathcal{V}_{\mu\to 2,2}(x) \approx \frac{2(3x^2-1)}{(1+x^2)^2} - \frac{\mu-2}{(1+x^2)^2}$$
$$\times \{2x(x^2-3)\arctan x$$
$$+ (3x^2-1)[2\gamma - 3 + \ln(1+x^2)]\}, \tag{23}$$

where $\gamma \approx 0.577216$ is Euler constant. This (along with numerical curves from Fig. 1) demonstrates the continuous transition from $\mu < 2$ to $\mu = 2$ in $\mathcal{V}_{\mu,2}(x)$.

VII. GAUSSIAN μ TARGETING FOR LÉVY FLIGHTS

In the previous publications [9,13,22] we have investigated various patterns of jump-type and diffusive behavior that would produce *a priori* selected, basically heavy-tailed PDFs in the large time asymptotics. While an association of jump type-processes with PDFs possessing a finite number of moments is rather natural, an observation of Ref. [13] that diffusion-type processes may as well admit such asymptotic PDFs, may be classified as "unnatural."

Here we proceed in the very same "unnatural" vein, asking for a Lévy-stable jump-type dynamics, whose asymptotic PDF would have a definite Gaussian form. Let us select the Gaussian target PDF

$$\rho_* = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}, \tag{24}$$

whose square root $\rho_*^{1/2}(x) \equiv f(x) = (2\pi\sigma^2)^{-1/4}\exp(-x^2/4\sigma^2)$ has the Fourier image $(\rho^*)^{1/2}(k) \equiv f(k) = (2\sigma^2/\pi)^{1/4}\exp(-k^2\sigma^2)$. That gives

$$\mathcal{V}_{\mu G}(x) = -\frac{\sigma^{-\mu}}{\sqrt{\pi}}e^{\frac{x^2}{4\sigma^2}}\Gamma\left(\frac{1+\mu}{2}\right){}_1F_1\left[\frac{1+\mu}{2},\frac{1}{2},-\frac{x^2}{4\sigma^2}\right], \tag{25}$$

where ${}_1F_1(a,b,x)$ is a hypergeometric function [38]. This μ family of semigroup potentials sets solution to the Lévy stable targeting problem, if the desired target has the Gaussian form.

Minor comments are necessary for a qualitative assessment of the above analytic result. The potential $\mathcal{V}_{\mu G}(x)$ (25) depends on two parameters: order of fractional derivative μ and variance σ. It can be seen from Eqs. (24) and (25) that the variance σ simply alters the width of the potential curve and does not influence its shape. The same is true for the factor $\sigma^{-\mu}$ in front of Eq. (25). That is why in Fig. 2 we report the shape of the potential (25) in normalized variables $z = x/(2\sigma)$ and $y_\mu = \sigma^\mu\mathcal{V}_{\mu G}(x)$. These universal curves are the same for any σ and depend on the single parameter μ. Note, that in these variables the $\mu = 2$ parabola assumes the form $y_2 = z^2 - 1/2$.

It is also seen from Fig. 2 that at small μ the potential y_μ is around -1 [we recollect that at $\mu = 0$ $\mathcal{V}_{\mu G}(x) \equiv -1$], while at larger x it has very steep growth like $\exp(z^2)$. These steep tails flatten as μ grows and around $\mu = 1.5$ the exponential growth of the potential is replaced by power-law z^μ so that at $\mu = 2$ we have the correct asymptotics z^2.

VIII. REVERSE ENGINEERING: ASYMPTOTIC μ TARGETS FOR LÉVY OSCILLATORS

Now we pass to a detailed discussion of a particular class of solvable examples of the reverse engineering problem which well illustrates the following general strategy (its full description is moved to another publication): given *a priori* a

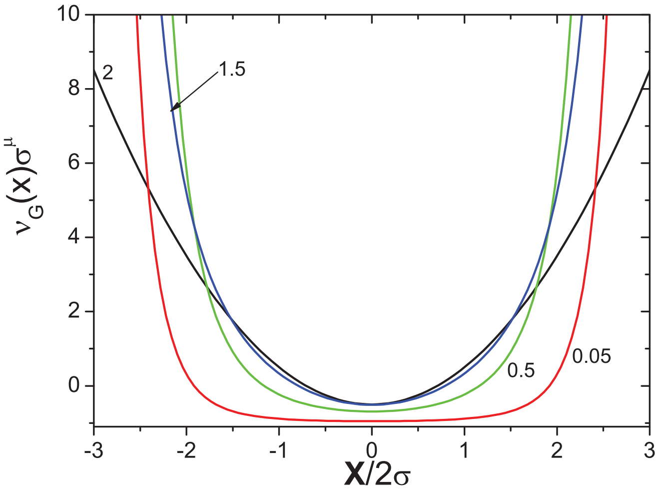

FIG. 2. (Color online) The potential (25) in normalized variables. Figures near curves correspond to μ values.

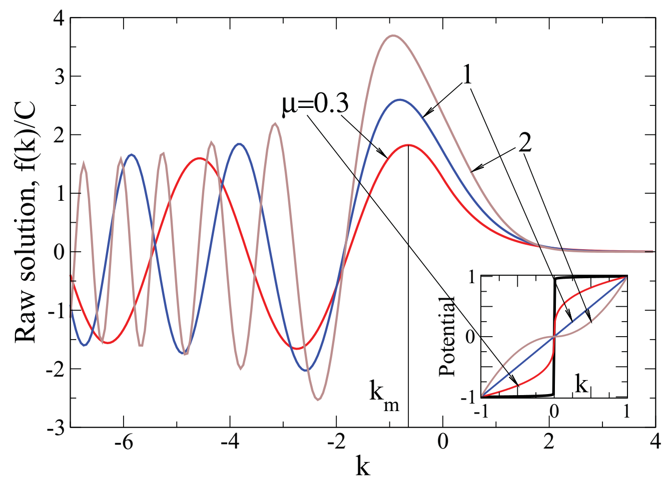

FIG. 3. (Color online) Raw solutions of Eq. (33) (main panel) and potential sign $k\ |k|^{\mu}$ (inset). Curves are μ-labeled. Arrows show the correspondence between potential and raw solution for given μ. Thick black line on the inset shows the potential for $\mu = 0.01$, which has almost rectangular shape. Solution for $\mu = 1$ corresponds to Airy function (35).

concrete semigroup with Lévy driver, infer an asymptotic PDF for the associated master equation (11).

Our main idea is to adopt an approach we have developed before [9] (see also [34,40]) to the Lévy oscillator with $\mathcal{V}(x) = x^2/2$ and arbitrary stability index μ.

We begin with the equation for a terminal PDF ρ_*, inferred from the μ-Lévy semigroup with a predefined harmonic potential

$$\mathcal{V}_\mu(x)\rho_*^{1/2} \equiv \frac{x^2}{2}\rho_*^{1/2} = -|\Delta|^{\mu/2}\rho_*^{1/2}, \quad 0 < \mu \leqslant 2. \tag{26}$$

We take Fourier images of both sides of Eq. (26) to obtain

$$u_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x^2}{2}f(x)e^{\imath kx}dx = -\frac{1}{2}\frac{\partial^2 f(k)}{\partial k^2}. \tag{27}$$

The right-hand side of Eq. (26) has the form $-|k|^{\mu} f(k)$ so that

$$\frac{\partial^2 f(k)}{\partial k^2} \equiv \frac{d^2 f(k)}{dk^2} = 2|k|^{\mu} f(k). \tag{28}$$

The idea to solve the Eq. (28) for arbitrary $0 < \mu \leqslant 2$ is borrowed from Ref. [40], where the solution for $\mu = 1$ had been obtained in terms of Airy functions. The method of Ref. [40] is based on the consideration of 1D Schrödinger problem with a potential being even function of the coordinate, which implies that the corresponding eigenfunctions should be either even or odd (see, e.g., [41,42]). In particular, the ground-state wave function should be even as it does not have nodes [41]. It can be shown that solution $f(k)$, defining the Fourier image of desired terminal PDF, corresponds to the ground-state wave function of the above Schrödinger problem. Generalizing the method of Ref. [40] for arbitrary μ, we can show that to obtain this function for even potential like $|k|^{\mu}$ we should consider instead of (28) the equation $\frac{d^2 f(k)}{dk^2} = 2\ \mathrm{sign}k\ |k|^{\mu} f(k)$ or

$$\begin{cases} \frac{d^2 f(k)}{dk^2} = 2k^{\mu} f(k), & k > 0 \\ \frac{d^2 f(k)}{dk^2} = -2(-k)^{\mu} f(k), & k < 0. \end{cases} \tag{29}$$

Now the scenario of obtaining the desired $f(k)$ is as follows. After finding the exponentially decaying solution of Eq. (29) for $k > 0$ and oscillatory one at $k < 0$, we should require the continuity of the function $f(k)$ and its derivative at $k = 0$. This is because the Eq. (29) is of the second order. After that we should find the position k_m of the first maximum of oscillating part and shift the solution to the right by k_m so that the first maximum of oscillatory part is at $k = 0$. Then "chopping" the rest of oscillating part and reflecting the obtained piece about the vertical axis to obtain the even "bell-shaped" function. The resultant solution in the k space should be Fourier-inverted and squared to yield the desired terminal PDFD in the x space.

To fulfill this scenario, we observe the following form of solutions of Eq. (29) for $k > 0$ and $k < 0$ [39]. Namely,

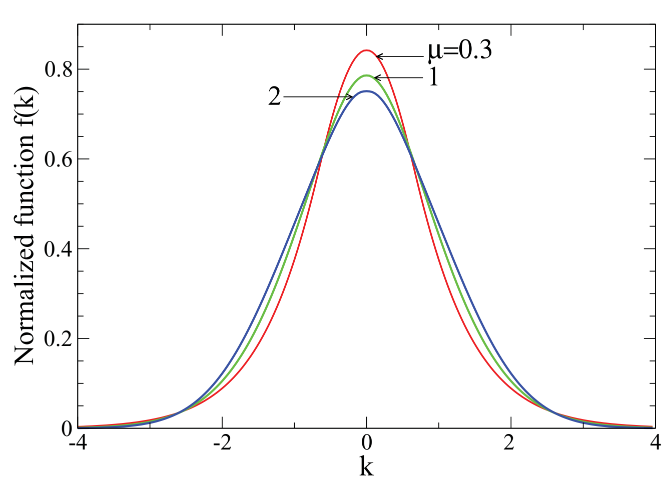

FIG. 4. (Color online) Normalized solutions for Fourier images of square roots of terminal PDFs in k space. Curves are μ-labeled.

for $k \geqslant 0$

$$f(k) = \sqrt{k}\left[C_{11} I_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right) + C_{12} K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right)\right], \tag{30}$$

while for $k < 0$

$$f(k) = \sqrt{|k|}\left[C_{21} J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22} N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right], \tag{31}$$

where $q = (\mu+2)/2$. Here $J_\nu(x)$ and $N_\nu(x)$ are Bessel functions and $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions, see Ref. [38]. At $x \to \infty$ $I_\nu(x)$ is exponentially growing function [38] while $K_\nu(x)$ is exponentially decaying [38]. On the other hand, as $x \to -\infty$ the functions $J_\nu(x)$ and $N_\nu(x)$ have "needed" oscillatory asymptotics [38]. This means that to have a localized PDF, we should leave the term with $K_{\frac{1}{2q}}$ in (30) only. Then $f(k)$ assumes the following form:

$$f(k) = \begin{cases} C_{12}\sqrt{k} K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right), & k \geqslant 0 \\ \sqrt{|k|}\left[C_{21} J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22} N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right], & k < 0. \end{cases} \tag{32}$$

Now we join (glue) the obtained solutions at $k = 0$ to secure a continuity of a function and its first derivative.

The gluing procedure yields

$$f(k) = C\sqrt{|k|}\begin{cases} K_\nu(u), & k \geqslant 0 \\ \frac{\pi}{2}\left[\cot\frac{\pi\nu}{2} J_\nu(u) - N_\nu(u)\right], & k < 0, \end{cases} \tag{33}$$

where $C \equiv C_{12}$,

$$\nu = \frac{1}{2q} \equiv \frac{1}{\mu+2}, \quad u = \frac{\sqrt{2}}{q}|k|^q \equiv \frac{2\sqrt{2}}{\mu+2}|k|^{1+\frac{\mu}{2}}. \tag{34}$$

We note here that for the Cauchy driver, i.e., $\mu = 1$ we obtain from (33) the result

$$f(k) = C\sqrt{k} K_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}k^{\frac{3}{2}}\right) = C\frac{\pi\sqrt{3}}{2^{\frac{1}{6}}}\mathrm{Ai}(2^{\frac{1}{3}}\, k), \tag{35}$$

known from our earlier publication [9].

The "raw" solutions (33) are plotted in the main panel of Fig. 3 for different values of μ. It is seen from the inset that for $\mu \to 0$ (thick black line corresponding to $\mu = 0.01$) the potential has the shape of almost rectangular barrier, corresponding to decaying solution (localized particle inside the barrier) at $k > 0$ and oscillating one (free particle) at $k < 0$ [41,42]. We note here that for potentials depicted in the inset to Fig. 3 the above kind of solution exist only if its eigenenergy lies between the limiting values of a barrier at $|x| \to \infty$ [41,42]. In this case the zeroth eigenenergy, which is the case for Eqs. (28) and (29), perfectly suits the problem under consideration not only for $\mu \to 0$, where the barrier is almost rectangular, but also at higher μ. This explains the fact that as the shape of barrier deviates from rectangular one at μ increase, the oscillations at $k < 0$ start to decay, the strongest one being at $\mu = 2$. Also, with the growth of μ, the period of the oscillations lowers, the minimum being achieved at $\mu = 2$ also.

Now we find the position k_m of the first maximum of oscillating part. Equating to zero the first derivative of an oscillating part of (33) we arrive at

$$N_{\nu-1}(u) - \cot\frac{\pi\nu}{2} J_{\nu-1}(u) = 0, \tag{36}$$

where ν and u are defined by (34). The roots of Eq. (36) can easily be obtained numerically for different μ.

The normalization of the obtained function can be achieved through the condition $C^2\int_{-\infty}^{\infty} f^2(k)dk = 1$ or

$$2C^2\left[\int_0^{-k_m} f_1^2(k)\,dk + \int_{-k_m}^{\infty} f_2^2(k)dk\right] = 1, \tag{37}$$

where f_1 and f_2 denote oscillatory and decaying parts of Eq. (33), respectively. Normalized solutions in the k space for different μ's are reported in Fig. 4. It is seen that for small k and on the tails, the distribution functions for higher μ's run below those for smaller μ's, while in the intermediate k range the situation is opposite.

The final step of the procedure is to invert the k-space solutions to the x space and square them to obtain the desired terminal PDF. For general μ this procedure can be accomplished only numerically.

Figure 5 displays both the inverted functions $f(k)$, corresponding to square roots of the inferred terminal PDFs (a) and those PDFs themselves (b). The opposite (if compared to this in the k space) tendency is seen in the x space, where the curve corresponding to lowest μ lies below all other curves in the small x region and has slowest decay. As μ grows, the central part of the curve rises and tails become steeper.

Panel (c) shows the microscopic Boltzmann potential $U(x) = -\ln[\rho_*(x)]$, corresponding to terminal PDFs from (b). This visualizes the microscopic potentials discussed in the Introduction.

Panel (d) of Fig. 5 reports a comparison between the shapes of functions $f(k)$ and $f(x)$. The situation here is the same as that for the Airy function, as discussed in [25]. Namely, the function in k space decays quicker then in x space and its value at the center is larger then that in x space. We plot here the exemplary case of $\mu = 0.5$, the situation for other μ is qualitatively the same.

IX. OUTLOOK

The next natural step in our μ-targeting procedure is to obtain (numerically) the dynamics of a function $\rho(x,t)$ for Lévy oscillators with different values of μ. This can be done both for the semigroup process (16) and for the Langevin-driven one (e.g., fractional Fokker-Planck dynamics). Those patterns of temporal behavior are inequivalent, although both processes may terminate at common PDFs with a predefined decay at infinities. The latter PDFs may have heavy tails, but generically admit an arbitrary (finite, eventually infinite) number of moments. A more general problem would be that of the existence of terminal PDFs, after passing from the master equation to the (fractional) Hamiltonian dynamics (12) with an arbitrary potential $\mathcal{V}$, in one, two, or three spatial dimensions.

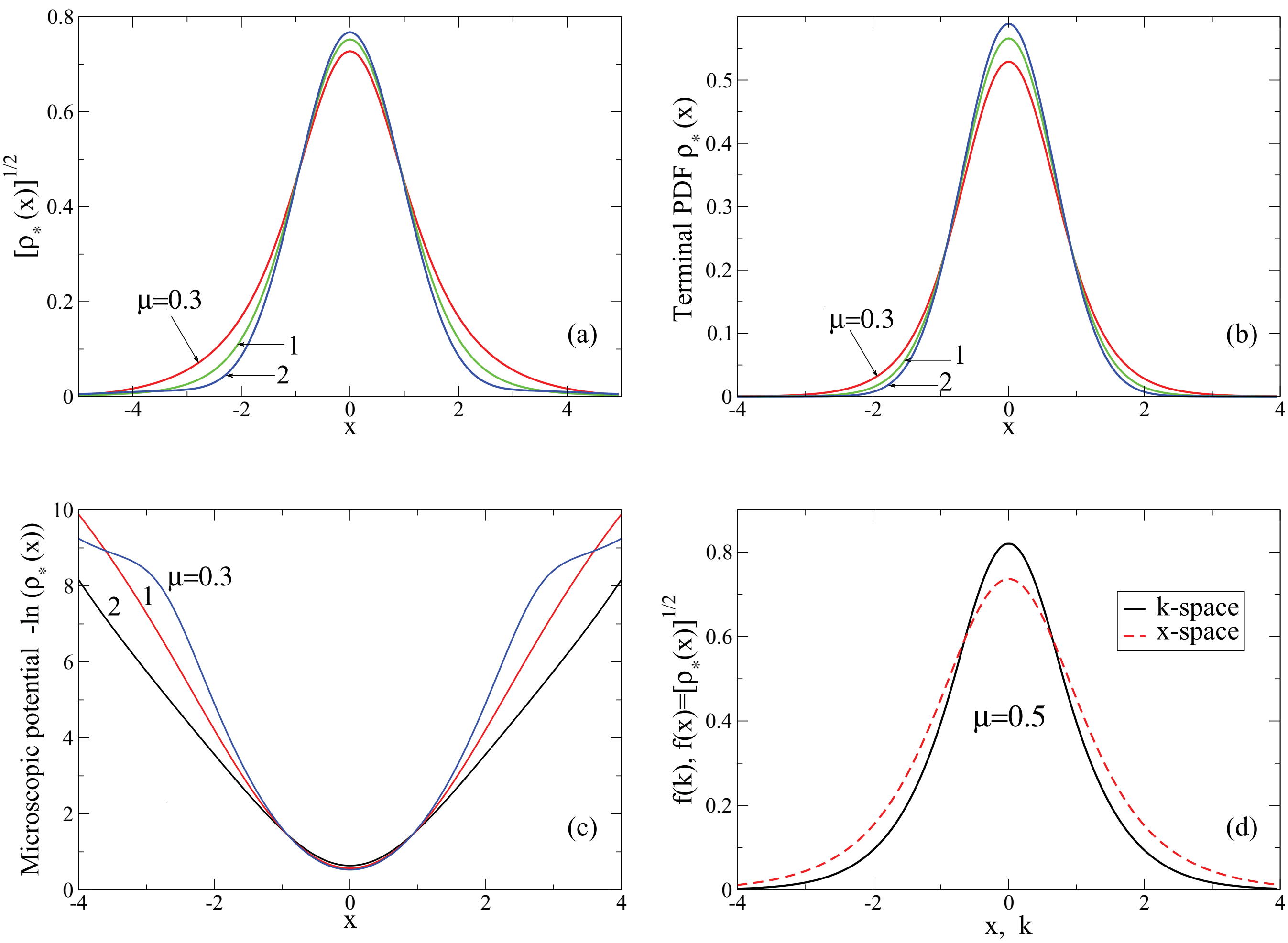

FIG. 5. (Color online) (a) Inverted Fourier images $[\rho_*(x)]^{1/2}$. (b) Desired terminal PDFs at different μ (figures near curves). (c) Shows the Boltzmann potential $U(x)$ obtained as a negative logarithm of the terminal PDF. (d) Compares the behavior of the functions in k and x spaces for $\mu = 0.5$.

In the case of $\mu = 2$, c.f. Eq. (22), the fractional Hamiltonian (12) may be formally replaced by an ordinary quantum-mechanical Hamiltonian operator. In the standard quantum mechanical setting (see, e.g., Refs. [41,42]) the above PDF existence problem is equivalent to the existence of bound states of a particle in a given potential. The quantum mechanical language is appropriate, because we can convert the parabolic equation of the Fokker-Planck type to the generalized Schrödinger equation.

The wave function of a bound state should be localized to ensure a normalization of its squared expression, i.e., the corresponding stationary PDF of the Fokker-Planck equation. It is known (see, e.g., Ref. [41]) that in the 1D case the bound state exists in the potential well $U(x)$ of not only finite but an infinitesimal depth. The only restriction is that the integral $\int_{-\infty}^{\infty} U(x)dx$ should exist. The latter condition is equivalent to the requirement that $U(x)$ should have the same asymptotics at infinities and potential zero point $U(\pm\infty) = 0$. In the 2D case, when the potential $U = U(x,y)$, the situation is similar to that in 1D one, while in 3D $[U = U(x,y,z)]$ the situation is to some extent opposite—if the potential well is not sufficiently deep (see Ref. [41] for details), the particle cannot be "captured," so that bound state does not exist. Confining potentials in 3D, where bound states exist, form the so-called Kato class of potentials.

The presence of fractional derivatives with $0 < \mu \leqslant 2$ alters the picture both in 1D (2D) and in 3D. In 1D they definitely "spoil" the bound states. It is not only that the PDFs (if in existence) may have heavier tails if compared to the conventional ($\mu = 2$) case. The PDFs in question may not exist at all, if a normalizability of the bound state is lost. In 3D and in equations with fractional derivatives there may typically be no normalizable bound states (and thus terminal PDFs), except for a carefully selected (Kato-)subclass of conceivable potentials.

Some peculiarities pertaining to the (non)existence of invariant PDFs in the case of Lévy drivers (Langevin-driven fractional dynamics) were discussed for the 1D case in Ref. [30]. We have encountered the same problem in connection with the Cauchy family of PDFs [9,13], see also Ref. [43] for a discussion of so-called infinite covariant densities.

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