Detection of weak signals through nonlinear relaxation times for a Brownian particle in an electromagnetic field

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The detection of weak signals through nonlinear relaxation times for a Brownian particle in an electromagnetic field is studied in the dynamical relaxation of the unstable state, characterized by a two-dimensional bistable potential. The detection process depends on a dimensionless quantity referred to as the receiver output, calculated as a function of the nonlinear relaxation time and being a characteristic time scale of our system. The latter characterizes the complete dynamical relaxation of the Brownian particle as it relaxes from the initial unstable state of the bistable potential to its corresponding steady state. The one-dimensional problem is also studied to complement the description.

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I. INTRODUCTION

Since its introduction in 1981 [1] stochastic resonance (SR) has been recognized as a paradigm for noise-induced effects in driven nonlinear dynamic systems. The phenomenon has been demonstrated in a variety of physical, chemical, and biological systems [1–6]. For any of these systems operating in strong noisy environments and subject to a periodic modulating signal, so weak as to be normally undetectable, the mechanism of SR appears when both the weak signal and noise enter in resonance, increasing the detectability of the weak signal and the transmission efficiency of information. On the other hand, among the noise-induced effects in driven nonlinear dynamics we can mention the problem of the time characterization of transient behavior of nonequilibrium phenomena, also of great interest in physics, chemistry, and biology [7,35]. Practically all of the above mentioned phenomena are essentially formulated in terms of the one-dimensional Brownian motion of particles in a potential field in a high-damping (large-viscosity) regime and admit a description in terms of the Fokker-Planck equation (FPE), master equation, and Langevin equation.

Alternative to the stochastic resonance, there exists another physical mechanism capable of detecting weak signals in the study of nonequilibrium phenomena. It consists in the decay of the unstable state of a given system driven by both stochastic fluctuations and some external force. Indeed, at the end of the 1980s and at the beginning of the 1990s it was theoretically and experimentally demonstrated in the transient dynamics of a laser with an external signal [12–15]. It was shown first by Vemuri and Roy [12] that very weak optical signals can be detected in the transient dynamics of the laser, using this system as a super-regenerative receiver in the same way as used in radar detectors [16]. The physical idea behind the proposal is that the weak signal is greatly amplified when used to trigger the decay of the unstable state. In 1989 a theoretical criterion related to the statistics of the mean first passage time (MFPT) distribution was proposed to detect weak optical signals in the switch-on process of a laser system [14]. The study focuses on the time characterization of the decay process of the laser's intensity, which relaxes from the initial value denoted by $I(0)$ to a given reference value $I_r = 0.02 I_{st}$ representing the absorbing barrier, with I_{st} being its steady-state value. The statistics of the MFPT has also been successfully applied to the efficient detection of large optical signals in a rotational laser system [17] and used for the detection of weak periodic signals in the transient dynamics of a class-A laser in which the resonance effect is known to exist [18]. Only very recently the results given in Refs. [14,17] have been extended to the study of the detection of weak and large electric fields in the dynamical relaxation of the unstable state of a Brownian charged particle in the presence of a constant electromagnetic field [19]. As shown in Ref. [19], for certain values of the relevant parameters the critical value of the parameter with weak detection is accounted for is the same as that calculated in Ref. [14], being in itself a surprising result.

The nonlinear relaxation time (NLRT), introduced long ago [7], is another time scale used to characterize the general relaxation process from arbitrary initial conditions to the corresponding steady states, and it parallels to some extent the MFPT approach. The decay of an unstable, metastable state is just a particular case, among others [8–15,21–27]. The most standard analysis of the decay of an unstable state assumes a one-dimensional Langevin equation with additive noise. Then the initial unstable state corresponds to a relative maximum of the effective potential, which gives the deterministic force. The study of the transient evolution of such a system has been focused on different descriptions, namely: the evolution of the statistical moments of the relevant variables in terms of the FPE [8–11], the description in terms of the MFPT distribution [14,20], and the inverse probability current also in the context of FPE [21–27].

The criteria proposed by Vemuri and Roy [12] to detect weak optical signals in the transient dynamics of a laser were

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given in terms of a dimensionless quantity named the receiver output (RO). This quantity was calculated as a function of the MFPT distribution and shown to be sensitive to the presence of the weak signal. The proposed method to measure the RO in the laser system is as follows: The laser works as a superregenerative receiver and is periodically switched on and off. The RO is defined as the ratio A_e/A_0, where A_e is the area under the curve of the time evolution of the mean output laser intensity in the presence of the electric field and A_0 is the area under the curve of the mean output laser intensity in the absence of this same field. In 1991 Jiménez-Aquino and Sancho [15] showed that the RO can also be calculated as a function of the NLRT and used to detect weak optical signals for the same laser system studied in Ref. [12]. The theoretical results of Ref. [15] were compared with those reported in Ref. [12] for the dye laser system only, showing excellent agreement.

Our contribution in this paper is to propose the RO as an alternative method to detect weak electric fields in the decay process of the unstable state of a Brownian charged particle under the influence of a two-dimensional bistable potential and in the presence of a constant electromagnetic field. The RO is calculated as a function of the NLRT, which is the characteristic time required by the particle to relax from its initial unstable state to the corresponding steady state of the bistable potential. It will also be shown, by means of this time scale, that the RO is sensitive to the presence of weak electric fields. The main difference between this time scale with the MFPT is that it characterizes the complete dynamical relaxation of the system. To achieve our goal, we choose a quench time denoted by T_c, wherewith the particle relaxes from the initial value $r^2(0) = 0$ to its steady-state value $r^2_{\rm st}$, where $r^2 = x^2 + y^2$ is the square modulus of the two-dimensional position vector $\mathbf{r} = (x,y)$. Thus, T_c should be taken with care to ensure that the particle relaxes very close to its steady state. To calculate the NLRT we choose its connection with the quasi-deterministic (QD) approach [8,14,20] because this provides a simple way to solve the problem of how to deal with an arbitrary nonlinear unstable potential without using a description in terms of the FPE. The QD approach is a good approximation because it gives a precise physical picture of the mechanism responsible for the decay of the unstable state. The physical mechanism is twofold: Small fluctuations change the initial condition in the surrounding of the unstable state, and afterwards the deterministic motion drives the system out of this state. This mechanism also holds for a charged Brownian particle in the presence of an additional electromagnetic field. In this case the decay process of the charged particle is accelerated by the electric force and rotationally evolved due to the action of the magnetic field. Our theoretical study is also given in the high-friction limiting case, so that the relaxation process will be described by the overdamped approximation (diffusive regime) of the Langevin equation.

This paper is organized as follows: In Sec. II we first study the problem in the one-dimensional case and use the QD approach to calculate the NLRT and the MFPT associated with the decay process of arbitrary unstable potential in the presence of a constant external force; the results for the symmetric bistable potential are obtained as a particular case. Some of the consequences of the effect of the noise delayed decay [24] are briefly discussed in this section. The problem of an electrically charged Brownian particle is studied in Sec. III, wherein we introduce the Langevin equation for a charged Brownian particle in a two-dimensional bistable potential under the action of a uniform electromagnetic field; the overdamped approximation of this equation is then formulated for crossed electric and magnetic fields. We also calculate the NLRT for the charged Brownian particle for arbitrary nonlinear unstable potentials in an appropriate space of coordinates wherein the QD is better understood. The MFPT, including nonlinear contributions, as well as the NLRT for the symmetric bistable potential, are also calculated. In Sec. IV we calculate the RO as a function of the NLRT for the bistable potential in the two-dimensional case. Our concluding remarks are given in Sec. V. We end our work with three appendices to further clarify our proposal. The physical picture of the QD approach is qualitatively explained in Appendix A. In Appendix B, we give an alternative calculation of how to deal with the nonlinearities of the potential profile. Finally, Appendix C deals with an explicit calculation of how to obtain the decay time in the one-dimensional case.

II. NLRT FOR BROWNIAN PARTICLE IN THE ONE-VARIABLE CASE

Consider a particle of mass m embedded in a thermal bath at temperature T initially located on the equilibrium unstable state ($x(0) = 0$) of a one-dimensional bistable potential $V(x) = (a/2)x^2 - (b/4)x^4$, with $a,b > 0$ and in the presence of a constant external force f_e. The Langevin equation in the diffusive regime is given by

$$\dot{x} = \bar{a}x - \bar{b}x^3 + \alpha^{-1} f_e + \alpha^{-1}\xi(t), \tag{1}$$

where $\bar{a} = a/\alpha$, $\bar{b} = b/\alpha$, and α being the friction constant. $\xi(t)$ is a fluctuating force that satisfies the property of Gaussian white noise with zero mean value $\langle \xi(t)\rangle = 0$ and correlation function $\langle \xi(t)\xi'(t')\rangle = 2\lambda\,\delta(t-t')$. The fluctuation-dissipation relation being satisfied is $\lambda = \alpha k_B T$, with k_B standing for the Boltzmann constant.

A. NLRT and QD approach

The dynamical characterization of the decay of the unstable state of the particle will be given by the NLRT, which focuses on the dynamic relaxation of the average $\langle x^2(t)\rangle$, where $\langle\cdots\rangle$ stands for realizations of the noise $\xi(t)$ and over initial conditions. The average $\langle x^2(t)\rangle$ evolves from an initial value $\langle x^2(0)\rangle$ to its steady-state value $\langle x^2\rangle_{\rm st}$. The NLRT was introduced long ago [7] and studied afterwards within the present context in Refs. [11,15]. Here we are interested in the following definition:

$$T = \int_0^\infty \frac{\langle x^2(t)\rangle - \langle x^2\rangle_{\rm st}}{\langle x^2(0)\rangle - \langle x^2\rangle_{\rm st}}\, dt. \tag{2}$$

This time scale, along with the QD approach, allows us to characterize, not only the complete dynamical relaxation of Eq. (1), but also the relaxation processes associated with arbitrary nonlinear unstable potentials, the bistable potential being just a particular case. For such a purpose, we introduce a more general definition of a deterministic nonlinear unstable state in terms of the scalar variable $\mathfrak{r} \equiv x^2$ [20]. The deterministic dynamics for this variable then reads as

$$\frac{d\mathfrak{r}}{dt} = f(\mathfrak{r}), \quad f(\mathfrak{r}) = \frac{\mathfrak{r}(\mathfrak{r}_{\rm st} - \mathfrak{r})}{C_0 + \mathfrak{r} g(\mathfrak{r})}, \tag{3}$$

where $C_0 = \mathfrak{r}_{\rm st}/2\bar{a}$ is the steady-state value and $g(\mathfrak{r}) > 0$ is a polynomial. The function $f(\mathfrak{r})$ has two roots: One is at $\mathfrak{r} = 0$, which corresponds to the unstable state such that $f'(\mathfrak{r})|_{\mathfrak{r}=0} > 0$, and the other root is at $\mathfrak{r} = \mathfrak{r}_{\rm st}$, corresponding to the stable state and thus $f'(\mathfrak{r})|_{\mathfrak{r}=\mathfrak{r}_{\rm st}} < 0$. The deterministic evolution of Eq. (1) without the external force must be compatible with Eq. (3) for a particular expression of $g(\mathfrak{r})$.

The connection between the NLRT and the QD approach can be achieved by assuming that $\mathfrak{r}(0) \equiv x^2(0) = h^2$ is a random variable that plays the role of an effective initial condition responsible for the decay of the unstable state toward its steady state characterized by the value $\mathfrak{r}(\infty) \equiv x^2(\infty) = \mathfrak{r}_{\rm st}$. Upon substitution of Eq. (3) into Eq. (2) and assuming fixed initial conditions such that $\langle \mathfrak{r}(0)\rangle = 0$ we get, in terms of $\mathfrak{r}$,

$$\begin{aligned} T &= \int_0^\infty \frac{\langle \mathfrak{r}(t)\rangle - \langle \mathfrak{r}\rangle_{\rm st}}{\langle \mathfrak{r}(0)\rangle - \langle \mathfrak{r}\rangle_{\rm st}} dt = \frac{1}{\mathfrak{r}_{\rm st}} \left\langle \int_{h^2}^{\mathfrak{r}_{\rm st}} \frac{\mathfrak{r}_{\rm st} - \mathfrak{r}}{f(\mathfrak{r})} d\mathfrak{r} \right\rangle \\ &= \frac{1}{2\bar{a}} \left\langle \ln\left(\frac{\mathfrak{r}_{\rm st}}{h^2}\right)\right\rangle + \frac{1}{\mathfrak{r}_{\rm st}} \left\langle \int_{h^2}^{\mathfrak{r}_{\rm st}} g(\mathfrak{r})\, d\mathfrak{r}\right\rangle. \end{aligned} \tag{4}$$

The logarithmic term is the universal and relevant contribution arising from the time characterization of the decay process in the linear regime of the nonlinear potential, wherein the stochastic fluctuations are dominant. The last term comes from the nonlinear contributions of the potential away from the initial unstable state. As stated in Appendix A, the QD approach tells us that in the nonlinear regime the dynamical evolution of the particle is practically deterministic and the stochastic fluctuations are not relevant, thus $h \to 0$. Under these circumstances the NLRT becomes

$$T = \frac{1}{2\bar{a}} \left\langle \ln\left(\frac{\mathfrak{r}_{\rm st}}{h^2}\right)\right\rangle + C_{\rm NL}, \tag{5}$$

where $C_{\rm NL}$ is a constant that is calculated through

$$C_{\rm NL} = \lim_{h\to 0} \frac{1}{\mathfrak{r}_{\rm st}} \int_{h^2}^{\mathfrak{r}_{\rm st}} g(\mathfrak{r})\, d\mathfrak{r}, \tag{6}$$

which accounts for nonlinear contributions and is a model-dependent quantity. We give in Appendix B an alternative calculation of this quantity. The time scale given by Eq. (5) characterizes the complete dynamical relaxation of arbitrary nonlinear unstable potentials in terms of the relaxing quantity $\langle \mathfrak{r}\rangle$. The first term of Eq. (5) can explicitly be calculated using the QD approach, which relies upon the linear approximation of Eq. (1) and reads

$$\dot{x} = \bar{a}x + \alpha^{-1} f_e + \alpha^{-1}\xi(t). \tag{7}$$

The solution of Eq. (7), assuming the initial condition $x(0) = 0$, is $x(t) = h(t)\, e^{\bar{a}t}$, where

$$h(t) = \alpha^{-1} \int_0^t e^{-\bar{a}s}[f_e + \xi(s)]\, ds. \tag{8}$$

According to the QD approach [8,14,20], the process $h(t)$ plays the role of an effective initial condition and, as time increases, becomes a Gaussian random variable (GRV). This is indeed the case since for small values of $\xi(t)$ and f_e,

$$\lim_{t\to\infty} \frac{dh(t)}{dt} = \lim_{t\to\infty} \alpha^{-1} e^{-\bar{a}t}[f_e + \xi(t)] \to 0. \tag{9}$$

Thus, at large times $h(\infty) = h$, where h is GRV. In this case the process $x(t)$ becomes a quasideterministic one such that

$$x^2(t) = h^2\, e^{2\bar{a}t}. \tag{10}$$

In this linear approximation, Eq. (10) can also be written, taking into account the whole process, as $x^2(t) = h^2\, e^{2\bar{a}t}\, \theta(t_i - t) + x_{\rm st}^2\, \theta(t - t_i)$, where $\theta(t)$ is the step function. After substitution of this expression into Eq. (2) and a time integration we get, in the linear approximation, that the NLRT is

$$T_{_L} = \frac{1}{2\bar{a}} \left\langle \ln\left(\frac{\mathfrak{r}_{\rm st}}{h^2}\right)\right\rangle - C_{_L}, \tag{11}$$

and $C_{_L} = (1/2\bar{a})[1 - \langle h^2\rangle/\mathfrak{r}_{\rm st}]$. This time scale can be calculated from the marginal probability density $P(h)$, which is the Gaussian distribution function given by

$$P(h) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(h - \langle h\rangle)^2/2\sigma^2}, \tag{12}$$

with $\sigma^2 \equiv \langle h^2\rangle - \langle h\rangle^2$ being the variance. From Eq. (8) it can be shown that

$$\langle h\rangle = \alpha^{-1} \int_0^\infty e^{-\bar{a}t}\, f_e\, dt = \frac{f_e}{a}, \tag{13}$$

$$\begin{aligned} \langle h^2\rangle &= \langle h\rangle^2 + \alpha^{-2} \int_0^\infty \int_0^\infty e^{-\bar{a}(t+t')} \langle \xi(t)\xi(t')\rangle\, dt\, dt' \\ &= \langle h\rangle^2 + \frac{\lambda}{a\alpha}, \end{aligned} \tag{14}$$

and therefore $\sigma^2 = \lambda/a\alpha = k_{_B}T/a$. To calculate the constant $C_{_L}$ we need to evaluate the mean value $\langle h^2\rangle$. It is clear that $\langle h^2\rangle = \sigma^2 + f_e^2/a^2$, which can be neglected for small noise and small amplitude of the external force. Hence, the constant $C_{_L}$ can be approximated by $C_{_L} = 1/2\bar{a}$. We show in Appendix C that the NLRT given by Eq. (11) reduces to

$$T_{_L} = T_{_L}^0 - \frac{e^{-\beta^2}}{\bar{a}} \sum_{m=1}^{\infty} \frac{\beta^{2m}}{m!} \sum_{k=1}^{m} \frac{1}{2k-1}, \tag{15}$$

where $\beta^2 = \langle h\rangle^2/2\sigma^2 = f_e^2/2a\, k_{_B}T$, and

$$T_{_L}^0 = \frac{1}{2\bar{a}} \left[\ln\left(\frac{a\, x_{\rm st}^2}{2\, k_{_B} T}\right) - \psi\left(\frac{1}{2}\right) - 1 \right] \tag{16}$$

is the linear approximation of the NLRT in the absence of the external force ($\beta = 0$) and $\psi(1/2) = -2 - \ln 2$ is the digamma function [36]. From Eqs. (5), (11), and (15) we finally get that the NLRT for arbitrary nonlinear unstable potentials in the one-dimensional case reads as

$$T_e = T_0 - \frac{e^{-\beta^2}}{\bar{a}} \sum_{m=1}^{\infty} \frac{\beta^{2m}}{m!} \sum_{k=1}^{m} \frac{1}{2k-1} + C_{\rm NL}, \tag{17}$$

where

$$T_0 = \frac{1}{2\bar{a}} \left[\ln\left(\frac{a\, x_{\rm st}^2}{2\, k_{_B} T}\right) - \psi\left(\frac{1}{2}\right) \right] \tag{18}$$

is the NLRT in the absence of the external force. The QD approach is valid only for $\beta^2 \leqslant 1$, which means that the amplitude of the external force is less or equal than that of the noise. In this case Eq. (17) can be approximated by

$$T_e = T_0 - \beta^2/\bar{a} + \beta^4/\bar{a} + C_{\rm NL}. \tag{19}$$

For a bistable potential the deterministic equation associated with Eq. (3) without the external force can be written as $d\mathfrak{r}/dt = 2\bar{a}\,\mathfrak{r}(\mathfrak{r}_{\rm st} - \mathfrak{r})/\mathfrak{r}_{\rm st}$, with $\mathfrak{r}_{\rm st} \equiv x_{\rm st}^2 = a/b$. It is clear that $g(\mathfrak{r}) = 0$, and thus the NLRT in this case is the same as Eq. (19) with $C_{\rm NL} = 0$.

B. MFPT and QD approach

The study of the decay time of arbitrary nonlinear unstable potentials in the presence of a constant external force by means of the MFPT initiates with the solution given by Eq. (10). In this case the variable x is in the interval $-R \leqslant x \leqslant R$, where R represents an absorbing barrier. The random passage time required by the particle to reach the value R^2 is clearly $t = (1/2\bar{a})\ln(R^2/h^2)$. We must notice the difference between this time scale with that given by Eq. (11). Following Appendix C, it can be shown that the MFPT $\tau_{_L} = \langle t \rangle$ is now

$$\tau_{_L} = \tau_{_L}^0 - \frac{e^{-\beta^2}}{\bar{a}} \sum_{m=1}^{\infty} \frac{\beta^{2m}}{m!} \sum_{k=1}^{m} \frac{1}{2k-1}, \tag{20}$$

where

$$\tau_{_L}^0 = \frac{1}{2\bar{a}} \left[\ln\left(\frac{a\,R^2}{2\,k_{_B}T} \right) - \psi\left(\frac{1}{2}\right) \right] \tag{21}$$

is the linear approximation of the MFPT in the absence of the external force ($\beta = 0$). Again, for $\beta^2 \leqslant 1$ we have $\tau_{_L} = \tau_{_L}^0 + \beta^2/\bar{a} - \beta^4/\bar{a}$. Now, to deal with nonlinear contributions we use again Eq. (3) and its connection with the QD approach such that $r^2(0) = h^2$, and thus

$$t = \int_{h^2}^{R^2} \frac{d\mathfrak{r}}{f(\mathfrak{r})} = \int_{h^2}^{R^2} \frac{C_0 + \mathfrak{r} g(\mathfrak{r})}{\mathfrak{r}(\mathfrak{r}_{\rm st} - \mathfrak{r})}. \tag{22}$$

After some easy algebra this time scale can be written as $\tau = (1/2\bar{a})\langle \ln(R^2/h^2)\rangle + K_{\rm NL}$, where $K_{\rm NL}$ takes into account the nonlinear effects and reads as

$$K_{\rm NL} = \lim_{h\to 0} \left[\frac{1}{2\bar{a}} \int_{h^2}^{R^2} \frac{d\mathfrak{r}}{\mathfrak{r}_{\rm st} - \mathfrak{r}} + \int_{h^2}^{R^2} \frac{g(\mathfrak{r})}{\mathfrak{r}_{\rm st} - \mathfrak{r}} \right]. \tag{23}$$

For the bistable potential $g(\mathfrak{r}) = 0$ and $K_{\rm NL} = (1/2\bar{a})\ln[M^2/(1 - M^2)]$, with $M^2 = R^2/x_{\rm st}^2$. Finally the MFPT, including the saturation term, is given by

$$\tau = \frac{1}{2\bar{a}} \left\{ \ln\left[\frac{a\,R^2\,M^2}{2\,k_{_B}T(1 - M^2)} \right] - \psi\left(\frac{1}{2}\right) \right\} + \frac{e^{-\beta^2}}{\bar{a}} \sum_{m=1}^{\infty} \frac{\beta^{2m}}{m!} \sum_{k=1}^{m} \frac{1}{2k-1}. \tag{24}$$

In the absence of the external force it is exactly the same as that calculated by Haake *et al.* [9] and also given by Eq. (16) in Ref. [24]. Notice that M^2 measures how close the particle is to its stationary state value, notwithstanding that it never reaches that value, contrary to what happens with the NLRT given by Eq. (17).

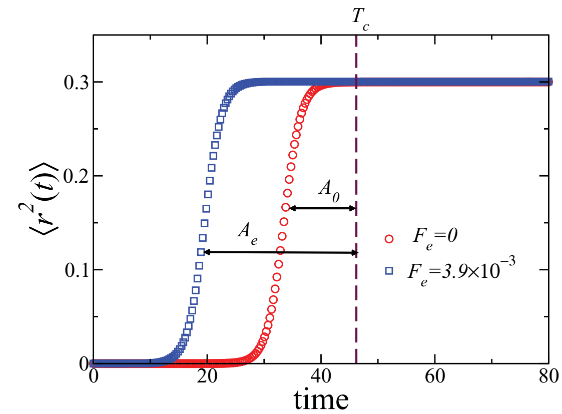

FIG. 1. (Color online) Time evolution of $\langle r^2(t)\rangle$ during the time interval $T_c = 46.2$ indicated by the vertical dashed line. Circles correspond to the zero-field case and squares to a nonzero-field value of $F_e = 4 \times 10^{-3}$. A_0 is the area under the curve wherein $F_e = 0$ and A_e is the area under the $F_e \neq 0$ curve. The parameter values employed in all our simulations are $a = 3 \times 10^2$, $b = 10^3$, $\alpha = 9 \times 10^2$, $C = 10^{-2}$, and $\lambda = 10^{-4}$ with an average over 2×10^4 different realizations.

On the other hand, when the absorbing barrier is removed, the point x can cross the point $x = \pm R$ any number of times and in any direction; in this case a study in terms of the inverse probability current [21–23] must be performed. This study leads to an unexpected effect called noise delayed decay (NDD), wherewith the stochastic fluctuations can considerably increase the decay time of unstable and metastable states. The method has been developed to calculate the NLRT for any fluctuation intensity and arbitrary potential profile [22,24]. In the particular case of small fluctuations, the NLRT coincides with the MFPT. Indeed, the NLRT defined in Ref. [24] has been calculated for the symmetric bistable potential and other potential profiles. For the symmetric bistable potential it has been shown that, for small noise intensity such that $q \ll \Phi(x_m)$, the NLRT given by Eq. (15) of Ref. [24] coincides with the MFPT calculated by Haake *et al.* [9] and given in Eq. (16) of the same Ref. [24], as expected. The comparison is shown in Fig. 9 of Ref. [24]. Herein we have also shown that the Haake *et al.* MFPT is the same as that given by our result (24) but without the external force, if we have the parameters $q = K_{_B}T$ and $x_m = \pm\sqrt{a/b}$, with $\Phi(x_m) = -a^2/4b$ as the depth of the potential profile. In conclusion, when we use the QD approach in the time characterization of the decay of the unstable state, this relaxation process is bounded by fixed and absorbing barriers, so that the inverse probability current becomes negligible. In fact, as is also shown in Fig. 1 of the next section, once the particle reaches its stationary state value the process stops at time T_c, being thus a quench time.

III. NLRT FOR BROWNIAN PARTICLE IN AN ELECTROMAGNETIC FIELD

We can go further by considering an electrically charged Brownian particle in a two-dimensional unstable potential in the presence of an electromagnetic field. Consider the above case when a particle of charge q and mass m is embedded in a thermal bath of temperature T and initially located on the equilibrium unstable state of a two-dimensional bistable potential $V(x,y) = -(a/2)(x^2 + y^2) + (b/4)(x^2 + y^2)^2$, where $a,b > 0$. $r^2 = x^2 + y^2$ is the square modulus of the position vector $\mathbf{r} = (x,y)$. The force derived from this potential reads as $\mathbf{F} = a\mathbf{r} - b\,r^2\mathbf{r}$. In addition, the particle is under the action of constant crossed electric and magnetic fields such that the latter points along the z axis, that is, $\mathbf{B} = (0,0,B)$ and the former lies on the x-y plane, i.e., $\mathbf{E} = (E_x,E_y)$. In this case the Lorentz force acting on the particle also lies on the x-y plane and is $\mathbf{F}_{L} = (q/c)\mathbf{u} \times \mathbf{B} + q\mathbf{E}$, where $\mathbf{u} = \dot{\mathbf{r}} = (u_x,u_y)$ is the planar velocity vector. The Langevin equation for the charged particle can be written as

$$m\dot{\mathbf{u}} = -\alpha\mathbf{u} + \frac{q}{c}\mathbf{u} \times \mathbf{B} + a\mathbf{r} - b\,r^2\mathbf{r} + q\mathbf{E} + \boldsymbol{\xi}(t), \tag{25}$$

where the two-dimensional fluctuating force $\boldsymbol{\xi}(t) = (\xi_x,\xi_y)$ also satisfies the property of Gaussian white noise with zero mean value $\langle\xi_i(t)\rangle = 0$ and correlation function $\xi_i(t)\xi_j(t')\rangle = 2\lambda\,\delta_{ij}\,\delta(t - t')$, with $i,j = x,y$. Again the noise intensity satisfies $\lambda = \alpha k_{B} T$. In the overdamped approximation the inertial term $m\dot{\mathbf{u}}$ is neglected, and the above Langevin equation reduces to

$$\dot{\mathbf{r}} = \widetilde{a}\,\mathbf{r} + \widetilde{W}\,\mathbf{r} - \widetilde{b}\,r^2\Lambda\,\mathbf{r} + \widetilde{q}\Lambda\,\mathbf{E} + \Lambda\,\boldsymbol{\xi}(t), \tag{26}$$

where $\widetilde{a} = a/\alpha_e$, $\widetilde{b} = b/\alpha_e$, and $\widetilde{q} = q/\alpha_e$, with $\alpha_e = \alpha(1 + C^2)$ acting as an effective friction coefficient; $C = qB/c\alpha$ is a dimensionless constant. The matrices $\widetilde{W}$ and Λ are defined as

$$\widetilde{W} = \begin{pmatrix} 0 & \widetilde{\Omega} \\ -\widetilde{\Omega} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & C \\ -C & 1 \end{pmatrix}, \tag{27}$$

with $\widetilde{\Omega} = \widetilde{a}C$. The stochastic process (26) is clearly a rotational one due to the second and third terms. To understand why the QD approach works well to describe the dynamical characterization of Eq. (26), a more appropriate dynamical description is required. This can be achieved by including the change of variable $\mathbf{r}' = e^{-\widetilde{W}t}\mathbf{r}$. In this coordinate space the Langevin Eq. (26) transforms into

$$\dot{\mathbf{r}}' = \widetilde{a}\,\mathbf{r}' - \widetilde{b}\,r'^2\Lambda\mathbf{r}' + \widetilde{q}\Lambda\,\mathbf{E}' + \Lambda\,\boldsymbol{\xi}'(t), \tag{28}$$

where $\mathbf{E}' = \mathcal{R}^{-1}(t)\mathbf{E}$, $\boldsymbol{\xi}'(t) = (1/\alpha_e)\mathcal{R}^{-1}(t)\boldsymbol{\xi}(t)$ and $\mathcal{R}(t) = e^{\widetilde{W}t}$ is an orthogonal rotation matrix such that the transpose is its inverse, that is, $\mathcal{R}^{T}(t) = \mathcal{R}^{-1}(t)$ and $\mathcal{R}^{-1}(t) = e^{-\widetilde{W}t}$ with

$$\mathcal{R}(t) = \begin{pmatrix} \cos\widetilde{\Omega}\,t & \sin\widetilde{\Omega}\,t \\ -\sin\widetilde{\Omega}\,t & \cos\widetilde{\Omega}\,t \end{pmatrix}. \tag{29}$$

In Eq. (28) the quantity $r'^2 = x'^2 + y'^2$ is the square modulus of vector $\mathbf{r}'$, and it satisfies $r'^2 = r^2$, which means that the modulus of vector $\mathbf{r}$ remains invariant under the transformation $\mathcal{R}^{-1}(t)$.

A. NLRT and QD approach

The dynamical characterization of the decay of the unstable state of the charged particle will now be given in terms of the relaxing quantity $\langle\widetilde{\mathfrak{r}}\rangle = \langle r^2\rangle$ or $\langle\widetilde{\mathfrak{r}}'\rangle = \langle r'^2\rangle$ since $r^2 = r'^2$. In this case the NLRT is the same as that defined in Eq. (2), except that x^2 is replaced by r^2 or by r'^2. It must be noticed that the $\widetilde{\mathfrak{r}}$ and $\widetilde{\mathfrak{r}}'$ variables also satisfy the same definition of the deterministic nonlinear unstable state given by Eq. (3). The deterministic evolution of Eq. (26) as well as of Eq. (28) without the electric field must be compatible with Eq. (3) for a particular expression of $g(\widetilde{\mathfrak{r}})$. In what follows, our theoretical description will be formulated in the transformed space of coordinates $\mathbf{r}'$ and $\widetilde{\mathfrak{r}}' = r'^2$. The NLRT in terms of $\widetilde{\mathfrak{r}}'$ variable is the same as Eq. (4) but replacing $\mathfrak{r}$ by $\widetilde{\mathfrak{r}}'$, $\mathfrak{r}_{\rm st}$ by $\widetilde{\mathfrak{r}}'_{\rm st}$, and $\mathfrak{r}(0) = h^2$ by $\widetilde{\mathfrak{r}}'(0) = h'^2$. Its connection with the QD approach leads to the same expressions given by Eqs. (5) and (6). To calculate the NLRT in this case, we use again the QD approach [14,20], which relies upon the linear approximation of Eq. (28), that is,

$$\frac{d\mathbf{r}'}{dt} = \widetilde{a}\mathbf{r}' + \widetilde{q}\Lambda\mathbf{E}' + \Lambda\,\boldsymbol{\xi}'(t). \tag{30}$$

The solution of Eq. (30), assuming the initial condition $\mathbf{r}'(0) = 0$, is $\mathbf{r}'(t) = \mathbf{h}'(t)\,e^{\widetilde{a}t}$, where

$$\mathbf{h}'(t) = \alpha_e^{-1}\int_0^t e^{-\widetilde{a}s}\Lambda\mathcal{R}^{-1}(s)[q\mathbf{E} + \boldsymbol{\xi}(s)]\,ds. \tag{31}$$

In a similar way as before, for small values of both electric and fluctuating forces, it can be shown that, as time increases, $d\mathbf{h}'(t)/dt \to 0$, and thus the process $\mathbf{h}'(t)$ becomes a GRV denoted as $\mathbf{h}'(\infty) = \mathbf{h}' = (h'_1,h'_2)$. The process $\mathbf{r}'(t)$ becomes a quasideterministic one and also satisfies the expression

$$r'^2(t) = h'^2\,e^{2\widetilde{a}t}, \tag{32}$$

where $|\mathbf{h}'|^2 \equiv h'^2 = h_1'^2 + h_2'^2$. Taking into account the whole process, Eq. (32) can also be written as $r'^2(t) = h'^2\,e^{2\widetilde{a}t}\,\theta(t_i - t) + r_{\rm st}'^2\,\theta(t - t_i)$, where $\theta(t)$ is the step function. Following the same steps as in the one variable case, the linear approximation of the NLRT is also

$$T_{L} = \frac{1}{2\widetilde{a}}\left\langle \ln\left(\frac{\widetilde{\mathfrak{r}}'_{\rm st}}{h'^2}\right)\right\rangle - \widetilde{C}_{L}, \tag{33}$$

and $\widetilde{C}_{L} = (1/2\widetilde{a})[1 - \langle h'^2\rangle/\widetilde{\mathfrak{r}}'_{\rm st}]$. Now, for the calculation of this time scale the marginal probability density $P(h')$ is required, and this quantity can be obtained from the joint probability density $P(h'_1,h'_2)$. This joint probability is in general given by the Gaussian distribution function [29]

$$P(h'_1,h'_2) = \frac{1}{2\pi(\det\sigma_{ij})^{1/2}} \times \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}(\sigma^{-1})_{ij}(h'_i - \langle h'_i\rangle)(h'_j - \langle h'_j\rangle)\right], \tag{34}$$

$\sigma_{ij} = \langle h'_i h'_j\rangle - \langle h'_i\rangle\langle h'_j\rangle$ being the correlation matrix. From Eq. (31) we have

$$\langle h'_i\rangle = \frac{q}{\alpha_e}\int_0^\infty e^{-\widetilde{a}s}\Lambda_{ik}\mathcal{R}^{-1}_{kl}(s)\,E_l\,ds, \tag{35}$$

$$\langle h'_i h'_j\rangle = \langle h'_i\rangle\langle h'_j\rangle + \frac{1}{\alpha_e^2}\int_0^\infty\int_0^\infty e^{-\widetilde{a}(s+s')}\Lambda_{ik}\Lambda_{jl} \times \mathcal{R}^{-1}_{km}(s)\mathcal{R}^{-1}_{ln}(s')\langle\xi_m(s)\xi_n(s')\rangle\,ds\,ds'. \tag{36}$$

First of all, the mean values $\langle h_i\rangle$ can be calculated by assuming without loss of generality that $\mathbf{E} = (E,E)/\sqrt{2}$, E being the modulus of this vector. This assumption yields $\langle h'_1\rangle =$

$\langle h_2' \rangle = qE/\sqrt{2}a$. Next, after integration of Eq. (36), we get $\langle h_i' h_j' \rangle = \langle h_i' \rangle \langle h_j' \rangle + (\lambda/a\alpha)\,\delta_{ij}$, with $\lambda/a\alpha = k_B T/a$. We conclude in this case that the variables h_i' are independent and $\sigma_{ij} = (k_B T/a)\,\delta_{ij}$ is clearly a diagonal matrix with elements $\sigma_{ii} = \sigma^2 = k_B T/a$. Hence, the joint probability density (34) reduces to

$$P(h_1',h_2') = \frac{1}{2\pi\sigma^2}\, e^{-[(h_1' - \langle h_1' \rangle)^2 + (h_2' - \langle h_2' \rangle)^2]/2\sigma^2}. \tag{37}$$

Following the same algebraic procedure proposed in Refs. [19,20], we obtain

$$P(h') = (h'/\sigma^2)\, I_0(p\, h'/2\sigma^2)\, e^{-(h'^2 + p'^2)/2\sigma^2}, \tag{38}$$

where $p'^2 = \langle h_1' \rangle^2 + \langle h_2' \rangle^2 = (qE)^2/a^2$ and $I_0(x)$ is the modified zeroth-order Bessel function [36]. With the help of Eq. (38) it is shown that $\langle h'^2 \rangle = 2\sigma^2 + p'^2$, which can be neglected for small noise intensity and small amplitude of the electric field. Thus, the constant $\widetilde{C}_L$ can be approximated by $\widetilde{C}_L = 1/2\widetilde{a}$. Using Eq. (38) we conclude, after some algebra, that the NLRT given by Eq. (33) can be written as

$$T_L^{em} = T_{0L}^{m} + \frac{1}{2\widetilde{a}} \sum_{m=1}^{\infty} \frac{(-1)^m\, \beta'^{2m}}{m m!}, \tag{39}$$

where $\beta'^2 = p'^2/\sigma^2 = (qE)^2/2a\, k_B T$. In this last expression

$$T_{0L}^{m} = \frac{1}{2\widetilde{a}}\left[\ln\left(\frac{a\, r_{\rm st}'^2}{2\, k_B T} \right) + \gamma - 1 \right] \tag{40}$$

is the linear approximation of the NLRT in the absence of the external electric field only ($\beta' = 0$) and $\gamma = 0.577$ is the Euler constant. If we use the identity $\sum_{m=1}^{\infty} \frac{(-1)^m z^m}{n m!} = -[E_1(z) + \gamma + \ln z]$ [36] the NLRT given by Eq. (39) can alternatively be written as

$$T_L^{em} = T_{0L}^{m} - \frac{1}{2\widetilde{a}}[E_1(\beta'^2) + \gamma + \ln(\beta'^2)]\,. \tag{41}$$

Equating Eq. (33) with (41) we conclude that

$$\frac{1}{2\widetilde{a}}\left\langle \ln\left(\frac{\widetilde{\mathfrak{r}}_{\rm st}'}{h'^2} \right) \right\rangle = \frac{1}{2\widetilde{a}}\left[\ln\left(\frac{a\, r_{\rm st}'^2}{2\, k_B T} \right) + \gamma \right] - \frac{1}{2\widetilde{a}}[E_1(\beta'^2) + \gamma + \ln(\beta'^2)]. \tag{42}$$

Substituting Eq. (42) into Eq. (5) we finally get the NLRT associated with the decay process of a charged Brownian particle from the unstable state of arbitrary nonlinear unstable potentials in the presence of a uniform electromagnetic field; it is given by

$$T_{em} = T_{0m} - \frac{1}{2\widetilde{a}}[E_1(\beta'^2) + \gamma + \ln(\beta'^2)] + C_{\rm NL}, \tag{43}$$

where

$$T_{0m} = \frac{1}{2\widetilde{a}}\left[\ln\left(\frac{a\, r_{\rm st}'^2}{2\, k_B T} \right) + \gamma \right] \tag{44}$$

is the NLRT in the absence of the external electric field only. In very similar way as done in the one variable case, we can show that the MFPT taking into account the nonlinear contributions to saturation reads as

$$\tau_{em} = \frac{1}{2\widetilde{a}}\left\{ \ln\left[\frac{a\, R^2\, M^2}{2\, k_B T(1 - M^2)} \right] + \gamma \right\} - \frac{1}{2\widetilde{a}}[E_1(\beta'^2) + \gamma + \ln(\beta'^2)]. \tag{45}$$

B. NLRT for the two-dimensional bistable potential

Once we have obtained the formal expression of the NLRT for any nonlinear unstable potential, we can now calculate the characteristic time associated with the Langevin dynamics (26) or (28) for the bistable potential. We first construct their corresponding deterministic equations in terms of the variables $\widetilde{\mathfrak{r}}$ or $\widetilde{\mathfrak{r}}'$. The deterministic equation associated with Eq. (26) without the electric field reads as

$$\frac{d\mathbf{r}}{dt} = \widetilde{a}\mathbf{r} - b\, r^2 \Lambda \mathbf{r} + \widetilde{W}\mathbf{r}. \tag{46}$$

In terms of the $\widetilde{\mathfrak{r}} = r^2$ variable it transforms into

$$\frac{d\widetilde{\mathfrak{r}}}{dt} = 2\widetilde{a}\widetilde{\mathfrak{r}} - 2\widetilde{b}\widetilde{\mathfrak{r}}^2 = \frac{2\widetilde{a}\,\widetilde{\mathfrak{r}}}{\widetilde{\mathfrak{r}}_{\rm st}}(\widetilde{\mathfrak{r}}_{\rm st} - \widetilde{\mathfrak{r}}), \tag{47}$$

where $\widetilde{\mathfrak{r}}_{\rm st} = \widetilde{a}/\widetilde{b} = a/b$. The deterministic equation associated with Eq. (28) without the electric field is now

$$\frac{d\mathbf{r}'}{dt} = \widetilde{a}\mathbf{r}' - b\, r'^2 \Lambda \mathbf{r}'. \tag{48}$$

In terms of the $\widetilde{\mathfrak{r}}' = r'^2$ variable we have

$$\frac{d\widetilde{\mathfrak{r}}'}{dt} = 2\widetilde{a}\widetilde{\mathfrak{r}}' - 2\widetilde{b}\widetilde{\mathfrak{r}}'^2 = \frac{2\widetilde{a}\,\widetilde{\mathfrak{r}}'}{\widetilde{\mathfrak{r}}_{\rm st}'}(\widetilde{\mathfrak{r}}_{\rm st}' - \widetilde{\mathfrak{r}}'), \tag{49}$$

and also $\widetilde{\mathfrak{r}}_{\rm st}' = a/b$. We can observe that Eqs. (47) and (49) are exactly as expected. Equation (47) is compatible with the general definition given by Eq. (3) if $g(\widetilde{\mathfrak{r}}) = 0$, and therefore the constant given by Eq. (6) is $\widetilde{C}_{\rm NL} = 0$. For this two-dimensional bistable potential, Eq. (43) reduces to

$$T_{em} = T_{0m} - \frac{1}{2\widetilde{a}}[E_1(\beta'^2) + \gamma + \ln(\beta'^2)], \tag{50}$$

and T_{0m} is the same as Eq. (44) with $r_{\rm st}'^2 = a/b$.

IV. RECEIVER OUTPUT

In an similar way as done in Ref. [15] we can also calculate the receiver output A_{em}/A_{0m} through the relaxation process of the charged Brownian particle in a bistable potential in the presence of a constant electromagnetic field. According to Fig. 1, the receiver output can be expressed in terms of the NLRT given by Eqs. (44) and (50). This relation can be achieved if the NLRT (4) is approximated by [15]

$$T = \int_0^{\infty} \frac{\langle \mathfrak{r}'(t) \rangle - \langle \mathfrak{r}' \rangle_{\rm st}}{\langle \mathfrak{r}'(0) \rangle - \langle \mathfrak{r}' \rangle_{\rm st}} dt \simeq \int_0^{T_c} \frac{\langle \mathfrak{r}'(t) \rangle - \langle \mathfrak{r}' \rangle_{\rm st}}{\langle \mathfrak{r}'^2(0) \rangle - \langle \mathfrak{r}' \rangle_{\rm st}} dt. \tag{51}$$

This approximation makes sense if $T_c \geqslant c\, T_{0m}$, where T_{0m} is the same as Eq. (44) and $c = 1.5$. If we make $\langle \mathfrak{r} \rangle_0 = 0$, it can be shown from Fig. 1 that the RO can be written as

$$\frac{A_{em}}{A_{0m}} = \frac{T_{em} - T_c}{T_{0m} - T_c} = 1 + \frac{T_{0m} - T_{em}}{T_c - T_{0m}}. \tag{52}$$

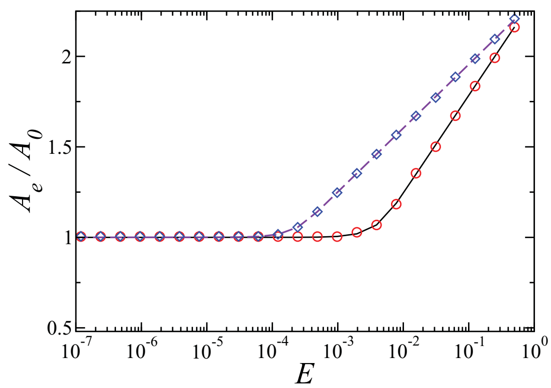

FIG. 2. (Color online) Receiver output A_e/A_0 as a function of the electric field E. The continuous line corresponds to the theoretical result (53) and the circles to the numerical simulation results, both for a noise intensity $\lambda = 10^{-4}$. The dashed line is again the plot of the theoretical expression (53), with diamonds corresponding to numerical results, for a smaller noise intensity of $\lambda = 10^{-7}$.

Thus, the RO is function only of T_e and T_0 and, according to Eqs. (44) and (50), it is finally given by the following expression:

$$\frac{A_{em}}{A_{0m}} = 1 + \frac{[E_1(\beta'^2) + \gamma + \ln(\beta'^2)]}{2\widetilde{a}\,(T_c - T_{0m})}. \qquad (53)$$

We plot in Fig. 2 the RO (53) as a function of the weak external force $F_e = E$ for a unit charge $q = 1$ and compare it with the numerical simulation results. As can be appreciated, both theoretical and numerical simulation results are in excellent agreement. The graph also shows that the RO, computed by means of the NLRT, is sensitive to the presence of very weak external electric fields for noise intensity values in a range spanning three orders of magnitude.

V. CONCLUDING REMARKS

Our contribution in this work is twofold: First, we have studied, in the one-dimensional case, the time characterization of the decay of unstable state of an arbitrary nonlinear unstable potential of a Brownian particle under the action of a constant external force. The quadratures given by Eqs. (5) and (22) are easily calculated through the Langevin scheme without employing the Fokker-Planck formalism, in which the nonlinear potential profiles are not necessarily easy to handle analytically. In the particular case of a symmetric bistable potential, the NLRT is given by Eq. (17) with $C_{\rm NL} = 0$ and the MFPT by Eq. (24). Both time scales have been calculated through the QD approach, which is valid only for small noise and small amplitude of the external force. In this approximation limit, and taking into account nonlinear potential effects, the MFPT given by Eq. (24), without the contribution of the external force, is a particular result whereupon a general one given by Agudov and Malakhov can be reduced for small noise intensity [see Eq. (16) and Fig. 9 in Ref. [24]].

Second, the formalism given in the one-dimensional case has been extended to characterize the dynamical relaxation of the unstable state of an arbitrary nonlinear unstable potential of a charged Brownian particle embedded in a constant electromagnetic field, which is described in a two-dimensional space of coordinates. Even though the dynamical evolution of the particle is rotational [see Eq. (26)] its time characterization can be achieved through the QD approach, being valid not only for both small noise and amplitude of the external force, but also in the linear approximation. This approach is better understood in the transformed space of coordinates $\mathbf{r}'$, where the Langevin dynamics is given by Eq. (28). In this space and in the linear approximation [see Eq. (30)] the Langevin dynamics can be seen as a rotational trajectory or not, depending on which intensity, that of the external signal or of the noise, is greater. For instance, if the amplitude of the former is less than or equal to the latter, then the rotational trajectory of the charged particle cannot be actually appreciated. But if the amplitude of the external signal is greater than the noise intensity, then the dynamical trajectory will be rotational (see Ref. [19]). As a particular case, the time characterization for the two-dimensional symmetric bistable potential are also calculated. The MFPT is given by Eq. (45) and the NLRT by Eq. (50).

Our proposal shows that the RO is useful to detect weak signals not only in laser systems [12,15] but also to detect weak electric fields in the dynamical relaxation of the unstable state of a Brownian charged particle embedded in a constant electromagnetic field. In a similar way as done with the laser system, it is proposed that the weak electric field is amplified when used to trigger the decay of the unstable state of the charged particle. As shown in Fig. 2, the RO is sensitive to the weak amplitude of $F_e = qE$ through the NLRT. Notwithstanding the notorious physical difference between the Laser system and that of a charged Brownian particle in an electromagnetic field, the behavior of the receiver output in both systems is very similar, which in itself is a curious result.

On the other hand, we would like to comment here that the detection process of weak periodic signals in the decay of unstable state of a charged particle in an electromagnetic field may exhibit a stochastic resonance-like phenomenon in a similar way as that studied in Ref. [31]. This corroboration will be the objective of future works.

Last, since the SR effect occurs in a wide range of physical, chemical, and biomedical systems, we think that our study may be extended to explore some of those phenomena where the dynamical relaxation of an unstable state is of main interest. For instance, the SR effect has been studied in the response of a single bistable neuron to a weak periodic signal by the addition of an optimal amount of noise [5,6]. We think that our proposal can be highly efficient to detect extremely weak external signals in other systems different from the one herein studied. For instance, it might be useful to investigate the response of a single neuron to a constant or periodic weak electric field when it undergoes the decay process from the unstable state of a bistable potential.

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APPENDIX A: QUALITATIVE ANALYSIS OF QD APPROACH

In classical Brownian motion the mean-square displacement (MSD), denoted as $\langle x^2(t)\rangle$, is usually analyzed in two limiting cases in terms of the magnitude of the observation times. (1) For short times such that $t \ll \tau$ with $\tau = m/\alpha$ the relaxation time is approximated by $\langle x^2(t)\rangle = Ct^2$, C being a constant. (2) For large times such that $t \gg \tau$, it becomes $\langle x^2(t)\rangle = 2Dt$, where $D = \lambda/\alpha^2 = k_B T/\alpha$ is the Einstein's diffusion coefficient. The limit of short times is well known as the ballistic regime, and it corresponds to the region in which the particle does not feel the presence of the heat bath; it moves as a free particle. For large times the particle enters in contact with the heat bath, which is the well-known diffusive regime.

For the decay process of an unstable state we can make a similar analysis for the particle initially located on the equilibrium unstable state $[x(0) = 0]$ of an unstable potential $V(x) = -(a/2)\,x^2$, corresponding to a linear approximation of the bistable potential, with $-R \leqslant x \leqslant R$ and R being an absorbing barrier. The overdamped Langevin equation is simply

$$\dot{x} = \bar{a}\,x + \alpha^{-1}\,\xi(t), \tag{A1}$$

where $\xi(t)$ is a Gaussian white noise. The solution of this equation is easily written as

$$x(t) = h(t)\,e^{\bar{a}t}, \quad h(t) = \int_0^t e^{-\bar{s}}\xi(s)\,ds. \tag{A2}$$

It is easily shown that $\langle h^2(t)\rangle = (\lambda/\alpha a)(1 - e^{-2\bar{a}t})$ and $\langle x^2(t)\rangle = (\lambda/\alpha a)(e^{2\bar{a}t} - 1)$. From this simple relation we can also study two limiting cases: namely, short and large times. (1) For short time such that $t \ll 1/2\bar{a}$ we get $\langle x^2(t)\rangle = 2Dt$, where D is the Einstein's diffusion constant. (2) For large times such that $t \gg 1/2\bar{a}$ we have $\langle x^2(t)\rangle = (D/\bar{a})e^{2\bar{a}t}$. This physically means the following: At the beginning of the decay process (short times), the particle is already in contact with the heat bath (diffusive regime). It is in this regime in which the decay process of the particle takes place due to the natural presence of noise. The process takes place even for small noise intensity. At large times, and assuming a very small noise intensity, the MSD is clearly dominated by the deterministic factor $e^{2\bar{a}t}$, and the particle evolves under the action of the regular force. Hence, in this approximation regime the dynamical evolution of the particle is practically deterministic, and it has been termed the quasideterministic approach [8]. We qualitatively conclude that the characteristic time required by the particle to reach the absorbing barrier R^2 is then $t = (1/2\bar{a})\ln(\bar{a}\,R^2/D)$, which is precisely the relevant contribution in the time characterization of the decay of an unstable state for small noise intensity.

APPENDIX B: ALTERNATIVE CALCULATION OF C_{NL}

The other way to calculate the quantity C_{NL}, which accounts for nonlinear contributions in the dynamical relaxation of the unstable state, can be given by taking the difference between the NLRT (4), expressed as a quadrature, and that corresponding to the relevant contribution. It is evaluated in the deterministic evolution of the particle where the fluctuations become negligible, that is,

$$C_{NL} = \lim_{h\to 0} \frac{1}{\mathfrak{r}_{st}} \int_{h^2}^{\mathfrak{r}_{st}} \left[\frac{\mathfrak{r}_{st} - \mathfrak{r}}{f(\mathfrak{r})} - \frac{1}{2\bar{a}\mathfrak{r}}\right] d\mathfrak{r}. \tag{B1}$$

Upon substitution of the function $f(\mathfrak{r})$ defined in Eq. (3), we get

$$C_{NL} = \lim_{h\to 0} \frac{1}{\mathfrak{r}_{st}} \int_{h^2}^{\mathfrak{r}_{st}} g(\mathfrak{r})\,d\mathfrak{r}. \tag{B2}$$

APPENDIX C: NLRT FOR ONE VARIABLE SYSTEM

The NLRT given by Eq. (11) can be written as

$$T_L = \frac{1}{2\bar{a}}\left[\ln\left(\mu^2 x_{st}^2\right) - \langle\ln(\mu^2 h^2)\rangle - 1\right], \tag{C1}$$

where $\mu^2 = 1/2\sigma^2$. The second term defined as $I \equiv \langle\ln(\mu^2 h^2)\rangle$ can be calculated with the help of Eq. (12). It can be written as $I = I_1 + I_2$, where

$$I_1 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{0} \ln(\mu^2 h^2)\,e^{-\mu^2(h-\langle h\rangle)^2}\,dh = \frac{e^{-\beta^2}}{\sqrt{\pi}} \int_0^{+\infty} \ln z^2\,e^{-z^2-2\beta z}\,dz, \tag{C2}$$

$$I_2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} \ln(\mu^2 h^2)\,e^{-\mu^2(h-\langle h\rangle)^2}\,dh = \frac{e^{-\beta^2}}{\sqrt{\pi}} \int_0^{+\infty} \ln z^2\,e^{-z^2+2\beta z}\,dz, \tag{C3}$$

with $z = \mu h$ and $\beta = \mu\langle h\rangle$. After some algebra and using some identities given in Ref. [36] we can show that

$$I = e^{-\beta^2}\psi\left(\frac{1}{2}\right) + e^{-\beta^2}\sum_{m=1}^{\infty}\frac{\beta^{2m}}{m!}\psi(m+1). \tag{C4}$$

But $\psi(m+1) = \psi(1/2) + 2\sum_{k=1}^{m}\frac{1}{2k-1}$ [36], so that

$$I = \psi\left(\frac{1}{2}\right) + 2e^{-\beta^2}\sum_{m=1}^{\infty}\frac{\beta^{2m}}{m!}\sum_{k=1}^{m}\frac{1}{2k-1}. \tag{C5}$$

Finally

$$T_L = \frac{1}{2\bar{a}}\left[\ln\left(\frac{x_{st}^2}{2\sigma^2}\right) - \psi\left(\frac{1}{2}\right) - 1\right] - \frac{e^{-\beta^2}}{\bar{a}}\sum_{m=1}^{\infty}\frac{\beta^{2m}}{m!}\sum_{k=1}^{m}\frac{1}{2k-1}, \tag{C6}$$

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