Order in a multidimensional system

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We show that any convex K-dimensional system has a level of order R that is proportional to its level of Fisher information I. The proportionality constant is $1/8$ the square of the longest chord connecting two surface points of the system. This result follows solely from the requirement that R decrease under small perturbations caused by a coarse graining of the system. The form for R is generally unitless, allowing the order for different phenomena, or different representations (e.g., using time vs frequency) of a given phenomenom, to be compared objectively. Order R is also invariant to uniform magnification of the system. The monotonic contraction properties of R and I define an arrow of time and imply that they are entropies, in addition to their usual status as informations. This also removes the need for data, and therefore an observer, in derivations of nonparticipatory phenomena that utilize I. Simple graphical examples of the new order measure show that it measures as well the level of "complexity" in the system. Finally, an application to cell growth during enforced distortion shows that a single hydrocarbon chain can be distorted into a membrane having equal order or complexity. Such membranes are prime constituents of living cells.

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I. BACKGROUND

Consider a system in a general state of thermodynamic nonequilibrium, e.g., possibly far from equilibrium. What is the level of order in the system?

The concept of "order" is commonly used in the sense of something that is highly structured or complex while lacking significant randomness. The brain of man is commonly considered the most ordered of any living creature's. But, is it and, if so, on what grounds and to what extent numerically? (This question is considered further below.) Before Darwin, nature used to commonly be regarded as harmonious and ordered. A cancerous organ is, when viewed microscopically, much less ordered in appearance than it was when healthy. Today more sophisticated properties of an ordered system are often mentioned, such as its spontaneous, coherent, or statistical natures. However, it is now known that man's nature is to often see order even where it does not exist. There is thus an obvious need to quantify "order" as a measurable quantity, but attempts to do so are seldom made, largely for the following reason.

Let us designate the order as quantity R. Intuition, and common language usage, suggest that the order R and the concept of disorder are, in some sense, polar opposites. And disorder has long been quantified in physics as the Boltzmann-Shannon entropy H, which must globally increase, by the second law of thermodynamics. On this basis, then, its opposite R must decrease. That is, R must be some mathematical, inverse function $R(H)$ of H. But then a problem arises. There are many possible inverses. For example, use of Ockham's razor would favor the simple negative inverse $R(H) = -H$, often called the "negentropy," but Ockham's razor cannot by itself prove anything physical. Also, on the same basis other choices exist, such as the simple reciprocal $1/H$, or $\exp(-H)$, or even some cross-entropy form. All of these are legitimate mathematical inverses to H.

Clearly, this approach gives too many candidate answers. Merely regarding R as the inverse of H does not suffice. If R is to be a unique measure, it cannot be merely defined by H, i.e., by the physics of the second law. Instead, it must be defined on its own, physically based grounds. What physical effect suffices to uniquely define R?

A previous paper [1] sought such a definition of R for, in particular, a one-dimensional (1D) system with signal probability amplitudes q_n, $n = 1, \ldots, N$. The system is, by hypothesis, weakly coarse grained. The term *coarse graining* describes any physical process that degrades the system by replacing its signal values with weighted mathematical projections of them. An example is when digitizing an analog signal or when, in C.T. scan imagery, a spatially continuous, signal absorptance specimen (say, a brain cross section) is sequentially projected at a finite sequence of angles. Because of the finite spacing between angles, these projections miss in-between details of the system and contain *in toto* less order than did the signal specimen. Coarse graining even demarks the transition from a quantum to classical universe [2–4]. Thus, in [1] the concept of coarse graining provides the physical grounds we sought for defining the concept of order on its own (i.e., independent of disorder). How may coarse graining be so used?

Intuition suggests that order must be lost or, at least, not gained when a system is coarse grained. Therefore, the order R is defined to either decrease or stay constant under any coarse graining operation. That is, during the infinitesimal time duration Δt required for the coarse graining,

$$\Delta R \leqslant 0 \text{ for } \Delta t > 0 \text{ or equivalently, } \Delta R \geqslant 0 \text{ for } \Delta t < 0. \tag{1}$$

Notice that although (of course) the physical arrow of time obeys positivity $\Delta t > 0$, the condition $\Delta R \geqslant 0$ for $\Delta t < 0$ (looking backward in time) in (1) turns out to be easier to work with.

On this basis, R was found [1] to relate linearly to the Fisher information [5, 6] I for a shift-invariant, 1D system $K = 1$, as

$$R = 8^{-1}L^2 I, \text{where } I \equiv 4\sum_{n=1}^{N}\left(\frac{dq_n}{dx}\right)^2 \to 4\int_a^b dx\left[\frac{du}{dx}\right]^2,$$
$$\text{with } u \equiv u(x) \equiv \Delta x^{-1/2}q(x) \quad (2)$$

in the continuous limit $\Delta x \to dx$, $x_n \to x$. Function $u(x)$ is the real amplitude density function of the 1D system of finite extension $L \equiv b - a$, with coordinate x obeying $a \leqslant x \leqslant b$. The following properties of expression (2) for R were found and discussed in [1]:

(1) It increases with system extension L, e.g. indicating increased order due to mere repetition of details (as in an apartment building, where each subsequent storey monotonically adds to the level of structural order). (2) It is unitless and hence permits meaningful comparisons of order for different types of scenarios, such as a bacterial culture and a hydrogen atom. (3) It has the fractal property of being invariant under linear system stretch $y \equiv mx$, m =const. (4) It increases as the square of the number of oscillations in a sinusoidal function $q(x)$, independent of their amplitude. Thus it is a measure of system complexity as well.

II. AIM

Here we seek the order in a generally K-dimensional system, described by probability values $p(x_1,\ldots,x_K) \equiv q^2(x_1,\ldots,x_K)$, with q a real amplitude function and K-dimensional rectangular coordinates $(x_1,\ldots,x_K)$. An example $K = 3$ of a system $\mathbf{q} = \mathbf{q}(x,y,z)$ of cubic shape is shown in Fig. 1. Hence we want to extend the premise (1) and answer (2) to a more general system of K dimensions. What is the unique measure of order for such a system? The system is defined by its sampled amplitude values $\mathbf{q}$ at N discrete, K-dimensional, rectangular pixel positions $[(x_{nk}, k = 1,\ldots,K), n = 1,\ldots,N]$. Thus x_{nk} represents the kth dimensional value of the nth pixel in the space. An example is a rectangular $K = 3$ space where x_{nk} designates the coordinate positions $(x_1,x_2,x_3)_n = (x,y,z)_n$ of the nth pixel. The pixels are numbered arbitrarily, e.g., row-wise, then columnwise.

As is usual, the spacing of increments $\Delta x_{nk} \equiv \Delta x$ are assumed equal in all K directions. Thus each vector pixel length

$$\Delta\mathbf{x} \equiv \underbrace{(\Delta x,\ldots,\Delta x)}_{\leftarrow K \text{ elements} \rightarrow} \text{ obeys } \Delta\mathbf{x}\cdot\Delta\mathbf{x} = K\Delta x^2. \quad (3)$$

Let the system surface generally bulge outward, i.e., be convex, so that a chord connecting any two surface points lies inside the system. Define L as the longest such chord in the Pythagorean sense. As in Fig. 1, let the origin O of coordinates lie at one end of this longest chord (call it the far-left one). Also, the other end (point O') of the chord has vector length $\mathbf{L} \equiv (L_k, k = 1,\ldots,K)$. Then the maximum chord length L obeys $L^2 = \sum_k L_k^2$.

Let the order R in the system depend upon the system amplitude law in some unknown way

$$R \equiv R(\mathbf{q}), \text{ with } \mathbf{q} \equiv [q(x_{nk}, k = 1,\ldots,K), n = 1,\ldots,N]. \quad (4)$$

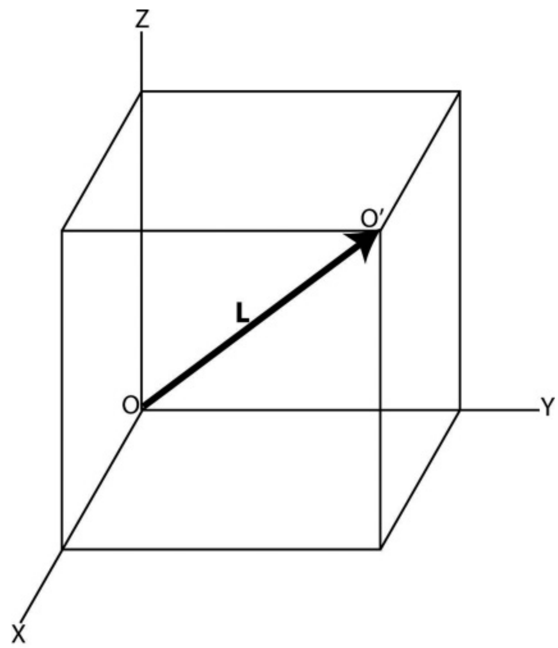

FIG. 1. A cubic system. Distance L is the maximum chord length connecting surface points.

The N amplitudes $\mathbf{q}$ at its K-dimensional pixel positions $\mathbf{x}$ are assumed to be known and fixed, defining the system.

A. Definition of the order

The order is defined to decrease under coarse graining. What order has this property? Let the system of amplitude values $\mathbf{q}$ be coarse grained by a second system, which perturbs the $\mathbf{q}$ by amounts $\Delta\mathbf{q}$. The second system is an effective observer, either in the familiar data taking sense or, more generally, as any physical system interacting with the first. For example, the second system might be the outside environment of the first. We assume that all perturbations Δp, $\Delta\mathbf{q}$, ΔR, etc., resulting from the interactive coarse graining, are small, i.e., the coarse graining is weak. Since the resulting system order R is to generally decrease for perturbations taking place over a small time interval $\Delta t > 0$, then $\Delta R \leqslant 0$ over that interval. Or, *ipso facto*, the change ΔR over the corresponding negative time increment $-\Delta t$ is positive, $\Delta R \geqslant 0$. That is, looking backward in time, the order increases. Since it is mathematically simpler to work with such positive changes, the analysis is carried through over this negative time increment. However, of course all applications and interpretations of the results assume the usual positive time increments. In summary, we postulate that R be a function of the q that satisfies

$$\Delta R \geqslant 0 \text{ for } \Delta t < 0 \,. \quad (5)$$

Cencov's famous inequality will be used to satisfy the requirement of decrease in order, including the effects of perturbations out to second order in the probabilities. The answer for $R(\mathbf{q})$ turns out to be unique, at least from heuristic considerations.

B. Strategy

There are two main steps to the analysis. By step 1, the dependence of ΔR upon solely the changed amplitudes $\Delta\mathbf{q}$ is established. This is by hypothesis independent of how the amplitudes and their changes are arranged spatially. Next, in step 2, the spatial arrangement is brought in by assuming that the $\mathbf{q}$ are analytic functions of all coordinates $\mathbf{x}$. The usual vector differential relation is then used:

$$\Delta\mathbf{q}_n = \sum_k \frac{\partial q(x_{nk})}{\partial x_{nk}}\Delta x_{nk} \equiv (\nabla\mathbf{q}\cdot\Delta\mathbf{x})_n, \quad \text{with } \nabla \equiv \hat{e}_1\frac{\partial}{\partial x_1}+\cdots+\hat{e}_K\frac{\partial}{\partial x_K}. \tag{6}$$

The $\hat{e}_k$ are unit vectors of corresponding coordinates $x_1,\ldots,x_K \equiv \mathbf{x}$. An example is the usual $K=3$ case of rectangular coordinates $\mathbf{x}=(x,y,z)$ with respective unit vectors $\hat{e}_1,\hat{e}_2,\hat{e}_3$.

III. STEP 1: USE OF EFFECTIVE 1D VECTOR OF CHANGES

It is convenient to first ignore the spatial coordinates of amplitudes $\mathbf{q}$. This is by describing $\mathbf{q}$ and its changes $\Delta\mathbf{q}$ by pixel number $n=1,\ldots,N$ in any fixed ordering, for example, two 1D vectors of length N. The corresponding order R and change in order ΔR due to $\mathbf{q}$ and $\Delta\mathbf{q}$ are respectively denoted as $R(\mathbf{q})$ and $\Delta R(\mathbf{q},\Delta\mathbf{q})$. These then depend only upon element numbers n in the vectors and not explicitly upon geometrical position $\mathbf{x}$ in the system. This first step of the two-step approach then follows precisely the $K=1$ dimensional analysis of amplitude changes in [1]. For completeness we repeat this but delete all discussions of secondary issues.

Hence here $R(\mathbf{q})$ is a simple vector function (of amplitudes $\mathbf{q}$) of length N, consider the effect upon R of perturbing the amplitudes $\mathbf{q}$ by small amounts $\Delta\mathbf{q}$. By Eq. (4) the order must likewise change. Depending upon the form of the order measure R and of the perturbations (which can be of either sign), R could go up or down. Using an ordinary Taylor series to second order in $\Delta\mathbf{q}$, this change is then

$$\begin{aligned}\Delta R &\equiv R(\mathbf{q}+\Delta\mathbf{q})-R(\mathbf{q}),\\ &= \Delta\mathbf{q}^T\nabla_q R + 2^{-1}\Delta\mathbf{q}^T\mathbf{M}\Delta\mathbf{q}+\cdots,\\ &\equiv \Delta R_1+\Delta R_2+\cdots, \text{ with } M_{mn}\equiv \partial^2 R/\partial q_m\partial q_n.\end{aligned} \tag{7}$$

Here $\Delta\mathbf{q}^T\nabla_q R \equiv \sum_n(\partial R/\partial q_n)\Delta q_n$, with $\nabla_q R$ the gradient of R, and $\mathbf{M}$ the Hessian matrix of elements M_{mn} As usual T denotes the transpose. We retain terms of the series only out to second order in the $\Delta\mathbf{q}$ since these are small.

Next, matrix $\mathbf{M}$ is Hermetian and Riemannian [1]. Hence it has positive eigenvalues

$$\lambda_n \geqslant 0\,,\quad n=1,\ldots,N. \tag{8}$$

This permits transformation

$$\Delta\mathbf{q}\equiv[\mathbf{B}]\Delta\mathbf{q}', \text{ such that } \Delta R_2 = 2^{-1}\sum_n \lambda_n \Delta q_n'^2, \tag{9}$$

of the changes $\Delta\mathbf{q}$ to new ones $\Delta\mathbf{q}'$ via a unitary matrix $[\mathbf{B}]$. Thus δR_2 is a 1D sum that replaces the cumbersome double sum $\Delta\mathbf{q}^T\mathbf{M}\Delta\mathbf{q}$ in (7).

Then by definition (7) of ΔR_2

$$\frac{\partial^2 R}{\partial q_m'\partial q_n'} = \lambda_n\delta_{mn}\,, \text{ so that } \frac{\partial R}{\partial q_n'} = \lambda_n q_n' + C_n, \tag{10}$$

where δ_{ij} is the Kronecker delta. The second equation follows an integration of the first, with C_n an arbitrary constant. We may now re-express the order R and its changes ΔR_1 and ΔR_2 in the shifted system. We show below that, effectively, the $C_n=0$.

In terms of the new changes $\Delta\mathbf{q}'$, using (9) and (10) give

$$\begin{aligned}\Delta R &= \Delta R_1+\Delta R_2,\\ \Delta R_1 &\equiv \Delta\mathbf{q}'^{\mathsf{T}}\nabla_q R \equiv \sum_n \Delta q_n'(\lambda_n q_n' + C_n),\\ \Delta R_2 &= 2^{-1}\sum_n \lambda_n\Delta q_n'^2.\end{aligned} \tag{11}$$

By postulate (5), over a negative time increment the order R must increase,

$$\Delta R = \Delta R_1 + \Delta R_2 \geqslant 0 \text{ for } \Delta t < 0 \tag{12}$$

to second-order in changes $\Delta q_n'$. It is convenient to first regard pixel length Δx and change ΔR as finite, and then take its continuous limit $\Delta x\to dx$. Hence we now ask, what order measure R obeys property (12)? By Eq. (11), requirement (12) becomes

$$\begin{aligned}\Delta R &\equiv \Delta R_1+\Delta R_2,\\ &= \sum_n(\lambda_n q_n' + C_n)\Delta q_n' + 2^{-1}\sum_n \lambda_n\Delta q_n'^2 \geqslant 0,\end{aligned} \tag{13}$$

for $\Delta t<0$. As was discussed, this requirement is equivalent to requiring a loss $-\Delta R$ of order in the usual positive time direction $\Delta t>0$.

A. Cencov's inequality

Is there a set of λ_i that, in any scenario of coarse graining, gives $\Delta R_1+\Delta R_2\geqslant 0$? Cencov's inequality [7–10] states that for a Hermitian metric such as $\mathbf{M}$, the required eigenvalues are

$$\lambda_n = 1,\ \ n=1,\ldots,N. \tag{14}$$

However, $\mathbf{M}$ is the metric for ΔR_2 and the overall sum $\Delta R_1 + \Delta R_2$ does not have a well-defined Hermitian metric (the form would have infinite diagonal terms in the limit $\Delta q_n'\to 0$). Hence, we simply evaluate ΔR under condition (14), so as to test whether it does indeed obey the required positivity (5).

B. Resulting order increase

By Eq. (14), requirement (13) becomes

$$\Delta R = \sum_n (q_n' + C_n)\Delta q_n' + 2^{-1}\sum_n \Delta q_n'^2 \geqslant 0, \text{ or} \tag{15}$$

$$\Delta R = \sum_n C_n\Delta q_n' + 2^{-1}\sum_n \Delta q_n^2 \geqslant 0, \text{ for } \Delta t\leqslant 0. \tag{16}$$

We used normalization condition $\sum_n q_n'^2 = 1$ which, when perturbed by the coarse graining, gives $2\sum_n q_n'\Delta q_n' = 0$. Also, the right-hand sum of unprimed Δq_n^2 arose from the primed sum of $\Delta q_n'^2$ by the identity

$$\Delta\mathbf{q}^{\mathsf{T}}\Delta\mathbf{q} \equiv [[\mathbf{B}]\Delta\mathbf{q}']^{\mathsf{T}}[[\mathbf{B}]\Delta\mathbf{q}'] = \Delta\mathbf{q}'^{\mathsf{T}}[\mathbf{B}]^{\mathsf{T}}[\mathbf{B}]\Delta\mathbf{q}' = \Delta\mathbf{q}'^{\mathsf{T}}[\mathbf{B}]^{-1}[\mathbf{B}]\Delta\mathbf{q}' = \Delta\mathbf{q}'^{\mathsf{T}}\Delta\mathbf{q}', \tag{17}$$

that is obeyed by transformation (9). Unitarity property $[\mathbf{B}\ ]^T = [\mathbf{B}]^{-1}$ was used.

IV. STEP 2: SPATIAL ANALYSIS IN K DIMENSIONS

According to plan, result (16) must now be properly expressed in K-dimensional position space. Thus in the preceding the nth positional pixel is actually a K-dimensional vector

$$\mathbf{x}_n \equiv (x_1, \ldots, x_K)_n \tag{18}$$

of scalar coordinates x_k . For example, in $K = 3$ space the pixel location is denoted as $(x,y,z)_n$, i.e., the rectangular coordinates of an nth three-space pixel. Hence to define the dependence of R upon spatial position, this additional description must be inserted into (16).

By (18) and identity (6), the first sum in (16) becomes

$$\sum_n C(\mathbf{x})_n \Delta q'(\mathbf{x})_n = \sum_n C(\mathbf{x})_n \sum_k \frac{\partial q'(\mathbf{x})_n}{\partial x_k}\Delta x_k \to \int dx_k \sum_n C(\mathbf{x})_n \frac{\partial q'(\mathbf{x})_n}{\partial x_k} \tag{19}$$

in the continuous limit. We see that the latter sum is not multiplied by any power of Δx, i.e., it is not infinitesimally small. This point is essential to the analysis below.

Again using Eq. (6), now in the second sum in (16), gives

$$\sum_n \Delta q^2(\mathbf{x})_n = \sum_n [\nabla q(\mathbf{x})_n \cdot (\Delta\mathbf{x})_n]^2 \leqslant K\Delta x^2 \times \sum_n \nabla q(\mathbf{x})_n \cdot \nabla q(\mathbf{x})_n \tag{20}$$

by use of the Schwarz inequality at each n. The factor $K\Delta x^2$ arises from uniform pixel spacing (3), giving at each n

$$(\Delta\mathbf{x})_n \cdot (\Delta\mathbf{x})_n = K\Delta x^2. \tag{21}$$

Now, using the definition of the Fisher channel capacity [11] of I, Eq. (20) becomes

$$\sum_n \Delta q^2(\mathbf{x})_n \leqslant 4^{-1}K\Delta x^2 I, \quad \text{where } I \equiv 4\sum_n \nabla q(\mathbf{x})_n \cdot \nabla q(\mathbf{x})_n. \tag{22}$$

It is convenient to restate inequality (22) as an equality by use of an efficiency coefficient η,

$$\sum_n \Delta q^2(\mathbf{x})_n = 4^{-1}\eta K\Delta x^2 I, \text{ where } 0 \leqslant \eta \leqslant 1. \tag{23}$$

As we shall see, η measures the fractional amount of a maximum possible level of order R_0 that can actually be observed. At this point it is important to distinguish between two kinds of order:

Using Eqs. (19) and (23) in (16) gives the requirement [later met at (32)]

$$\Delta R = \sum_k \int dx_k \sum_n C(\mathbf{x})_n \frac{\partial q'(x)_n}{\partial x_k} + 8^{-1}\eta K\Delta x^2 I \geqslant 0, \quad \text{for } \Delta t \leqslant 0. \tag{24}$$

A. Intrinsic order R_0 vs received order R

Let amplitudes $q(\mathbf{x})$ define a system A. Next, consider a system B that coarse grains A, i.e., it interacts with it in some fashion (system B might simply be an observer.) Now there are two levels of order to consider. First, there is the level of order R in A that is present prior to the coarse graining. This is defined by the system law $q(\mathbf{x})$. Since it precedes coarse graining it is undegraded, showing the intrinsic level of order in A. Call this R_0. Next, there is the level R that is actually utilized by B (say, the observer). This level follows, in time, the coarse graining, so that postulate (1) requires

$$R \leqslant R_0 \text{ for } \Delta t > 0. \tag{25}$$

According to the size of η, some systems are intrinsically able to display more, or less, a fraction of their total possible level R_0 of order. The latter amounts to a kind of efficiency condition on the system amplitudes $q(\mathbf{x})$, in analogy with that of classical estimation theory (see below). This situation is also in analogy with the connection $I = \kappa J$ between information I and J in EPI (extreme physical information) theory [11], where κ measures the efficiency with which information is transferred from a source level J to data at level I. In comparison, our coefficient η measures the efficiency with which order is transferred from the subject system to the system that coarse grains it (e.g., an observer).

V. R AS A FUNCTION OF COORDINATES x

Since the system is fixed, information I in Eq. (24) is a fixed constant, i.e., not a function of $\mathbf{x}$. Then (24) is effectively an expansion for ΔR out to second order in changes $\Delta\mathbf{x} \to d\mathbf{x}$ in the continuous limit. Likewise, in this limit $dR = dR(\mathbf{x})$. Integrating this will give $R = R(\mathbf{x})$, a function of the K coordinates $\mathbf{x}$, e.g., the $K = 3$ case gives $R(x,y,z)$. That R is a function of the coordinates is not surprising, since Eq. (4) defines R as a function of the amplitudes $q(\mathbf{x})$. The order function $R(\mathbf{x})$ thereby measures the local order pointwise at coordinates $\mathbf{x}$. It is a (second-order) density of the order. What is this density function?

A. Plan

Result (24) is a power series expansion to second order in the small scalar Δx. Such a series is only a good approximation if the system is small, so we assume that the length L_k of the system in each dimension k is small. We may therefore evaluate the coefficients of the powers in (24) by matching it up with an ordinary Taylor series [Eq. (28)] for $R(\mathbf{x}_n)$ in powers of distances Δx from the origin pixel $O \equiv \mathbf{0}$ to the position $\mathbf{x}_n$. Therefore $\Delta x = |\mathbf{x}_n - \mathbf{0}|$, and Δx is assumed to be small.

It follows from (21) that

$$|\mathbf{x}_n - \mathbf{0}|^2 \equiv |\Delta\mathbf{x}|^2 = K\Delta x^2, \tag{26}$$

which is likewise small.

B. Implementation

The reason for choosing the origin of coordinates at the leftmost point $\mathbf{0}$ of the maximum chord will now become clear. Let the function $R(\mathbf{x}_n)$ be analytic about this boundary point $\mathbf{0}$. Then we may express R to a second-order power series about it. For simplicity, we suppress subscript n from all quantities $\mathbf{x}_n$ and $\Delta\mathbf{x}_n$ when it is superfluous. The power series is accordingly,

$$\begin{aligned} R(\mathbf{x}) &= R(\mathbf{0}) + \sum_k \left.\frac{\partial R}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} \Delta x_k + 2^{-1}\sum_{kl} \left.\frac{\partial^2 R}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\mathbf{0}} \Delta x_k \Delta x_l \\ &= R(\mathbf{0}) + \Delta x \sum_k \left.\frac{\partial R}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} + 2^{-1}\Delta x^2 \sum_{kl} \left.\frac{\partial^2 R}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\mathbf{0}} \end{aligned} \tag{27}$$

after again using the equality (3) of the Δx_k. Or, since $R(\mathbf{x}) - R(\mathbf{0}) \equiv \Delta R(\mathbf{x}) \equiv \Delta R$,

$$\Delta R = \Delta x \sum_k \left.\frac{\partial R}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} + 2^{-1}\Delta x^2 \sum_{kl} \left.\frac{\partial^2 R}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\mathbf{0}}. \tag{28}$$

Equations (24) and (28) represent the same thing, namely, ΔR expressed as a power series in powers Δx. Therefore terms with corresponding powers of Δx in the two series must be equal. This gives rise to the following identities:

1. *Terms* $(\Delta x)^0$

The first right-hand term in (24) implicitly has a multiplier $(\Delta x)^0 \to (dx)^0 = 1$, i.e., is a constant in Δx. By comparison, the right-hand side of (28) has no constant term in $(dx)^0$. Therefore, the first right-hand term of (24) must be zero. For an arbitrary system $q'(\mathbf{x})$, this requires that

$$C(\mathbf{x})_n = 0, \quad \text{all} \quad n. \tag{29}$$

2. *Terms* $(\Delta x)^1$

Equation (24) has no term in $(\Delta x)^1$, whereas the first term of (28) has this form. Therefore its sum obeys

$$\sum_k \left.\frac{\partial R}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} = 0. \tag{30}$$

(Notice that this requirement on the sum does not require each element $\left.\frac{\partial R}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} = 0, k = 1,\ldots,K$. Such a result would express a condition of system equilibrium, whereas, at the outset, the system is defined as being in any general state of nonequilibrium.)

3. *Terms* $(\Delta x)^2$

The terms that are quadratic in Δx in (24) and (28) are equal if

$$2^{-1}\sum_{kl} \left.\frac{\partial^2 R}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\mathbf{0}} = 8^{-1}\eta K I. \tag{31}$$

VI. ANSWER

Using identities (29) in (24) gives

$$\Delta R = 8^{-1}\eta K \Delta x^2 I \geqslant 0, \text{ for } \Delta t \leqslant 0. \tag{32}$$

Because every factor in the product obeys positivity, the positivity requirement (1) on ΔR is now fulfilled.

Likewise, using (30) and (31) in (27) gives

$$\begin{aligned} R(\mathbf{x}) &= R(\mathbf{0}) + 8^{-1}\eta(K\Delta x^2)I, \\ &= R(\mathbf{0}) + 8^{-1}\eta|\mathbf{x} - \mathbf{0}|^2 I \end{aligned} \tag{33}$$

by (26). Equation (33) states that in second-order approximation, the local value of the order R at point x must increase with its squared distance from the fixed boundary point $\mathbf{0}$. It also increases with the system's Fisher information I. This gives the two key results in the next subsection.

A. Order associated with distance inside system

(1) Finite order is only associated with finite distance from the boundary point $\mathbf{0}$, so that the amount of order within interval $(\mathbf{0},\mathbf{0})$ is zero,

$$R(\mathbf{0}) \equiv 0. \tag{34}$$

(2) The total level of order R for the system shown in Fig. 1 is therefore the total amount of local order that is included from the system origin point O to the maximum chord position $\mathbf{x} = \mathbf{L}$ at point O'. Combining (33) and (34) gives $R(\mathbf{x}) = 8^{-1}\eta|\mathbf{x}|^2 I$, so that

$$R(\mathbf{L}) \equiv R = 8^{-1}\eta|\mathbf{L}|^2 I = 8^{-1}\eta L^2 I. \tag{35}$$

This is the main result of the paper. Special cases follow.

The answer for the maximum chord length L depends upon the shape of the system. For example, for a cubic system in K dimensions, the length components $L_k \equiv l$, $k = 1,\ldots,K$. Then by the Pythagorean theorem, $|\mathbf{L}|^2 \equiv L^2 = Kl^2$, so that (35) gives

$$R_{cube} = 8^{-1}\eta K l^2 I, \quad \text{where} \quad 0 \leqslant \eta \leqslant 1. \tag{36}$$

Thus for a 1D system $K = 1$, result (36) checks with the result (2), since there $l = L$ and also $\eta = 1$ (the latter since for the 1D case the first dot product in (20) is simply Δq^2 without need for the Schwarz inequality). Or, for a case $K = 3$ of ordinary space, the order (36) is $K = 3$ times larger than for the corresponding 1D system, although once I is evaluated it can go as $K^2 = 9$ (see (45) below).

Another case of interest is a spherical system of diameter d. Here the maximum chord length $L = d$, its diameter. Then the result (35) gives

$$R_{sphere} = 8^{-1}\eta d^2 I. \tag{37}$$

B. Is the measure unique?

Heuristic indications are that the expression (35) for R is unique. The arguments are as in [1] and rest on the requirement at the outset that the system function $q(\mathbf{x})$ be in a general state of nonequilibrium. For brevity we do not repeat the proof.

C. Order and information as entropies

Result (35) holds for a generally convex system of any shape. Taking its differential gives $\Delta R = 8^{-1}\eta L^2 \Delta I$. Then positivity requirement (1) indicates that not only does the order R decrease after a coarse graining, but so, likewise, does the information I,

$$\Delta I \leqslant 0 \text{ for } \Delta t > 0, \text{ or conversely, } \Delta I \geqslant 0 \text{ for } \Delta t < 0, \quad (38)$$

i.e., looking backward in time. Thus, for the general system, the order changes monotonically, thereby defining an "arrow of time." This is also the defining property of an entropy. Hence, the information channel capacity I is both a Fisher information and a Fisher entropy. It is both a property of the system and of any data from it.

D. Nonparticipatory phenomena

Many physical and biological effects may be derived [11–17] on the basis of extremizing the Fisher information I. It is presumed in these derivations that the observer has data at hand from the unknown effect. This fits in nicely with Wheeler's concept of "participatory phenomena" [18]. However, such effects as the timewise evolution of the wave function occur unseen to any observer. They are nonparticipatory. Then how, in principle, could extremizing I be justified as a means for deriving such effects? Result (38) is that I is an entropy, which is a system property, not a data property. Therefore, the use of I in these derivations would allow nonparticipatory effects to be derived as well. This emphasizes a dichotomy in nature: Its laws express both order in a system *and* in its data.

E. Efficiency of the order

Equation (35) indicates that that the level R of order acquired by observing (i.e., interacting) system B is maximized when $\eta = 1$, i.e., when the order is

$$R_0 = 8^{-1} L^2 I, \text{ with } R = \eta R_0 \quad (39)$$

more generally. Are there systems for which such maximum order may be attained? Recall that η arose from use of the Schwarz inequality (20). The latter becomes an equality when the components of its dot product are parallel. In (20) there is a total of N dot products, one for each pixel n. Therefore, the condition for equality is that for each n there is a proportionality constant b_n such that

$$\nabla q(\mathbf{x})_n = (\Delta \mathbf{x})_n b_n,\ b_n \equiv b(\mathbf{x})_n. \quad (40)$$

Since subscript n denotes a pixel $\mathbf{x}$, it is now superfluous and, so, may be dropped. Also, using equal spacings (3), the condition (40) becomes

$$\frac{\partial q}{\partial x_k} = b(\mathbf{x})\Delta x \quad (41)$$

for any integrable function $b(\mathbf{x})$.

By inspection, a solution is

$$q(\mathbf{x}) = h\left(\Delta x \sum_k x_k\right)^{\alpha},\ \alpha = 1 \text{ or } 2 \quad (42)$$

for any differentiable function h. Its use in (41) identifies $b(\mathbf{x}) = h'$ for $\alpha = 1$ or $b(\mathbf{x}) = 2\mathbf{x}h'$ for $\alpha = 2$. The prime denotes a derivative. Equation (42) seems to be the general solution as well. An example of solution (42) is the K-dimensional, Gaussian function $q(\mathbf{x}) = A\exp[-\Delta x \sum_k {x_k}^2]$, with A a normalization constant. Its substitution into (41) verifies this, with $b(\mathbf{x}) = -2q(\mathbf{x})\mathbf{x}$.

F. Partial coarse graining

Partial coarse graining, as opposed to full coarse graining in all the preceding, occurs when there are a finite number of linear constraints upon the system amplitudes $q(\mathbf{x})$. As in the Appendix of [1], the order of a partially coarse grained system may readily be found. Among other results, it is shown that partial coarse graining results in a smaller loss of order than does full coarse graining. Also, the value of the partial order is the unconstrained order value R reduced by a sum of weighted Fisher informations $I(F_k)$ [19]. Essentially the order is reduced because the linear constraints upon $q(\mathbf{x})$ smooth out its fluctuations, and order form (35) decreases with decreasing fluctuation.

G. Nature independent of representation

Since, at the outset, q is a unitless probability amplitude (whose square is a unitless probability p), by Eq. (22) the information I has units of x_k^{-2}. Then the result (35) shows that the order R is unitless. This is a powerful result. It has the following ramifications.

First, principle (1) and representation (35) for R thereby hold for all choices of representation of the independent variables $\mathbf{x}$, whether they be coordinates of time, position, energy, potential, etc. The results are independent of the particular coordinate representation.

Second, principle (1) and results (35), (39) hold for any choice of the dependent variable, i.e., the amplitude function $q(\mathbf{x})$ or equivalent density function $\rho(\mathbf{x}) \equiv q^2(\mathbf{x})/\Delta x^3$. But given a system, which of its density functions should be used? Do we want the order in the structural density of the mass, or charge, electric current, magnetic flux, or some other observable density that shows structure? Each is specified by a generally different density function $\rho(\mathbf{x})$, and therefore has a generally different level of order R. (It would be interesting if some of these agreed.) It is notable that any such choice obeys principle (1) and therefore holds, regardless of the type of coarse graining that the system may be subjected to.

Any relative error is unitless. The Cramer-Rao [5,6] result for the minimum rms error is $e_{\min} = 1/\sqrt{I}$. Multiplying this equation by L^{-1} and using (39) gives the minimum relative error $\epsilon_{\min} \equiv e_{\min}/L$ as obeying $\epsilon_{\min} = (\sqrt{2}/4)(1/\sqrt{R_0})$. This is again unitless and shows that order R_0 relates to the *relative* error just as I relates to the *absolute* error.

H. Fractal property

Equations (35) and (39) are the principle results of the paper. They give the total order measure as proportional to the shift-invariant Fisher information *and* to the square of the maximal system extension. These joint effects indicate a strong, basic distinction between order R and information I as system measures. This is conveniently seen when the system is degraded by the particular coarse graining effect of uniform system stretch. Here the coordinates $\mathbf{x}$ are linearly magnified to form new coordinates $\chi_k = mx_k, m = $const.$, k = 1, \ldots, K$. By Jacobian transformation, the information I_χ in the magnified system is $I_X m^{-2}$. Thus for a stretch $m > 1$ the information goes down. By comparison, the order (35) for the original system is $R_X = 8^{-1}\eta L^2 I_X$, so that the stretched system has an order $R_\chi = 8^{-1}\eta(mL)^2 I_\chi = 8^{-1}\eta(mL)^2 I_X m^{-2} = R_X$ once again! Uniform magnification does not affect the order (also see end of Sec. IX).

VII. EXAMPLES: SINUSOIDAL STRUCTURE IN K DIMENSIONS

A simple and instructive class of systems is a K-dimensional cube of side l containing a probability density function (PDF) $\rho(\mathbf{x})$, which is a product of sinusoids $\sin^2(n\pi x_k/l)$ over the K dimensions. Thus each dimension contains n oscillations or fringes. Here it is simplest to work with problems on the continuous space $\mathbf{x}$, described by PDFs $\rho(\mathbf{x})$ rather than the discrete space and its probabilities $p(\mathbf{x})_n$, used previously.

A. Fisher I and order R

The normalized, corresponding amplitude function $u(\mathbf{x})$ is its square root [see the last of Eqs. (2)],

$$u(\mathbf{x}) = (l/2)^{-K/2}\prod_{k=1}^{K}\sin(n\pi x_k/l), \quad 0 \leqslant x_k \leqslant l. \tag{43}$$

Its gradient is easily evaluated, giving for the Fisher information [11]

$$I \equiv 4\int d\mathbf{x}\nabla u \cdot \nabla u \equiv 4\sum_{k=1}^{K}\int d\mathbf{x}\left(\frac{\partial u}{\partial x_k}\right)^2 = (2n\pi K/L)^2. \tag{44}$$

Using result (44) in (39), and the fact that here chord length L obeys $L^2 = Kl^2$, gives an intrinsic order

$$R_0 = (n\pi K)^2/2. \tag{45}$$

This extremely simple result relates the concepts of order and complexity. It shows that the amount of order is independent of the extension(s) l of the system, and it rapidly increases with the number n of oscillations of the sine wave and the dimensionality K, i.e., with system complexity. By comparison, I is seen, in (44), to depend upon the number per unit length n/L of the oscillations. This is a vital difference. Also, the K dependence indicates that a three-dimensional (3D) system has intrinsically nine times the level of order, or complexity, as a corresponding 1D system. As mentioned before, the result (45) also is unitless.

B. Case $K = 2$ with $n = 5$ ripples in each direction

Let side $l = 1$, with $n = 5$ ripples in each direction. The PDF $\rho(x,y) \equiv |u(x,y)|^2 = 4\sin^2(5\pi x)\sin^2(5\pi y)$ by (43). It is plotted in Fig. 2.

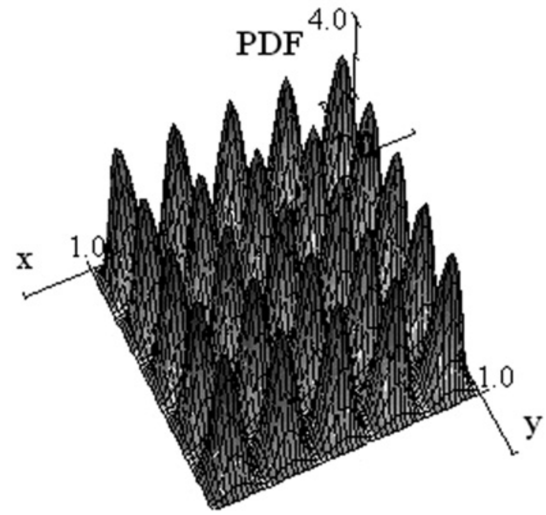

FIG. 2. Sinusoidal PDF with $n = 5$ ripples in each of $K = 2$ dimensions. Note the visually high level of complexity that accompanies the high level of order R.

Visually, the complexity of detail is quite high. By (45) the intrinsic order level is $R_0 = (5\pi \times 2)^2/2 = 493.48$, or about six times larger as for a case $n = 2$. Hence R_0 measures the level of system order in the sense of its degree of complexity as well.

VIII. WEIGHTED SINUSOIDAL STRUCTURE WITHIN A 3D CYLINDRICAL SYSTEM

Cylindrical shapes are of interest in biology, as in the cases of (a) an *E coli* bacterium or (b) many simple carbon-based polymers (e.g., narrow linear forms, thin membranes, etc.). What are their levels of order? We consider in particular case (a), noting that the results have obvious application to cases (b) as well.

Consider a general cylinder of length $\mathcal{L}$ and cross-sectional diameter l. Using cylindrical coordinates r,z,θ, let its internal details be described by an amplitude function $u\,(r,z,\theta)$ of M basis functions ψ_m that are weighted by respective coefficients a_m,

$$u(r,z,\theta) = A\sum_{m=1}^{M}a_m\psi_m(r,z), \text{each } \psi_m \equiv \sin(2m\pi r/l) \times \sin(m\pi z/\mathcal{L}) \tag{46}$$

with $A^2 = (8\mathcal{L}^{-1}l^{-2})/(\sum_m a_m^2)$, the normalization factor for the PDF $\rho(r,z,\theta) \equiv u^2(r,z,\theta)$. Each basis function ψ_m has m sinusoidal ripples, both within length $\mathcal{L}$ and across each diameter l. The results will hold for any set of weights a_m, so that the analysis is general in this regard. Note that $a_m^2 = p_m$, the probability of occurrence of state ψ_m.

A. Information and order

It is straightforward, if tedious, to calculate the information (44)

$$I \equiv 4\int_{-\pi}^{\pi}d\theta\int_0^{l/2}dr\,r\int_0^{L}dz\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right] \tag{47}$$

for the amplitude function (46), which immediately simplifies since the third integrand $r^{-2}(\partial u/\partial\theta)^2$ is zero identically. The result is $I = (\pi^3/4)(4l^{-2} + \mathcal{L}^{-2})\langle m^2\rangle$, with expectation $\langle m^2\rangle = \sum_m a_m^2 m^2 / \sum_m a_m^2$. Also, for this geometry the maximum chord length is $L^2 = \mathcal{L}^2 + l^2$. Then by (39), the intrinsic order is

$$R_0 = (\pi^3/32)(1+\epsilon^2)(1+4\epsilon^{-2})\langle m^2\rangle, \text{where } \epsilon \equiv l/\mathcal{L} \tag{48}$$

defines the eccentricity of the cylinder. Its shape for $\epsilon \ll 1$ is that of a fiber, or for $\epsilon \gg 1$ that of a disk or membrane.

B. Change in order from fiber to disk

A typical *E coli* bacterium is of diameter $l = 0.5\ \mu$m and length $\mathcal{L} = 2.0\ \mu$m, therefore of eccentricity $\epsilon = 0.25$, a fiber. However, the shapes of bacteria and other single-celled creatures are often variable, depending upon their environment (see below) and stage of development. A fundamental question is how the level of order changes as the cell shape parameter ϵ is increased (say, due to longitudinal compression), while maintaining a fixed internal structure defined by fixed values of n, the a_m, and $\langle m^2\rangle$. For simplicity, in (48) we can therefore set $\langle m^2\rangle(\pi^3/32) \equiv 1$. The resulting plot of order R_0 vs eccentricity ϵ is shown in Fig. 3.

This shows that in general the same value of order R_0 is met by two widely disparate shapes, a fiber and a disk. For example, the order 69.1 in the undisorted *E coli* (for $\epsilon = 0.25$ on the curve) is equalled by an equivalent disk of eccentricity 8.0. Also, by allowing a change in mass by some factor α to take place during the compression, the disk has dimensions $l = 0.707\alpha^{1/3}\ \mu$m, $\mathcal{L} = 0.088\alpha^{1/3}\ \mu$m. Obviously their quotient gives the same eccentricity, that of a disk, for any mass factor α. Integer values $\alpha = 2,3,\ldots$ connote mitosis events. The change in mass is effected by energy flow, either from (for $\alpha > 1$) or into ($\alpha < 1$) the environment. These results have the following biological significance:

(1) It was recently found [20] that *E coli* or other bacteria can squeeze through practically any opening. When forced to pass through an extremely narrow slit it typically takes on a completely flattened shape. This corresponds to the disk shape in the above. Also, the mass change factor obeys $\alpha > 1$ due to cell growth by mitosis during the distortion process. Such distortions of bacteria are also envisioned [20] as naturally occurring underground, where bacteria are constrained to live in spaces of about a micrometer.

(2) One model for the origin of life is the formation of many linear chains of hydrocarbons – the fiber case above – that eventually combine into a thin membrane which is the protype for a cell membrane [21]. For example, bacterial cell walls are membranes of peptidoglycan, which is made in part from polysaccharide chains [22]. But from the preceding analysis the membrane could have instead resulted from distortion of a single such chain (or fiber), since both a chain and corresponding membrane can have equal complexity R. Single chains are known to be readily formed by UV radiation [23] and are long known to exist widely in interstellar space, even away from galactic centers [24]. These suggest that life could exist widely in space.

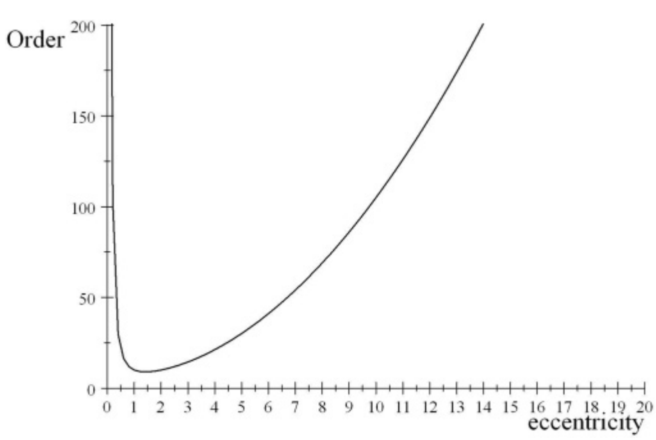

FIG. 3. Level of order R_0 in the internal structure of a cylindrical system vs its eccentricity ϵ. In general, two different ϵ values – corresponding to a cylinder or a disk – give rise to the same level of R_0, for values $R_0 \geqslant \sqrt{2}$.

IX. SUMMARY, AND TWO QUESTIONS

By postulating that the order of a system should decrease under weak coarse graining, we have quantified that the total order R obeys Eq. (39). This basically traces from expansion (7), to second order in small amplitude changes $\Delta\mathbf{q}$, for the change ΔR resulting from the coarse graining. The second derivative matrix $\mathbf{M}$ defined in (7) is found to be Hermitian and, by (8), this guarantees positive eigenvalues λ_i in a suitably transformed system (9) to changes $\Delta\mathbf{q}'$. Use of the Cencov choice (14) of unit eigenvalues in Eq. (13) and steps (15)–(23) then give an intermediary result (24) for ΔR. This is the sum of an integral of an unknown function $C(x)$ and a term linear in Fisher information I. Then by Taylor expansion (28), whose powers of Δx must match those of (24), the term in $C(x)$ is found to be zero [see (29)] and the mixed partial derivatives (31) are found to be proportional to I. It results that $\Delta R \propto I$. Finally, the assumption that chord length L is small enough to ignore cubic and higher-order powers of Δx_k gives the order (35) as $R = 8^{-1}\eta L^2 I$, with η an efficiency coefficient obeying $0 \leqslant \eta \leqslant 1$. Complete efficiency $\eta = 1$ is found to hold for amplitude functions of the form (42), an example being a K-dimensional Gaussian law.

Properties of the new measure R are examined. It is found to be unitless, invariant to linear system stretch (a fractal property), and to be an entropy. The system property (38) indicates that Fisher I is likewise an entropy, independent of the presence of data. This therefore allows past I-based derivations of physical law [11, 13] to describe both participatory and non-participatory phenomena (where no data are available). Finally, hydrocarbon-based polymers shaped like fibers or disks can have equal levels of order. This suggests that a two-dimensional (2D) cell membrane can readily form from a single 1D hydrocarbon chain, permitting the widespread occurrence of life in the cosmos.

We end with two questions of interest.

(1) Suppose that we want to test the hypothesis that man's brain is more ordered or complex than that of any other living creature. We use Eq. (39) to compute R_0, and decide to

compare R_0 values on the basis of mass density distributions ρ. We choose to compare human brain complexity with that of the creature with the largest brain, the sperm whale. The neuronal density in cetaceans is usually as high as in humans. The lamination or number of cortical layers is also, as in humans, advanced [25,26]. On this basis, it seems reasonable to estimate the Fisher information level I as roughly equal in the two brains. However, this estimate is compromised by the lack of knowledge of the complexity of cortical patterns, the specialization of cortical arrangements, and the amount of communication between cells in the whale. Hence the comparison will not be definitive. With the two I values about equal, Eq. (39) indicates that the factor L^2 is decisive to the comparison. The whale's brain weighs about 9.0 kg vs 1.3 kg in a human. Assuming both brains are roughly the same shape, the ratio of lengths $L \approx (9.0/1.3)^{1/3}$, so that the ratio of factors L^2 is about $(9.0/1.3)^{2/3} = 3.6$. On this basis the whale brain is more complex, although the question remains open because of the unknowns mentioned above. Note, furthermore, that an advantage in the complexity of the brain's mass distribution does not necessarily associate with intelligence. Instead, one author [26] associates intelligence with a brain's neural distribution (and qualities): "These data... demonstrate that there is no neural basis for the often-asserted high intellectual abilities of cetaceans."

(2) Regard our universe as a closed system. Due to Hubble expansion, the universe is apparently expanding at an ever-accelerating rate. Thus its fate in the distant future is often imagined to be an irreversibly stretched system from which recovery via a "big crunch" could never occur. This implies that due to the expansion, its order is ever approaching zero. But, what does the measure R predict? Order R was found to remain constant under uniform system stretch. Then, is Hubble expansion uniform over our system, i.e., all of space, taking into account its possible curvature and quantum gravitational effects? If so, the expansion would leave the order unaffected at all times! So, how does Hubble expansion affect our ultimate fate? A paper on this question is in preparation.

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