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Scaling of cluster heterogeneity in percolation transitions

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We investigate a critical scaling law for the cluster heterogeneity H in site and bond percolations in d-dimensional lattices with $d = 2, \ldots, 6$. The cluster heterogeneity is defined as the number of distinct cluster sizes. As an occupation probability p increases, the cluster size distribution evolves from a monodisperse distribution to a polydisperse one in the subcritical phase, and back to a monodisperse one in the supercritical phase. We show analytically that H diverges algebraically, approaching the percolation critical point p_c as $H \sim |p - p_c|^{-1/\sigma}$ with the critical exponent σ associated with the characteristic cluster size. Interestingly, its finite-size-scaling behavior is governed by a new exponent $\nu_H = (1 + d_f/d)\nu$, where d_f is the fractal dimension of the critical percolating cluster and ν is the correlation length exponent. The corresponding scaling variable defines a singular path to the critical point. All results are confirmed by numerical simulations.

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Percolation is a geometric phase transition for connectivity [1,2]. Suppose that a fraction p of sites or bonds is occupied in an infinite lattice. When p is less than a threshold p_c, any site is connected to others up to a finite distance via occupied sites or bonds. As p increases, clusters (sets of connected sites) grow until there emerges a spanning, giant, or infinite cluster to which a finite fraction of sites belong.

The percolation transition is manifested in several quantities. The order parameter m, given by the fraction of sites belonging to the giant cluster, becomes nonzero beyond the transition. The mean cluster size S, defined as the average size of finite clusters which include a randomly selected site, diverges at the transition. Their scaling properties, such as the exact critical exponents in two dimensions [3] and the upper critical dimensionality $d_u = 6$ [1], are well understood.

Recently, the cluster heterogeneity H, defined as the number of distinct cluster sizes, has been suggested as a useful indicator of a percolation transition [4]. Consider a cluster size distribution function n_s that is defined as the number of clusters of size s per site. When $p = 0$, the distribution is monodisperse with $H = 1$. As p increases, clusters nucleate and aggregate into larger ones. Consequently, in the subcritical phase, the cluster size distribution becomes broader. In the supercritical phase, finite clusters are absorbed into the giant cluster to decrease H. Hence one expects that the cluster heterogeneity may be maximal at the transition.

Lee *et al.* [4] found that the maximum heterogeneity points indeed converge to a percolation threshold in the thermodynamic limit in the study of the so-called explosive percolation [5]. Exploiting the finite-size-scaling (FSS) property at the maximum heterogeneity points, they could clarify the nature of the explosive percolation transition. However, the scaling property of H by itself has not been fully understood even in the ordinary random percolations. In this Rapid Communication, we establish the critical scaling law for H with emphasis on its FSS theory.

We begin with a brief review for the scaling theory. For a detailed review, we refer readers to Refs. [1] and [2]. The ordinary percolation exhibits a continuous transition [6]. The correlation length diverges algebraically as $\xi \sim |\epsilon|^{-\nu}$, where $\epsilon \equiv p - p_c$ and ν is the correlation length exponent. At $p = p_c$, the giant cluster is a fractal characterized with a fractal dimension d_f. Physical quantities have a singular dependence on ϵ. For example, the order parameter scales as $m \sim \epsilon^{\beta}$ for $p \geqslant p_c$ with the order parameter exponent β. The mean cluster size S diverges as $S \sim |\epsilon|^{-\gamma}$ with the susceptibility exponent γ. The cluster size distribution function $n_s(p)$ for finite s scales as

$$n_s(p) \sim s^{-\tau} e^{-s/s_c}, \tag{1}$$

where τ is called the Fisher exponent and s_c is the characteristic cluster size. It diverges as $s_c \sim |\epsilon|^{-1/\sigma}$ with a critical exponent σ. There exist scaling relations among those critical exponents. So, any exponent can be written in terms of two independent ones, say ν and d_f. The scaling relations read $\beta = \nu(d - d_f)$, $\gamma = \nu(2d_f - d)$, $\sigma = 1/(\nu d_f)$, and $\tau = 1 + d/d_f$.

In finite systems of linear size L, the scaling laws are modified because the correlation length ξ is limited by L. The FSS hypothesis assumes that a finite-size effect comes into play through the ratio between L and ξ [7]. This leads to the FSS ansatz for an observable $Q(p, L)$:

$$Q(p, L) = L^{X_Q} \mathcal{F}_Q(\epsilon L^{1/\nu}), \tag{2}$$

where X_Q is a scaling exponent and $\mathcal{F}_Q(x)$ is a scaling function for Q [7]. The FSS ansatz claims that a finite system suffers from a finite-size effect in the region $|\epsilon| < L^{-1/\nu}$, whereas it behaves as an infinite one elsewhere. The aim of this Rapid Communication is to find a scaling law for H.

First, we present numerical data for H in site and bond percolations in two-dimensional (2D) square and triangular lattices, which shows that the standard FSS form (2) is not valid for H. Next, we derive the central result of Eq. (14) and present numerical data in $d = 2, \ldots, 6$ dimensional hypercubic lattices to confirm it. This is followed by our summary and conclusion.

We have performed Monte Carlo simulations of site and bond percolations in 2D square and triangular lattices of $L \times L$ sites under the periodic boundary condition using the Newman and Ziff algorithm [8]. This algorithm allows an efficient and fast measurement of a quantity Q as a function of the number n of occupied sites or bonds. A corresponding quantity as a function of the occupation probability p is then obtained by the convolution $Q(p) = \sum_n B(N, n, p) Q(n)$, where N is the

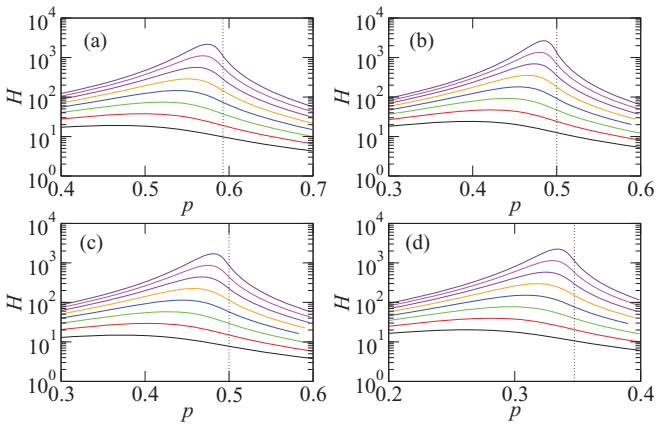

FIG. 1. (Color online) Cluster heterogeneity in site percolations in (a) and (c), and bond percolations in (b) and (d) on 2D square lattices in (a) and (b) and triangular lattices in (c) and (d). The dotted lines represent the critical percolation threshold. Lattice sizes are $L = 2^5, \ldots, 2^{12}$ for the square lattices and $L = 27 \times 2^0, \ldots, 27 \times 2^7$ for the triangular lattices. The larger the value of L, the higher the value of H.

total number of sites or bonds and $B(N,n,p) \equiv \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$ is the binomial distribution function [8,9].

The cluster heterogeneity, averaged over $N_S = 10^5$ samples, is presented in Fig. 1. In all cases, the curves have a diverging peak (maximum cluster heterogeneity) at a position denoted by (p^*, H^*). The peak position seems to approach from below the critical point, which is $p_c = 0.592746$ for the square lattice site percolation [8], 1/2 for the square lattice bond percolation and the triangular lattice site percolation, and $2\sin(\pi/18)$ for the triangular lattice bond percolation [10]. FSS properties are analyzed in Fig. 2. We find that $(p_c - p^*)$ and H^* scale algebraically with L with apparently universal exponents.

It is noteworthy that $(p_c - p^*)$ does not scale as $L^{-1/\nu}$ with the correlation length exponent $\nu = 4/3$ in two dimensions. Such a scaling would be natural from the FSS hypothesis

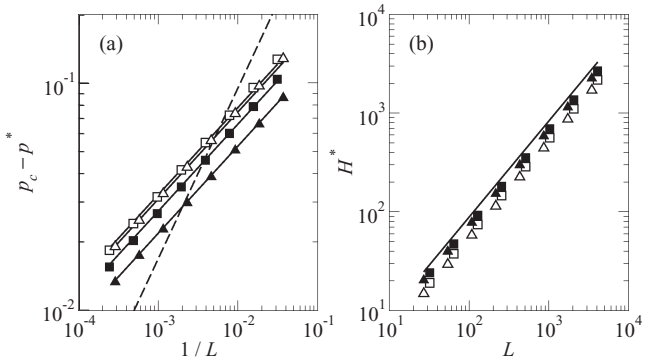

FIG. 2. FSS of $(p_c - p^*)$ in (a) and H^* in (b) for the site (open symbols) and the bond (filled symbols) percolations on 2D square (square symbols) and triangular (triangle symbols) lattices. All straight lines are guides to the eye. The slopes of the solid lines are explained in the text. In (a), the dashed line has a slope of $1/\nu = 3/4$.

of Eq. (2). However, Fig. 2(a) excludes such a possibility definitely. It calls for an appropriate FSS theory for H.

In order to characterize the cluster heterogeneity, one needs to specify the size of each cluster. Jan *et al.* [11] considered a FSS behavior of the rth largest cluster size at the critical point where the cluster size distribution follows a power law $n_s \sim s^{-\tau}$. The rth cluster size s_r is estimated from the relation $r \sim L^d \int_{s_r} ds\, s^{-\tau}$, which yields

$$s_r \sim r^{-\frac{1}{\tau-1}} L^{\frac{d}{\tau-1}} = r^{-\frac{1}{\tau-1}} L^{d_f}. \tag{3}$$

We extend their idea to the off-critical region to derive the FSS theory for H.

For $p \lesssim p_c$, the cluster size distribution function is given by Eq. (1) [12]. Then, the size of an rth largest cluster is obtained from

$$r = L^d \int_{s_r} ds\, n_s \sim L^d s_c^{1-\tau} \Gamma(1-\tau, s_r/s_c), \tag{4}$$

where the function $\Gamma(u,x) \equiv \int_x^\infty dt\, t^{u-1} e^{-t}$ is the incomplete gamma function. The characteristic size itself displays a FSS behavior [2]:

$$s_c \sim \begin{cases} |\epsilon|^{-1/\sigma} & \text{for} \quad |\epsilon| \gg L^{-1/\nu}, \\ L^{d_f} & \text{for} \quad |\epsilon| \ll L^{-1/\nu}. \end{cases} \tag{5}$$

We now compare the average size of the rth and the $(r+1)$th clusters. Using Eq. (4), one finds that

$$\Delta_r s = s_r - s_{r+1} \sim L^{-d} s^\tau e^{s/s_c}. \tag{6}$$

The distribution is *dense* when $\Delta_r s < 1$ and *sparse* when $\Delta_r s > 1$. The two regions are separated at $s = s_0$, satisfying

$$L^{-d} s_0^\tau e^{s_0/s_c} = O(1). \tag{7}$$

The rank of a cluster of size s_0 is given by

$$r_0 \sim L^d s_c^{1-\tau} \Gamma(1-\tau, s_0/s_c). \tag{8}$$

Note that $L^d n_{s_0} = O(1)$. This implies that there are at least $O(1)$ clusters of all sizes $s < s_0$. On the other hand, there are r_0 clusters in the region $s > s_0$, whose sizes are distinct because $\Delta_r s > 1$. Therefore we find that

$$H \simeq s_0 + r_0. \tag{9}$$

To obtain the solution of Eq. (7) for s_0, we first assume that $s_0 \ll s_c$. Then, the exponential term is negligible and the solution is given by

$$s_0 \sim L^{d/\tau}. \tag{10}$$

Using the asymptotic behavior $\Gamma(u, x \to 0) \sim -\frac{1}{u} x^u$, one also finds that $r_0 \sim s_0$. This solution is self-consistent when $L^{d/\tau} \ll |\epsilon|^{-1/\sigma}$ or

$$|\epsilon| \ll L^{-1/\nu_H}, \tag{11}$$

where the exponent variable is

$$\nu_H = \frac{\tau}{d\sigma} = \frac{\tau}{\tau-1}\nu = \left(1 + \frac{d_f}{d}\right)\nu. \tag{12}$$

In the opposite case $s_0 \gg s_c$, the leading-order solution of Eq. (7) is given by

$$s_0 \sim s_c \ln L. \tag{13}$$

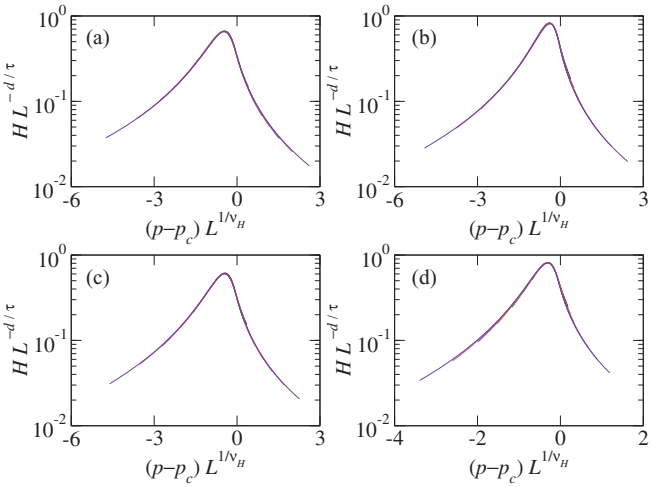

FIG. 3. (Color online) Scaling analysis of H for the site [(a) and (c)] and bond [(b) and (d)] percolations on 2D square [(a) and (b)] and triangular [(c) and (d)] lattices. The same data sets are used as in Fig. 1.

From the asymptotic behavior $\Gamma(u, x \to \infty) \sim x^{u-1}e^{-x}$, one also finds that $r_0 \sim s_c$. This solution is self-consistent when $|\epsilon| \gg L^{-1/\nu_H}$.

Using the solution for s_0 and r_0, we can summarize the scaling property of H with the FSS form

$$H(p,L) = L^{d/\tau}\mathcal{F}_H(\epsilon L^{1/\nu_H}). \tag{14}$$

The scaling function has a limiting behavior

$$\mathcal{F}_H(x) \sim \begin{cases} |x|^{-1/\sigma} \ln |x| & \text{for} \quad |x| \gg 1, \\ \text{constant} & \text{for} \quad |x| \ll 1, \end{cases} \tag{15}$$

so that $H \sim (\ln L)|\epsilon|^{-1/\sigma}$ for $|\epsilon| \gg L^{-1/\nu_H}$ and $H \sim L^{d/\tau}$ for $|\epsilon| \ll L^{-1/\nu_H}$.

Remarkably, the FSS exponent ν_H for the cluster heterogeneity is distinct from the correlation length exponent ν. This explains why $(p_c - p^*)$ does not scale as $L^{-1/\nu}$ in Fig. 2(a). Instead, the numerical data are consistent with the scaling $(p_c - p^*) \sim L^{-1/\nu_H}$ with $\nu_H = 187/72$ in two dimensions (see Table I). The solid lines in Fig. 2(a) have a slope of $1/\nu_H$. The maximum heterogeneity is expected to scale as $H^* \sim L^{d/\tau}$ with $d/\tau = 182/187$ in two dimensions. The solid line in Fig. 2(b) has this slope and is in agreement with the numerical data. The FSS scaling form is tested in Fig. 3, where we replot the data in Fig. 1 according to Eq. (14). All the data from different sizes collapse perfectly onto a single curve.

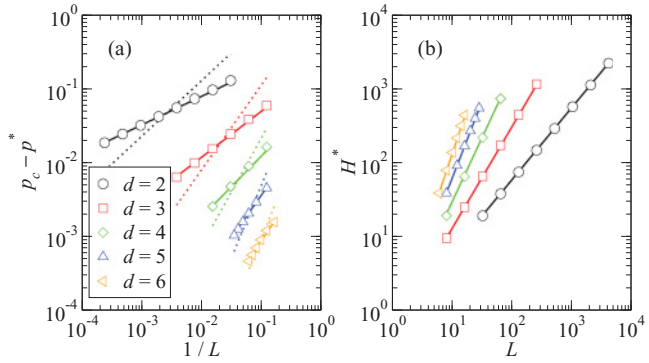

FIG. 4. (Color online) (a) $(p_c - p^*)$ vs $1/L$. The solid lines have a slope of $1/\nu_H$, while the dotted lines have a slope of $1/\nu$. (b) H^* vs L with the solid lines having a slope of d/τ.

We examine whether the FSS form in Eq. (14) for H is valid universally in higher dimensions up to the upper critical dimension $d_u = 6$. We have performed extensive numerical simulations of site and bond percolations to measure H in hypercubic lattices in $d = 2, \ldots, 6$ dimensions under the periodic boundary condition. The system sizes are $L = 2^3, \ldots, 2^8$ in three dimensions, $L = 2^3, \ldots, L^6$ in four dimensions, $L = 8, \ldots, 28$ with $\Delta L = 4$ for five dimensions, and $L = 6, \ldots, 16$ with $\Delta L = 2$ for six dimensions. The number of samples is $N_S = 10^5$ in all cases.

In Fig. 4(a), $(p_c - p^*)$ is plotted against $1/L$ for the site percolations. Also drawn are the straight lines corresponding to the scalings $L^{1/\nu}$ and L^{1/ν_H}. The parameter values used in the plot are collected from the literature and listed in Table I. In all dimensions, $(p_c - p^*)$ scales with the exponent $1/\nu_H$ instead of $1/\nu$. Figure 4(b) shows the plot of H^* against L, which also confirms the scaling $H^* \sim L^{d/\tau}$. We have obtained the same result for the bond percolations, for which data are not shown here.

We have shown that the cluster heterogeneity diverges at the percolation critical point and that it satisfies the FSS form in Eq. (14). The FSS is governed not by the correlation length exponent ν but by the new exponent $\nu_H = (1 + d_f/d)\nu$. Consequently, the maximum cluster heterogeneity points follow the scaling $[p_c - p^*(L)] \sim L^{-1/\nu_H}$. This is contrasted with the standard $L^{-1/\nu}$ scaling of the effective critical points obtained from the order parameter data and the average cluster size data.

Since $\nu_H > \nu$, the maximum heterogeneity points $p^*(L)$ constitute a singular path of $\epsilon \sim -L^{-1/\nu_H}$ in the subcritical phase to the critical point. Although they converge to p_c in the

TABLE I. Critical points and critical exponents in the site percolations in d-dimensional hypercubic lattices. The mean field (MF) exponent values are given for $d = 6$.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
p_c (site)	0.592746 [8]	0.3116077 [13]	0.196889 [16]	0.140765 [17]	0.109017 [17]
ν	4/3	0.875 [14]	0.689 [15]	0.51 [18]	1/2 (MF)
d_f	91/48	2.523 [14]	3.05 [16]	3.54 [16]	4 (MF)
$\nu_H = (1 + d_f/d)\nu$	187/72	1.611	1.21	0.87	5/6 (MF)
$d/\tau = d/(1 + d/d_f)$	182/187	1.370	1.73	2.07	12/5 (MF)

$L \to \infty$ limit, the system along the path remains outside the critical region defined by the condition $|\epsilon| \ll L^{-1/\nu}$. Physical quantities along a singular path may exhibit a peculiar FSS behavior.

Consider, for example, the order parameter m, which follows the FSS form in Eq. (2) with $X_m = -\beta/\nu$. In the subcritical phase, it scales with L as $m \sim L^{-d}$ in the leading order [19], which requires that the scaling function should have a limiting behavior $\mathcal{F}_m(x \ll -1) \sim |x|^{-(d\nu-\beta)}$. So the order parameter m^*, evaluated at the maximum heterogeneity points, follows the scaling $m^* \sim L^{-\beta/\nu}\mathcal{F}_m(-L^{-1/\nu_H+1/\nu}) \sim L^{-\beta/\nu-d_f(1-\nu/\nu_H)}$, instead of the standard critical scaling $m^* \sim L^{-\beta/\nu}$. In fact, this is a manifestation of the crossover phenomena near a multicritical point [20] in the context of the FSS [21]. In general, when there exist multiple relevant scaling fields for a critical point, scaling behaviors become dependent on a path approaching it. Since $1/L$ is one such relevant scaling variable [7,22], one can naturally expect a nontrivial FSS along a singular path. The maximum cluster heterogeneity condition indeed allows us to access a singular path to the percolation critical point.

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