Dynamics of a nonautonomous soliton in a generalized nonlinear Schrödinger equation

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We solve a generalized nonautonomous nonlinear Schrödinger equation analytically by performing the Darboux transformation. The precise expressions of the soliton's width, peak, and the trajectory of its wave center are investigated analytically, which symbolize the dynamic behavior of a nonautonomous soliton. These expressions can be conveniently and effectively applied to the management of soliton in many fields.

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I. INTRODUCTION

Since Zabusky and Kruskal first introduced the concept of the soliton in 1965 [1], studies on solitons have been done in many fields, including hydrodynamics [2,3], quantum field theory [2], plasma physics [4], nonlinear optics [5–9], and the Bose-Einstein condensate [10–14]. The classical soliton concept was developed for nonlinear and dispersive systems. Time only played the role of an independent variable. Moreover, time has not appeared explicitly in the nonlinear evolution equation, which is seen as autonomous [6]. However, in most real experiments, solitons cannot be autonomous, which is quite different from the conventional soliton concept. For example, there is (i) the test of solitons in nonuniform media with time-dependent density gradients [15]; (ii) the test of the core medium of the real fibers, which cannot be homogeneous, fiber loss is inevitable, and dissipation weakens the nonlinearity [16]; and (iii) the formation of solitons in Bose-Einstein condensates, which tunes the interaction near Feshbach resonance and provides a good example for a nonautonomous system as well [17,18].

For the nonautonomous system, three questions are asked: Do solitons still exist and maintain their identities through nonlinear interaction in time-dependent external potentials? In which condition can a soliton exist? How can the dynamical behaviors of nonautonomous solitons be controlled? These are the main points physicists want to know. In most cases, the dynamics of nonautonomous solitons are governed by the nonlinear Schrödinger equation (NLSE). Thus, it is more meaningful to solve the generalized NLSE, which can be used conveniently to study many kinds of nonautonomous systems. Serkin *et al.* presented soliton solutions for the generalized NLSE and proved that solitons can exist in a nonautonomous system [6]. Luo *et al.* provided some ways to manage the soliton in a nonautonomous system [7].

In this paper, we present one family of analytical nonautonomous soliton solutions for the generalized nonautonomous NLSE [Eq. (1)] by making use of the Darboux transformation method from a trivial seed solution. The precise expressions of the soliton's width, peak, and the motion of its center are investigated analytically. In general terms, we choose dispersion, nonlinearity, gain (or loss), and external potentials as arbitrary time-dependent functions. From their analytical expressions, it can be seen how these factors affect the dynamical properties of solitons. This will provide us with explicit ways to control the evolution of solitons. We also present the condition to manage dispersion, nonlinearity, and the gain term to keep the amplitude of the nonautonomous soliton unchanged, which could be used to improve the quality of soliton transmission in optical communication.

II. THE DARBOUX TRANSFORMATION AND BRIGHT SOLITON SOLUTION

For one-dimensional generalized nonautonomous systems, the dynamics of nonautonomous solitons can be governed by the NLSE, in which the parameters of dispersion, nonlinearity, gain (or loss), and external potentials are all dependent on time. The related dimensionless nonautonomous NLSE can be written as

$$i\frac{\partial\psi(x,t)}{\partial t}+\Omega(t)\frac{\partial^2\psi(x,t)}{\partial x^2}+2R(t)|\psi(x,t)|^2\psi(x,t)+V(x,t)\psi(x,t)+i\frac{G(t)}{2}\psi(x,t)=0,\qquad(1)$$

where $\Omega(t)$ and $R(t)$ are the dispersion and nonlinearity management parameters, respectively. $V(x,t)$ denotes the external potential applied and $G(t)$ is the dissipation [$G(t)>0$] or gain [$G(t)<0$]. The general form of Eq. (1) includes many special cases discussed in the literature, and its analytical solitonlike solution was recently termed the nonautonomous soliton. Based on the Painlevé analysis [19], it is known that the most generalized form follows, which can be solved analytically. $V(x,t)=M(t)x^2+f(t)x$, where $M(t)x^2$ means a time-dependent harmonic trap and $f(t)x$ stands for an arbitrary time-dependent linear potential. $\Omega(t)$, $R(t)$, and $G(t)$ are not allowed to be space-dependent according to the Painlevé analysis. Zhao *et al.* have found a transformation from some nonautonomous to standard NLS equations [20]. However, it still lacks the Lax pair of the generalized NLSE, and this is significant when deriving the Lax pair, which can be used to generate many different soliton solutions.

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To solve the NLSE, we assume that the solution of Eq. (1) is

$$\psi(x,t) = Q(x,t)\exp\left[iC(t)x^2+\int\left(2\Omega(t)C(t)-\frac{G(t)}{2}\right)dt\right], \tag{2}$$

where $C(t)$ is introduced to help one find some ways to simplify the NLSE. Inserting Eq. (2) into Eq. (1), the following equation could be derived:

$$\begin{aligned} &i\frac{\partial Q}{\partial t}+2R(t)\exp\left[\int[4\Omega(t)C(t)-G(t)]dt\right]|Q|^2Q \\ &+f(t)xQ+\Omega(t)\frac{\partial^2 Q}{\partial x^2}+4i\Omega(t)C(t)x\frac{\partial Q}{\partial x} \\ &+4i\Omega(t)C(t)Q+\left[-\frac{dC(t)}{dt}x^2-4\Omega(t)C^2(t)x^2 \right.\\ &\left.+M(t)x^2\right]Q=0, \end{aligned} \tag{3}$$

where Q denotes $Q(x,t)$. To simplify the above equation, we choose the relation of nonlinearity, external potentials, and the gain as $R(t)=g\Omega(t)\exp\{\int[G(t)-4\Omega(t)C(t)]dt\}$ (g is a real number and $g\neq 0$), where $C(t)$ satisfies the condition $4\Omega(t)C^2(t)+\frac{dC(t)}{dt}=M(t)$. Then Eq. (3) becomes

$$\begin{aligned} &i\frac{\partial Q}{\partial t}+\Omega(t)\frac{\partial^2 Q}{\partial x^2}+4i\Omega(t)C(t)x\frac{\partial Q}{\partial x}+f(t)xQ \\ &+4i\Omega(t)C(t)Q+2g\Omega(t)|Q|^2Q=0. \end{aligned} \tag{4}$$

Then, the corresponding Lax pair of the equation for $Q(x,t)$ can be assumed as follows:

$$\begin{aligned} \partial_x\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}&=M\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}\zeta(x,t) & p(x,t)\\ q(x,t) & -\zeta(x,t)\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \\ \partial_t\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}&=N\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}A & B\\ C & -A\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \end{aligned} \tag{5}$$

where $\zeta(x,t)$ is a spectrum parameter and

$$A=\sum_{j=0}^{2}a_j(x,t)\zeta(x,t)^{2-j},$$

$$B=\sum_{j=0}^{2}b_j(x,t)\zeta(x,t)^{2-j},$$

$$C=\sum_{j=0}^{2}c_j(x,t)\zeta(x,t)^{2-j}.$$

If we assume the parameter $\zeta(x,t)$ is only time-dependent as $\zeta(t)$, and $\zeta_t=\lambda(t)\zeta(t)+k(t)$ (the footnote t or x means partial derivation of t or x), then from the compatibility condition $M_t-N_x+[M,N]=0$, the following relations can be given:

$$a_0(x,t)=\alpha_0(t),\ b_0(x,t)=0,\ c_0(x,t)=0,$$

$$a_1(x,t)=\lambda(t)x,\ b_1(x,t)=p(x,t)\alpha_0(t),$$

$$c_1(x,t)=q(x,t)\alpha_0(t),$$

$$b_2(x,t)=\frac{\alpha_0(t)}{2}p_x+p(x,t)a_1(x,t),$$

$$c_2(x,t)=-\frac{\alpha_0(t)}{2}q_x+q(x,t)a_1(x,t),$$

$$a_2(x,t)=-\frac{\alpha_0(t)}{2}p(x,t)q(x,t)+k(t)x,$$

and

$$p_t=\frac{\partial b_2(x,t)}{\partial x}+2p(x,t)a_2(x,t),$$

$$q_t=\frac{\partial c_2(x,t)}{\partial x}-2q(x,t)a_2(x,t),$$

which are usually named for the developing equation.

Finally, assuming $p=\sqrt{g}Q$ and $q=-\sqrt{g}\bar{Q}$ (hereafter the overbar denotes the complex conjugate), we can derive the evolution equation of Q as follows:

$$\begin{aligned} &iQ_t-i\frac{a_0(t)}{2}Q_{xx}-i\lambda(t)xQ_x-i[\lambda(t)+2k(t)x]Q \\ &-iga_0(t)|Q|^2Q=0. \end{aligned} \tag{6}$$

Comparing Eq. (6) with Eq. (4), one can know that $\alpha_0(t)=i2\Omega(t)$, $\lambda(t)=-4\Omega(t)C(t)$, and $k(t)=if(t)/2$. From the above relations, which are derived from the compatibility condition, one can derive the following expressions:

$$A=2i\Omega(t)\zeta(t)^2-4\Omega(t)C(t)x\zeta(t)+ig\Omega(t)|Q|^2+if(t)x/2,$$

$$B=2i\sqrt{g}\Omega(t)\zeta(t)Q+i\sqrt{g}\Omega(t)Q_x-4\sqrt{g}\Omega(t)C(t)xQ,$$

$$C=-2i\sqrt{g}\Omega(t)\zeta(t)\bar{Q}+i\sqrt{g}\Omega(t)\bar{Q}_x+4\sqrt{g}\Omega(t)C(t)x\bar{Q},$$

$$\zeta_t=-4\Omega(t)C(t)\zeta(t)+if(t)/2.$$

In this way, the Lax pair is finally given. Corresponding to the Lax pair, the Darboux transformation can be presented as

$$p'(x,t)=p_0(x,t)+\frac{2[\zeta(t)+\zeta(t)^*]\sigma(x,t)^*}{[1+|\sigma(x,t)|^2]}, \tag{7}$$

where $\sigma(x,t)=\frac{\Phi_2}{\Phi_1}$, and Φ_1 and Φ_2 are the solution of the Lax pair with $p=p_0$. It is obvious that $Q=0$ is the solution of Eq. (4); we can choose $p_0=\sqrt{g}Q=0$ as the seed solution to derive soliton solutions. With the seed solution p_0, one can solve the Lax pair to get Φ_1 and Φ_2 as

$$\Phi_1(x,t)=\exp\left[\zeta(t)x+\int 2i\Omega(t)\zeta(t)^2dt\right],$$

$$\Phi_2(x,t)=A_c\exp\left[-\zeta(t)x-\int 2i\Omega(t)\zeta(t)^2dt\right],$$

where $\zeta(t)=b(t)+id(t)$, and

$$b(t)=\alpha\exp\left[\int -4\Omega(t)C(t)dt\right],$$

$$d(t)=\left[\int\frac{f(t)}{2}e^{\int 4\Omega(t)C(t)dt}dt+\beta\right]\exp\left[\int -4\Omega(t)C(t)dt\right],$$

where A_c, α, and β are arbitrary real numbers. Then, $\sigma(x,t)$ can be given as

$$\sigma(x,t)=A_c\exp\left[-2\zeta(t)x-\int 4i\Omega(t)\zeta(t)^2dt\right].$$

Performing the Darboux transformation Eq. (7), one can get a new solution $Q(x,t)$ of Eq. (4). From Eq. (2), the analytical solution of Eq. (1) can be presented as

$$\psi(x,t) = \frac{2[\zeta(t) + \zeta(t)^*]\sigma(x,t)^*}{\sqrt{g}[1 + |\sigma(x,t)|^2]} \exp\theta(x,t), \tag{8}$$

where

$$\theta(x,t) = iC(t)x^2 + \int [-G(t)/2 + 2\Omega(t)C(t)]\,dt.$$

As a result, we obtain one family of soliton solutions of Eq. (1) in a generalized nonautonomous system. If the Darboux transformation is performed from some nontrivial seed solution, similar results can be attained [10]. When $C(t) = 0$, $\Omega(t) = 1$, and $G(t) = 0$, similar soliton solutions in an arbitrary time-dependent linear potential can be studied [11]. Furthermore, we can calculate the nonautonomous soliton's peak, width, and the motion of its wave center from the soliton solution [Eq. (8)]. We define the maximum value of density as the soliton's wave center and its half-value as the soliton's width; thus, their evolution can be given as follows [with $C(t)$ a real function]:

The width of the nonautonomous soliton is

$$W(t) = \frac{1}{2b(t)} \ln(3 + 2\sqrt{2}), \tag{9}$$

the evolution of its peak is

$$|\psi|^2_{\max} = \frac{4b(t)^2}{g} \exp\left[\int 4\Omega(t)C(t) - G(t)dt\right], \tag{10}$$

and the trajectory of its wave center is

$$x_c(t) = \frac{\ln A_c}{2b(t)} + \frac{\int 4\Omega(t)b(t)d(t)dt}{b(t)}. \tag{11}$$

Based on these expressions, soliton management can be realized. To demonstrates this, we will study the evolution of solitons in nonlinear fibers.

III. OPTICAL SOLITONS IN NONLINEAR FIBER

The optical soliton has become an intensely studied subject with the development of modern technology. In the ideal situation, it is well known that propagation of optical solitons in a single mode fiber is governed by the standard NLSE [21]. However, in a real fiber, in general, the core medium is not homogeneous [16]. There are always some nonuniformities due to many factors, e.g., variations in the lattice parameters of the fiber medium and the fiber geometry (diameter fluctuations). These nonuniformities influence various effects, including loss (or gain), dispersion, and phase modulation. Considering the inhomogeneities in fiber [22], the dynamics of the optical pulse propagation can be governed by the following inhomogeneous NLSE:

$$i\frac{\partial\Psi}{\partial Z} + \Omega(Z)\frac{\partial^2\Psi}{\partial T^2} + 2R(Z)|\Psi|^2\Psi + M(Z)T^2\Psi + i\frac{G(Z)}{2}\Psi = 0, \tag{12}$$

where Z is the normalized distance and T is the retarded time. $\Omega(Z)$ is the group velocity dispersion parameter, $R(Z)$ is related to the Kerr nonlinearity, and $M(Z)$ and $G(Z)$ are inhomogeneous parameters related to phase modulation and loss (or gain). In this case, Ψ [which denotes $\Psi(T,Z)$] is the complex envelope of the electrical field in a comoving frame. When $M(Z) = \beta^2$ and $G(Z) = 2\beta$ (β is a real constant), the dynamics of solitons on a continuous wave background have been discussed by Li [22].

From the generalized soliton solution [Eq. (8)] and its compatible condition, we can know that $R(Z) = g\Omega(Z)\exp[\int G(Z) - 4\Omega(Z)C(Z)dZ]$ and $C(Z)$ should satisfy the condition $4\Omega(Z)C^2(Z) + \frac{dC(Z)}{dZ} = M(Z)$; thus, the soliton solution of the inhomogeneous fiber can be presented as

$$\Psi(T,Z) = \frac{4\alpha A_c \exp\theta'(T,Z)}{\sqrt{g}[1 + A_c^2\exp\varphi(T,Z)]}, \tag{13}$$

where

$$\begin{aligned}\theta'(T,Z) &= iC(Z)T^2 + \int [-G(Z)/2 - 2\Omega(Z)C(Z)]dZ \\ &\quad - 2(\alpha - i\beta)T\exp\left[\int -4\Omega(Z)C(Z)dZ\right] \\ &\quad + \int 4i\Omega(Z)(\alpha - i\beta)^2\exp\left[\int -8\Omega(Z)C(Z)dZ\right]dZ\end{aligned}$$

and

$$\begin{aligned}\varphi(T,Z) &= -4\alpha T\exp\left[\int -4\Omega(Z)C(Z)dZ\right] \\ &\quad + \int 16\alpha\beta\Omega(Z)\exp\left[\int -8\Omega(Z)C(Z)dZ\right]dZ.\end{aligned}$$

The evolution of a soliton under many different nonautonomous ways could be shown from the generalized solution. It is well known that dispersion management has been done in nonlinear fiber (for related works, see Refs. [23–26]). As usual, periodic dispersion management is performed, namely $\Omega(Z) = l\cos(\omega Z)$, if the gain term is chosen as $G(Z) = h\cos(\omega_2 Z)$ and the chirp parameter $C(Z) = C_0$. We show the dynamics of solitons in Fig. 1 from the general solution. To get solitons to evolve as in Fig. 1, nonlinearity management can be performed as $R(Z) = 2gl\cos(\omega Z)\exp[h\sin(\omega_2 Z)/\omega_2 - 4lC_0\sin(\omega Z)/\omega]$, and the inhomogeneous parameter could be designed as $M(Z) = 4C_0^2 l\cos(\omega Z)$. From the density plot Fig. 1(b), we can see that it is a "breather" soliton. It is well known that the classical soliton comes from the balance between dispersion and nonlinear effects. Then, we can know that this "breathing" feature that comes from this balance is destroyed periodically. Many different soliton shapes can be achieved through manipulation of dispersion and gain terms. However, the generalized solution cannot present explicit ways to manipulate solitons directly. For soliton application, it is desirable to know how to design related management parameters for certain properties of solitons.

From the explicit expressions of width, peak, and wave central position [see Eqs. (9), (10), and (11)], we can derive the corresponding expressions of the temporal optical soliton. Its width is

$$W(Z) = \frac{\exp[\int 4\Omega(Z)C(Z)dZ]}{2\alpha} \ln(3 + 2\sqrt{2}), \tag{14}$$

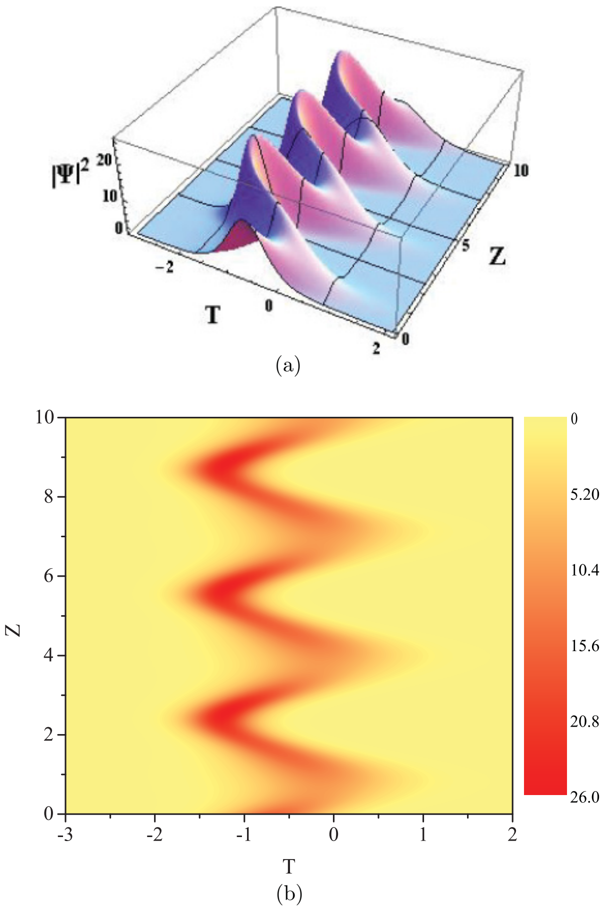

FIG. 1. (Color online) (a) The dynamics of a chirped bright nonautonomous soliton under periodic dispersion management with gain. (b) The density plot of (a) with the same parameters. It is obvious that the soliton is "breathing" because its width and peak oscillate with propagation distance. The parameters are $\alpha = 1, \beta = 0.2, C_0 = 0.1, \Omega(Z) = 2\cos(2Z), g = 0.25, A_c = 2,$ and $G(Z) = 0.5\cos(4Z)$.

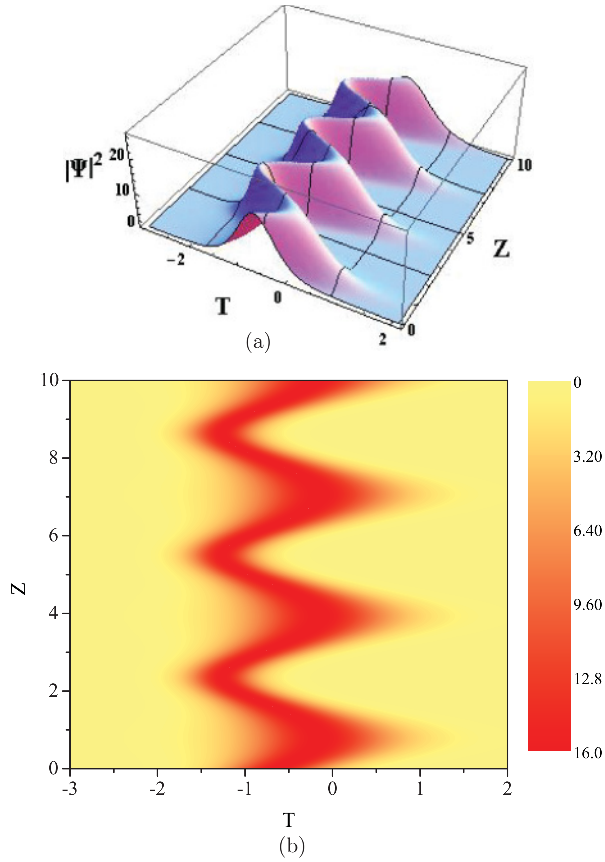

FIG. 2. (Color online) (a) The dynamics of a chirped bright nonautonomous soliton under the balance condition between periodic dispersion management, nonlinearity, and gain. (b) The density plot of (a) with the same parameters. It is shown that the soliton's peak is a constant $4\alpha^2/g$. The parameters are $\alpha = 1, \beta = 0.2, C_0 = 0.1, \Omega(Z) = 2\cos(2Z), g = 0.25, A_c = 2,$ and $G(Z) = -0.8\cos(2Z)$.

its peak is

$$|\Psi|^2_{\max} = \frac{4\alpha^2}{g}\exp\left[\int -4\Omega(Z)C(Z) - G(Z)dZ\right], \quad (15)$$

and its wave central position is

$$T_c(Z) = \frac{\ln A_c}{2\alpha}\exp\left[\int 4\Omega(Z)C(Z)dZ\right] + \frac{\int 4\beta\Omega(Z)\exp[\int -8\Omega(Z)C(Z)dZ]dZ}{\exp[\int -4\Omega(Z)C(Z)dZ]}. \quad (16)$$

From the above expressions, soliton management can be realized theoretically. The evolution of a soliton's shape (trajectory) can be controlled through designing related experimental parameters based on investigating Eqs. (14) and (15) [Eq. (16)]. It is known that the gain term only affects a soliton's peak through observing the equations. When we need a certain property of solitons, the explicit functions can give us some hints to design the modulations. This has a significant potential in the application of solitons. For example, to achieve a soliton with a stable peak, the related operation can be presented as $G(Z) = -4\Omega(Z)C(Z)$ from Eq. (15), which could be seen as the balance condition between dispersion, nonlinearity, and the gain term. In this condition, the peak of the soliton will be a constant $4\alpha^2/g$. For comparison, we show the evolution of solitons under the balance condition with the same periodic dispersion management in Fig. 2). Moreover, based on the integrable condition, related parameters can be chosen precisely. To obtain the solitons in Fig. 2, the nonlinear parameter could be $R(Z) = 0.5\cos(2Z)\exp[-0.8\sin(2Z)]$, the gain term is $G(Z) = -0.8\cos(2Z)$, and the parameter $M(Z)$ could be chosen as $M(Z) = 0.08\cos(2Z)$ when the dispersion term is $\Omega(Z) = 2\cos(2Z)$ and the chirp parameter $C(Z) = C_0(0.1)$. Therefore, this provides an appropriate way to improve the optical soliton transmission quality.

IV. DISCUSSION

We present a series of bright nonautonomous soliton solutions of the generalized NLSE. The evolution of a soliton's width, peak, and the trajectory of its wave center has been investigated analytically. As an example, we discussed the evolution of a bright optical soliton in inhomogeneities fiber. The shape of the bright soliton can be controlled by modulating the chirp parameter, dispersion, and gain term. The result is that the gain only affects the amplitude of the soliton. Moreover, the temporal "breathing" soliton under periodic dispersion management is demonstrated. A certain way to manage dispersion, nonlinearity, and the gain term is found to keep the amplitude of the nonautonomous soliton unchanged, which can be used to improve the quality of soliton transmission. We believe that these results will stimulate experiments to manipulate solitons in many fields, e.g., the Bose-Einstein condensate and spatial or temporal optical solitons in nonlinear optics.

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