Nonstationary stochastic charge fluctuations of a dust particle in plasmas

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Stochastic charge fluctuations of a dust particle that are due to discreteness of electrons and ions in plasmas can be described by a one-step process master equation [T. Matsoukas and M. Russell, J. Appl. Phys. **77**, 4285 (1995)] with no exact solution. In the present work, using the system size expansion method of Van Kampen along with the linear noise approximation, a Fokker-Planck equation with an exact Gaussian solution is developed by expanding the master equation. The Gaussian solution has time-dependent mean and variance governed by two ordinary differential equations modeling the nonstationary process of dust particle charging. The model is tested via the comparison of its results to the results obtained by solving the master equation numerically. The electron and ion currents are calculated through the orbital motion limited theory. At various times of the nonstationary process of charging, the model results are in a very good agreement with the master equation results. The deviation is more significant when the standard deviation of the charge is comparable to the mean charge in magnitude.

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I. INTRODUCTION

One reason for stochastic fluctuations of the electrical charge on a dust particle in plasmas is that ions and electrons (plasma particles) are absorbed onto the surface of the dust particle at random times. This stochastic behavior is an internal noise [1] type which occurs in systems with discrete nature. Another reason for fluctuations of charge on dust particles could be the randomness in the plasma variables, which are an external noise type. This paper is concerned with the former reason. It is emphasized that, in particular, the effect of such fluctuations is important to the dynamics of the dust particle, single or in a group [2–10].

Stochastic charge fluctuations of dust particles in plasmas have been studied through various approaches in the past two decades. The first study is due to Cui and Goree [11], who develop a method in which, first, the time interval between absorption of the plasma particles (either ions or electrons) varies randomly following an exponential distribution and, second, whether the arriving plasma particle at every time step is an electron or ion is also determined through random criteria. Although not mentioned at the time, this method in fact solves a master equation, later proposed by Matsoukas and Russell [12], to model charging with a one-step stochastic process [1]. The method of Cui and Goree [11] could be considered as a special case of a Monte Carlo method developed by Gillespie [13] to solve the master equations. Matsoukas and Russell [12] derive analytical solutions for the average and variance of the charge at the stationary state through a Fokker-Planck equation that they obtained by expanding the master equation. In a later work, Matsoukas and Russell [14] develop a linear Fokker-Plank equation [1] through the linearization of the currents around the average charge at the stationary state. Moreover, they derive an analytical solution for the distribution function of the charge at the nonstationary state for cases in which the initial mean charge is within the linear range of the currents close to the equilibrium charge (stationary mean charge). Khrapak *et al.* [15] develop a model utilizing a Langevin equation to describe the charge fluctuations at stationary states. The Langevin equation can be considered statistically equivalent to the Fokker-Planck equation proposed by Matsoukas and Russell [14]. All of the works mentioned above except that of Khrapak *et al.* [15] consider the charging mechanism through which ions and electrons are collected on the dust particle from a plasma. Khrapak *et al.* [15] also study charge fluctuations with charge mechanisms of thermionic emission and UV irradiation.

The main objective of the present study is to develop a model for the description of the stochastic charge fluctuations of the dust particle, which is valid at both stationary and nonstationary states, and it is applicable to an arbitrary initial mean charge. Section II is concerned with the development of a Fokker-Planck equation from the master equation through the system-size expansion method of Van Kampen [1], along with the linear noise approximation for dust particle charging. In Sec. III results obtained by the model for a charging mechanism based on the orbital-motion-limited (OML) theory are discussed and they are compared to the results obtained by solving the master equation. Finally, concluding remarks are made in Sec. IV.

II. MODEL DEVELOPMENT

The charging process of the dust particle is assumed to be Markovian; hence, the master equation [1]

$$\frac{dP(N,t)}{dt} = \int [W(N|N')P(N',t) - W(N'|N)P(N,t)]dN' \qquad (1)$$

can be utilized to develop a model for charging. In Eq. (1), $P(N,t)$ is the probability density function of elementary charge on the dust particle at time t, and $W(N|N')$ is the transition probability per unit time where with N' elementary charges, a jump occurs to N elementary charges. A negative N or N' means that electrons are carried by the dust particle

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while a positive one indicates that ions are carried by the dust particle.

Furthermore, it is assumed that only one plasma particle at a time is absorbed onto the dust particle. This assumption leads to the modeling of charging with a one-step (birth-death) process [1,12], which is a subclass of the Markov processes. With known electron and ion currents to the dust particle, it is possible to construct the transition probability per unit time,

$$W(N|N') = I_e(N')\,\delta(N - N' + 1) + I_i(N')\,\delta(N - N' - 1), \tag{2}$$

where $I_i(N')$ and $I_e(N')$ denote the ion and electron currents, respectively. The delta functions seen in (2) are to render jumps of only ± 1 (one-step process). Via the substitution of Eq. (2) in the master equation (1), the master equation for the one-step process of dust particle charging can be derived:

$$\frac{dP(N,t)}{dt} = I_e(N+1)P(N+1,t) + I_i(N-1)P(N-1,t) - [I_i(N) + I_e(N)]P(N,t). \tag{3}$$

This equation, given for the first time by Matsoukas and Russell [12], cannot be exactly solved except for a special case of linear $I_i(N)$ and $I_e(N)$ functions [1].

At this stage of the model development, the system-size expansion method of Van Kampen [1] is utilized. This method, accompanied by the linear noise approximation, is a systematic method to approximate the master equation of a system of an internal noise type with a linear Fokker-Planck equation. A Gaussian function with a time-dependent mean and variance can be obtained as an exact solution to this linear equation.

The first step in employing the system-size expansion method is to express the transition probability per unit time in Eq. (1) as a function of the state of the dust particle in terms of the charge normalized by the system size Ω, while expressing it as a function of the size of the jumps in terms of the charge. That is to say,

$$W(N|N') = W(N'; N - N') = W(\Omega n'; r) = \Phi(n'; r), \tag{4}$$

where $r = N - N'$ is the jump size and $n' = N'/\Omega$. It is noted that N' is an extensive variable denoting the current state of the dust particle whereas n' is its associate intensive variable independent from the system size Ω. This means that the transition probability per unit time is expressed through a function $\Phi(n'; r)$ with the starting point described via the intensive variable n', while the jump is described via the extensive variable r.

Now a change of variable is performed,

$$N = \Omega\phi(t) + \Omega^{1/2}\xi, \tag{5}$$

where $\phi(t)$ is a function of time to be determined. This equation is of key importance in which the macroscopic time change of the system is described through $\phi(t)$, whereas the microscopic behavior is described through ξ. It is the ansatz of Van Kampen [1], which in our problem models the macroscopic behavior of the dust particle charge, i.e., the location of the charge peak in the phase space, through the first term on the right-hand side of Eq. (5) and the scale of charge fluctuations through the second term.

Substituting for N from (5) in $P(N,t)$ leads to

$$P(N,t) = P(\Omega\phi(t) + \Omega^{1/2}\xi, t) = \Pi(\xi, t), \tag{6}$$

which, in turn, is substituted with P in the master equation (1) while Eq. (4) is used for the transition probabilities. The resulting integro-differential equation terms are expanded in powers of $\Omega^{-1/2}$. Setting the coefficients of the two lowest powers of $\Omega^{-1/2}$ to zero leads to [16]

$$\frac{d\phi(t)}{dt} = \widetilde{\alpha}_1(\phi), \tag{7}$$

$$\frac{\partial \Pi}{\partial t} = -\widetilde{\alpha}_1'(\phi)\frac{\partial \xi \Pi}{\partial \xi} + \frac{1}{2}\widetilde{\alpha}_2(\phi)\frac{\partial^2 \Pi}{\partial \xi^2}, \tag{8}$$

where $\widetilde{\alpha}_k(n) = \Omega^{-1}\alpha_k(n)$, where

$$\alpha_k(n) = \int r^k \Phi(n; r)\,dr, \tag{9}$$

denotes the moments of jumps.

Equation (8) is a linear Fokker-Planck equation, which is obtained through the system-size expansion method followed by the linear noise approximation. It has an analytical solution in the form of a Gaussian function, whose average and variance are governed by [1]

$$\frac{d}{dt}\langle \xi \rangle = \widetilde{\alpha}_1'(\phi)\langle \xi \rangle, \tag{10}$$

$$\frac{d}{dt}\langle\langle \xi^2 \rangle\rangle = 2\widetilde{\alpha}_1'(\phi)\langle\langle \xi^2 \rangle\rangle + \widetilde{\alpha}_2(\phi). \tag{11}$$

where $\langle\langle \xi^2 \rangle\rangle = \langle (\xi - \langle \xi \rangle)^2 \rangle$. For a specified initial condition in the form of $P(N,0) = \delta(N - N_0)$, which means that, initially, the mean charge of the dust particle is known with no fluctuations, it is concluded that $\langle \xi(0) \rangle = \langle\langle \xi^2(0) \rangle\rangle = 0$, leading to $\langle \xi(t) \rangle = 0$ according to Eq. (10). Thus, using Eqs. (5) and (7), it can be said that

$$\frac{d\langle n(t) \rangle}{dt} = \widetilde{\alpha}_1(\langle n(t) \rangle), \tag{12}$$

which represents the macroscopic equation for the dust particle charge.

Now using Eq. (2), valid for the one-step process, one may express Φ in Eq. (4) as

$$\Phi(n; r) = I_e(\Omega n)\delta(r + 1) + I_i(\Omega n)\delta(r - 1), \tag{13}$$

and the moments in Eq. (9) as

$$\alpha_k(n) = I_i(\Omega n) + (-1)^k I_e(\Omega n), \tag{14}$$

From Eqs. (5), (12), and (11), the equations of charge mean and variance are obtained as

$$\frac{d\langle N \rangle}{dt} = I_{i-e}(\langle N \rangle), \tag{15}$$

$$\frac{d\langle\langle N^2 \rangle\rangle}{dt} = 2I_{i-e}'(\langle N \rangle)\langle\langle N^2 \rangle\rangle + I_{i+e}(\langle N \rangle). \tag{16}$$

where $I_{i\pm e}(N) = I_i(N) \pm I_e(N)$. Equations (15) and (16) are in a closed form that statistically describe charge fluctuations of the dust particle when the charging process is nonstationary. The initial conditions required to solve this set of equations are a known initial charge $\langle N(0) \rangle = \langle N \rangle_0$ with a zero variance $\langle\langle N^2(0) \rangle\rangle = 0$.

The stationary (equilibrium) values of the charge mean and variance can be readily obtained via Eqs. (15) and (16). At the stationary state, the charge statistics do not change with time, and therefore the mean and variance at this state are the values rendering the right-hand sides of Eqs. (15) and (16) vanish, i.e.,

$$I_{i-e}(\langle N\rangle_s) = 0, \tag{17}$$

$$\langle\langle N^2\rangle\rangle_s = -\frac{1}{2}\frac{I_{i+e}(\langle N\rangle_s)}{I'_{i-e}(\langle N\rangle_s)}, \tag{18}$$

where the subscript s indicates the stationary value. Equations (17) and (18) are the same given by Matsoukas and Russell [12], which are obtained through a stationary-state analysis.

The substitution of $\xi = \Omega^{-1/2}(N - \langle N\rangle)$ in Eq. (8) leads to

$$\frac{\partial P(N,t)}{\partial t} = -\frac{\partial}{\partial N}[I_{i-e}(\langle N\rangle) + I'_{i-e}(\langle N\rangle)(N - \langle N\rangle)]P + \frac{1}{2}I_{i+e}(\langle N\rangle)\frac{\partial^2 P}{\partial N^2}, \tag{19}$$

where $\langle N\rangle$ is governed by Eq. (15). It is noted that the use of the stationary mean charge $\langle N\rangle_s$ instead of $\langle N\rangle$ in Eq. (19) leads to the Fokker-Planck equation given by Matsoukas and Russell [14] as the first term in the brackets of the first term on the right-hand side vanishes at the stationary state.

It is also possible to develop a Langevin equation for the dust particle charge using Eqs. (8) or (19). According to Ito's formula [16], Eq. (19) is statistically equivalent to

$$dN = [I_{i-e}(\langle N\rangle) + I'_{i-e}(\langle N\rangle)(N - \langle N\rangle)]dt + [I_{i+e}(\langle N\rangle)]^{1/2}dW, \tag{20}$$

where W is the Wiener process. If $\langle N\rangle$ is substituted by $\langle N\rangle_s$, the first term in the brackets of the first term on the right-hand side vanishes and the Langevin equations proposed by Matsoukas and Russell [14] and Khrapak *et al.* [15] for charging at the stationary state are obtained.

III. RESULTS

In this study it is assumed that the dust particle is spherical with a radius of R. Moreover, the OML theory [17,18] is utilized to model the charging mechanism. The OML theory is valid when the Debye shielding length is much larger than the particle radius, while it is much smaller than the mean-free-path length. In the OML theory the current of electrons (ions) to the dust particles is [12]

$$I_{e(i)}(N) = \Gamma J_{e(i)}(\Omega^{-1}N), \tag{21}$$

where

$$J_e(n) = \begin{cases} \exp(n), & n \leqslant 0, \\ 1 + n, & n > 0, \end{cases} \tag{22}$$

$$J_i(n) = \begin{cases} \gamma(1 - \widehat{T}n), & n \leqslant 0, \\ \gamma\exp(-\widehat{T}n), & n > 0, \end{cases} \tag{23}$$

where

$$\gamma = \frac{1}{\widehat{n}}\left(\frac{\widehat{M}}{\widehat{T}}\right)^{1/2}, \tag{24}$$

where nondimensional parameters of the plasma are defined as $\widehat{n} = n_e/n_i$, $\widehat{T} = T_e/T_i$, and $\widehat{M} = M_e/M_i$, where $n_{e(i)}$ is the concentration of electrons (ions), $T_{e(i)}$ is the temperature of the electrons (ions), and $M_{e(i)}$ is the mass of an electron (ion). In Eq. (21)

$$\Omega = \frac{4\pi\epsilon_0 R k_B T_e}{e^2}, \tag{25}$$

is the size of the system and

$$\Gamma = n_e\pi R^2\left(\frac{8k_B T_e}{\pi M_e}\right)^{1/2}, \tag{26}$$

where e is the electrical charge of an electron, k_B is the Boltzmann constant, and ϵ_0 is the vacuum primitivity.

Equations (12) and (11) can be expressed in the following forms:

$$\frac{d\langle n\rangle}{d\tau} = J_{i-e}(\langle n\rangle), \tag{27}$$

$$\frac{d\langle\langle \xi^2\rangle\rangle}{d\tau} = 2J'_{i-e}(\langle n\rangle)\langle\langle \xi^2\rangle\rangle + J_{i+e}(\langle n\rangle), \tag{28}$$

where $J_{i\pm e}(n) = J_i(n) \pm J_e(n)$, $\langle n\rangle = \Omega^{-1}\langle N\rangle$, $\langle\langle \xi^2\rangle\rangle = \Omega^{-1}\langle\langle N^2\rangle\rangle = \Omega\langle\langle n^2\rangle\rangle$, and

$$\tau = \Omega^{-1}\Gamma t. \tag{29}$$

In the present study, Eqs. (27) and (28) are numerically solved with LSODA, a variant version of the Livermore solver for ordinary differential equations (LSODE) package [19,20], which is utilized by the NDSolve function of MATHEMATICA, a product of Wolfram Research, Inc.

Figure 1 displays the difference of the ion and electron currents $J_{i-e}(n)$, which dictates how the mean charge evolves in time according to Eq. (27). The nonlinearity of $J_{i-e}(n)$ is evident, as can be seen in this figure, although it linearly varies according to $J_{i-e}(n) = \gamma(1 - \widehat{T}n)$ at the asymptotic limit of $n \to -\infty$ and according to $J_{i-e}(n) = -(1 + n)$ at the asymptotic limit of $n \to \infty$. These asymptotic forms justify the observation made in Fig. 1 in which the electron temperature does not influence $J_{i-e}(n)$ at larger values of n,

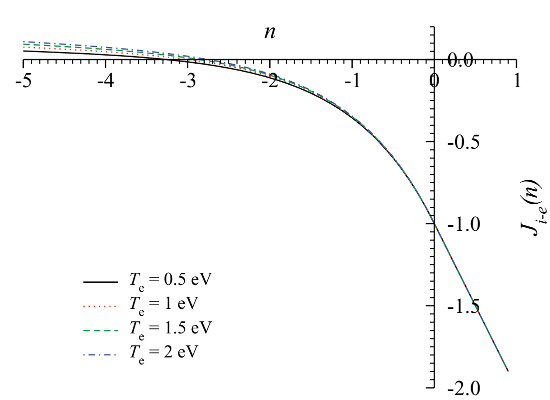

FIG. 1. (Color online) The difference of ion and electron currents $J_{i-e}(n) = J_i(n) - J_e(n)$: $T_i = 600$ K and $\widehat{n} = 1$.

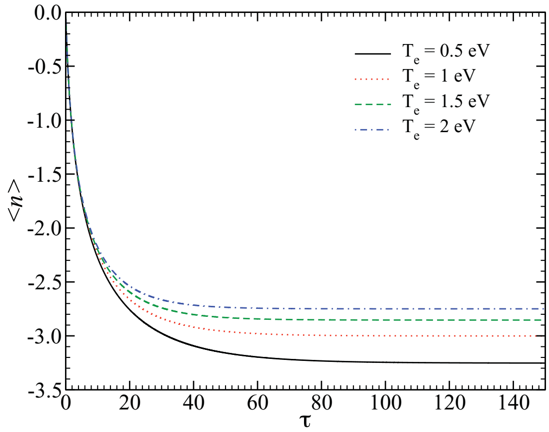

FIG. 2. (Color online) Time evolution of the charge mean: $T_i = 600$ K and $\widehat{n} = 1$.

whereas it influences $J_{i-e}(n)$ at smaller values of n. The location of the equilibrium points are stable and negative, which is evident as the curves cross the n axis at negative locations. $J_{i-e}(n)$ approximately varies according to $\gamma(1 - \widehat{T}n)$ at the region close to the equilibrium point. This form of functionality of $J_{i-e}(n)$ is the reason that the equilibrium point is further shifted toward $n = 0$ with an increase of the electron temperature, as seen in Fig. 1.

The time progress of $\langle n \rangle$ is shown in Fig. 2 for different electron temperatures. The initial condition is $\langle n \rangle_0 = 0$ since at this study, it is assumed that the dust particle is initially uncharged. As seen in this figure, no significant difference is observed between cases with different electron temperatures for times up to approximately $\tau = 3$, after which the difference between cases starts becoming significant. With the increase of τ, $\langle n \rangle$ increases until it asymptotically reaches the equilibrium mean charge value, which is $\langle n \rangle_s$, satisfying $J_{i-e}(\langle n \rangle_s) = 0$.

The time progress of the charge variance $\langle\langle n^2 \rangle\rangle$ is displayed in Fig. 3. The initial condition for the variance is $\langle\langle n^2 \rangle\rangle_0 = 0$ as no charge fluctuations initially exist. For a typical case of T_e seen in this figure, with the increase of τ, $\langle\langle n^2 \rangle\rangle$ increases until it asymptotically reaches a stationary-state charge variance. It is seen that $\langle\langle n^2 \rangle\rangle$ initially undergoes a sharp increase until $\tau = 2$, after which its rate of change decreases until it flattens.

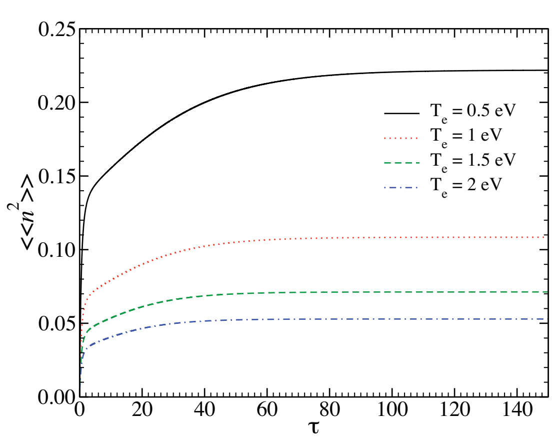

FIG. 3. (Color online) Time evolution of the charge variance: $T_i = 600$ K, $\widehat{n} = 1$, and $R = 10$ nm.

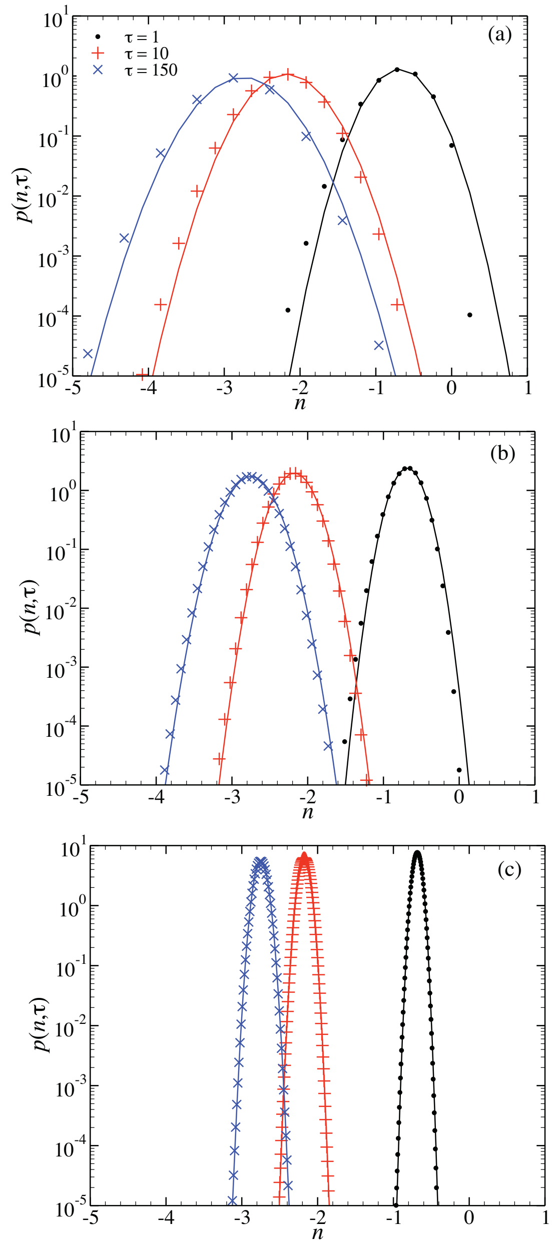

FIG. 4. (Color online) Charge distribution obtained by the linear noise approximation (continuous lines) and the master equation (discrete points) for $T_e = 2$ eV, $T_i = 600$ K, and $\widehat{n} = 1$: (a) $R = 3$ nm; (b) $R = 10$ nm; and (c) $R = 100$ nm.

It is observed in this figure that at a given τ of the nonstationary state, the larger is the electron temperature, the smaller is the variance.

To validate the developed model based on the linear noise approximation, the probability density functions obtained by this model are compared to those obtained by solving the master equation (3). As mentioned earlier, it is not possible to solve this equation exactly as the currents are nonlinear functions of the dust particle charge, so it is solved numerically. Equation (3) represents coupled differential equations with unknowns $\ldots, P(N-1,t), P(N,t), P(N+1,t), \ldots$ that can be solved progressively in time for every $P(N,t)$ in a domain $[N_{\min}, N_{\max}]$, where boundary values $N_{\min}$ and $N_{\max}$ are

TABLE I. Mean, standard deviation $\sigma = \langle\langle n^2\rangle\rangle^{1/2}$, and skewness $S = \langle\langle n^3\rangle\rangle/\sigma^3$ of dust particle charge obtained by the master equation (ME) and the linear noise approximate (LNA) models at different τ's; $T_e = 2$ eV, $T_i = 600$ K, $\widehat{n} = 1$, and $R = 3$ nm.

τ	Model	$\langle n\rangle$	σ	S
1	ME	−0.700	0.306	−0.249
	LNA	−0.685	0.300	0
10	ME	−2.23	0.371	−0.061
	LNA	−2.18	0.368	0
150	ME	−2.81	0.423	−0.039
	LNA	−2.75	0.420	0

specified in such a way that they are sufficiently away from the initial charge and the stationary charge values. The coupled differential equations for N's in Eq. (3) are also solved with the LSODA method [19,20] by utilizing the NDSolve function of MATHEMATICA.

Figure 4 shows $p(n,\tau) = \Omega P(N,t)$, where t and τ correlation is given in Eq. (29) for various cases. The initial condition is $p(n,t) = \delta(n)$. It can be seen that the deviation between the linear noise approximation and the master equation results is more significant for the dust particle with $R = 3$ nm. In order to make a better comparison between the master equation and linear noise approximation results, various statistics obtained from $p(n,\tau)$ are tabulated for this dust particle radius in Table I. At all three τ values, the difference between the two models is below 3% for the mean and the difference is much less for the standard deviation. The negative skewness of the master equation, seen in this table, is more noticeable in Fig. 4(a). While the magnitude of the skewness decreases with an increase of τ, it is significantly larger at $\tau = 1$ than that at other τ values.

IV. CONCLUDING REMARKS

In this study, model equations are developed to describe stochastic charge fluctuations of a dust particle suspended in a plasma at nonstationary states. Such fluctuations could be important to scenarios in which the initial transient behavior of the charge statistics is of interest and/or electron and ion currents are time dependent via time-varying properties of the plasma.

The proposed models here are based on the system-size expansion of Van Kampen [1], which is applied on the master equation and is valid for the cases where the size of fluctuations is relatively small compared to the charge mean. Furthermore, the probability density function of the dust particle charge is determined to be Gaussian after the linear noise approximation is made and a linear Fokker-Planck equation is obtained. In order to test the models its predicted results have been compared versus the results that have been obtained by directly solving the master equation. It is observed that, except for a small dust particle at the early transient stage, with a small particle mean there is excellent agreement between the Fokker-Planck equation and the master equation results. The discrepancy between the results obtained for the small dust particle is observable through a relatively larger skewness calculated from the master equation. This discrepancy could be due to the fact that the charge standard deviation and mean are comparable in magnitude, which is in contradiction to the large system-size assumption made in the system-size expansion method.

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