Critical behavior and correlations on scale-free small-world networks: Application to network design

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We analyze critical phenomena on networks generated as the union of hidden variable models (networks with any desired degree sequence) with arbitrary graphs. The resulting networks are general small worlds similar to those à la Watts and Strogatz, but with a heterogeneous degree distribution. We prove that the critical behavior (thermal or percolative) remains completely unchanged by the presence of finite loops (or finite clustering). Then, we show that, in large but finite networks, correlations of two given spins may be strong, i.e., approximately power-law-like, at any temperature. Quite interestingly, if γ is the exponent for the power-law distribution of the vertex degree, for $\gamma \leqslant 3$ and with or without short-range couplings, such strong correlations persist even in the thermodynamic limit, contradicting the common opinion that, in mean-field models, correlations always disappear in this limit. Finally, we provide the optimal choice of rewiring under which percolation phenomena in the rewired network are best performed, a natural criterion to reach best communication features, at least in noncongested regimes.

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I. INTRODUCTION

In the last decade, it has been recognized that at the base of many complex systems, as diverse as those observed in nature and in technological and social sciences, there is an ubiquitous presence of networks having certain universal topological features upon which the functionality of the system largely depends [1]. Essentially, there are two basic topological features in these complex networks: scale free and small world. The former refers to the fact that the distribution of the links among the nodes is strongly heterogeneous, in particular, many networks have a power-law distribution for the vertex degree $\mathcal{P}(k) \sim k^{-\gamma}$; the latter refers to the fact that two randomly chosen nodes are at a distance that, for $\gamma > 3$, scales with the system size N as slowly as $\log(N)/\log(b)$, with b being the mean branching of the network, while for $\gamma \leqslant 3$ (where b diverges), the average distance scales as $\log[\log(N)]$ or even slower and the network is called ultrasmall [2]. Network models have then been profusely studied over the years and many fundamental results are by now well established and widespread [3,4]. The main assumption under these studies has been the treelike hypothesis, thanks to which the generating function technique and the Bethe-Peierls (BP) method [5] can be applied to get exact solutions for the percolative and thermal (at least in the ferromagnetic case) properties as well as the communication features of the system. However, the treelike assumption is almost never satisfied in real-world networks. For example, networks of friends, networks of neurons, the WWW, and the Internet are just a few examples in which the average clustering coefficient C [1,4] is finite. More precisely, whereas networks having a hierarchical structure share a k degree-dependent clustering coefficient of the form $C(k) \sim k^{-\alpha}$, with $\alpha \sim 1$, so that the most connected (and most important) nodes are not clustered, there are other networks having $C(k) \sim O(1)$ for almost any k and for which clustering is important for all nodes. The former class includes, e.g., some social networks, language networks, the WWW, and the Internet at the autonomous system level, whereas the latter class includes the Internet at the router level, the power grid, but also the brain. As discussed in Ref. [6], the reason for this difference is related to the fact that, in the second class, wiring is expensive (economically or biologically) and the network, rather than hierarchically organized, is geographically organized. Finally, we recall that even in pure scale-free networks characterized by an exponent $\gamma \leqslant 3$, the treelike assumption is not true; such networks in fact contain many large cliques [7].

In the last few years, there has been important progress for the modeling of networks with loops [8–12]; however, such progress was essentially confined to the cases in which the random graph can be seen as a treelike hypergraph, or with a weak transitivity (i.e., with a small overlap between clusters). It is then of fundamental importance to understand, in general, as to what the role is of the loops in complex networks from the point of view of collective behavior. In the presence of loops, how do the critical surface and the correlation functions change? Are the analytical results accumulated over 10 years of research in complex networks robust with respect to the presence of loops? And, if yes, to what extent? We point out that the crucial question concerns the loops of finite length. In fact, in classical random graphs ($\gamma = \infty$) and in complex networks ($2 < \gamma < \infty$), there are no finite loops (at least for $\gamma > 3$; for a detailed discussion of the case $\gamma \geqslant 3$, see the next section) since the length of the loops scales as $\log(N)$ and, as a consequence, one can say that the treelike approximation in these models becomes exact for $N \to \infty$. But, in networks in which there are loops of finite length (for any N), due to the fact that the correlation length in all these models remains finite, we can not neglect the short-loops even near the critical point and, as a consequence, the (exact) solution to these issues is nontrivial.

In this paper, we address the above questions via the introduction of heterogeneous small-world networks, a natural generalization of the "classical" small-world networks [13] that were introduced as intermediate systems lying between

loopylike (e.g., finite-dimensional lattices) and treelike networks. In the classical ($\gamma = \infty$) small-world networks, we have a homogeneous distribution of links among the sites, but also a finite clustering coefficient, so that treelike based techniques such as the BP can not be used to solve, e.g., an Ising model defined on them. By using a completely different approach, it is, however, possible to solve exactly the homogeneous small-world models, at least in the paramagnetic phase (P) getting their exact critical surface and behavior [14]. If J_0 is the coupling associated to a given graph $(\mathcal{L}_0, \Gamma_0)$, which in particular may have short-range links and short loops of any kind, and J is the coupling associated to a number $Nc/2$ of additional uniformly spread long-range links (so that they alone would form a classical random graph [15]), then, for any $c > 0$, the mean-field equation for these models is given by

$$m = m_0(\beta J_0; ctm + \beta h), \quad t \stackrel{def}{=} \tanh(\beta J), \tag{1}$$

where $m_0(\beta J_0; \beta h)$ is defined as the average magnetization of the model in the absence of the long-range connections, with a short-range coupling J_0 and in the presence of a generic external field h at temperature $T = 1/\beta$. Equation (1) is a natural generalization of the celebrated Curie-Weiss mean-field equation $m = \tanh(\beta J m + \beta h)$, which is valid for $J_0 = 0$ and $c = N$. It is easy, however, to check that, for any $J_0 \geqslant 0$, the critical behavior of Eq. (1) is classical, regardless of the local topology and clustering coefficient of the network (see also Ref. [16]).

In this paper, we face the natural extension of Eq. (1) toward a large class of heterogeneous small-world networks generated by using hidden variables [17–20], which have the slight inconvenience that the resulting network has some small degree-degree correlation [21], but also the great advantage that the model is analytically solvable even in the presence of loops (at least within our effective field theory). After deriving the equation for the order parameter and the critical surface (thermal or bond percolative), we analyze the connected correlation functions in general and we show that, in these models, even for $J_0 = 0$, there are finite size corrections as strong as $1/N^{\delta}$, with $\delta = (\gamma - 2)/(\gamma - 1)$ for $\gamma > 3$ and $\delta = 0$ for $\gamma \leqslant 3$, contradicting the common opinion that, in mean-field models, the connected correlation functions always disappear in the thermodynamic limit. Then, we prove that the critical behavior (thermal or bond percolative) on these networks is never affected by the presence of a local non-treelike structure, provided that the connectivity associated to such loopy structures is nonheterogeneous. This latter result already has been presented and discussed in a previous paper [22], but limiting the proof to an infinitesimal coupling J_0, while postponing to this paper the general proof as well as the derivation of the mean-field equation. Meanwhile, in [23], we find that a similar robustness theorem has been proved for both static and growing networks embedded in a metric space when $\gamma \leqslant 3$.

In Tables I and II, we summarize the state of the art reached about the analytical behavior of the Ising model built on classical and complex random graphs. We stress that we mention only the cases and the models where exact analytical calculations have been possible without any pretension to be exhaustive [in particular, we do not mention here the hierarchical models (random or not), where some exact analytical results are also possible].

Finally, as an application, given the desired degree sequence and the graph $(\mathcal{L}_0, \Gamma_0)$, we find that the equation for the critical surface leads to an optimization problem consisting in finding the rewiring of the additional links that provide the minimal percolating point, a criterion that amounts to finding the rewiring that provides the best communication performance at the minimal cost in the absence of congestion. This optimization problem, in general, is an NP-hard problem; however, we provide heuristic solutions, the effectiveness of which depends on how much the network is structured in communities (if any), and we show that the use of the formula for the critical surface is always exponentially (in N) convenient with respect to a direct inspection of the network, even in the worst case scenario in which there is no community structure at all.

II. RANDOM ISING MODELS BUILT ON HETEROGENEOUS SMALL-WORLD NETWORKS

A. The model

The family of models that we consider is built as follows. Let $(\mathcal{L}_0, \Gamma_0)$ be any graph, $\mathcal{L}_0$ and Γ_0 being the set of vertices $i = 1, \ldots, N$ and links (i, j), $i < j$, respectively. Let us consider the Ising model defined on the graph $(\mathcal{L}_0, \Gamma_0)$ with a fixed coupling J_0 and in the presence of an arbitrary external field $\{h_i\}$:

$$H_0 = -J_0 \sum_{(i,j)\in\Gamma_0} \sigma_i \sigma_j - \sum_i h_i \sigma_i. \tag{2}$$

We will call this *the pure model*. Let us now consider the model obtained by removing randomly some links of the graph $(\mathcal{L}_0, \Gamma_0)$ and by adding new links as follows. Let us indicate with $c_{0;i,j} = 0, 1$ the adjacency matrix of the new graph in which some links of Γ_0 have been removed. Given an ensemble $\mathcal{C}$ of random graphs $\boldsymbol{c}$, $\boldsymbol{c} \in \mathcal{C}$, the links of which are determined by the adjacency matrix elements $c_{i,j} = 0, 1$, we define our *heterogeneous small-world model* through the following Hamiltonian:

$$H_{\boldsymbol{c}_0, \boldsymbol{c}, \boldsymbol{J}_0, \boldsymbol{J}} \stackrel{def}{=} -\sum_{(i,j)\in\Gamma_0} c_{0;i,j} J_{0;i,j} \sigma_i \sigma_j - h \sum_i \sigma_i - \sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j. \tag{3}$$

The variables $c_{i,j}$ specify whether a "long-range" link between the sites i and j is present ($c_{i,j} = 1$) or absent ($c_{i,j} = 0$), whereas the variables $c_{0;i,j}$ specify whether a link $(i, j) \in \Gamma_0$ has been removed ($c_{0;i,j} = 0$) or not ($c_{0;i,j} = 1$). The $J_{i,j}$'s are the random couplings of the given link (i, j) and similarly for the $J_{0;i,j}$'s for the links of Γ_0. All the above random variables are assumed to be independent. For the $J_{0;i,j}$'s and the $J_{i,j}$'s, we will not assume any particular distribution, while for the $c_{0;i,j}$'s and the $c_{i,j}$'s, we assume, respectively, the probabilities

$$p_0(c_{0;i,j}) = (1 - p)\delta_{c_{0;i,j},1} + p\delta_{c_{0;i,j},0}, \tag{4}$$

$$p_{ij}(c_{ij}) = f(p_i, p_j)\delta_{c_{ij},1} + [1 - f(p_i, p_j)]\delta_{c_{ij},0}, \tag{5}$$

TABLE I. Critical behavior of the unweighted magnetization $\overline{m} = \sum_i \overline{\langle \sigma_i \rangle}/N$ ($\tau \equiv 1 - T/T_c$), the susceptibility χ, and the equation for the critical temperature of the Ising model built on several network models: the classical random graph; the configuration model [i.e., the maximally random graph under the constraint that the degree distribution is a given one, $P(k) \sim k^{-\gamma}$]; the static model [a hidden variable model with weights $p_i \propto i^{-\mu}$, where $\mu \in [0,1)$ is such that $P(k) \sim k^{-\gamma}$]; the classical small-world network built by overlapping the classical random graph with additional links associated to an arbitrary graph $(\mathcal{L}_0, \Gamma_0)$; the heterogeneous small-world networks [[22] (J_0 infinitesimal) and this paper (J_0 arbitrary)] built by overlapping hidden variable models with additional links associated to an arbitrary graph $(\mathcal{L}_0, \Gamma_0)$; the spatial network model for $\gamma < 3$ (where self-similarity applies). We use the notation $c = \langle k \rangle_P$.

Ising on the "classical random graph" (Ref. [15])	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
$P(k)$ Poissonian {no finite loops; shortest loops scale as $O[\log(N)]$}	$\propto \tau^{1/2}$	$\propto \tau^{-1}$	$ct_c = 1$
Ising on the "configuration network" (Ref. [24])	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
$\gamma > 5$, $\langle k^4 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto \tau^{1/2}$	$\propto \tau^{-1}$	$t_c = \frac{\langle k \rangle_P}{\langle k^2 \rangle_P - \langle k \rangle_P}$
$\gamma = 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto 1/\ln \tau^{-1}$		
$3 < \gamma < 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto \tau^{1/(\gamma-3)}$		
$\gamma = 3$, $\langle k^2 \rangle_P = \infty$ (finite loops)	$\propto e^{-2T/\langle k \rangle_P}$	$\propto T^{-1}$	$\beta_c \to 0$
$2 < \gamma < 3$, $\langle k^2 \rangle_P = \infty$ (finite loops)	$\propto T^{-1/(3-\gamma)}$		
Ising on the "static network" (hidden variables) (Ref. [32])	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
$\gamma > 5$, $\langle k^4 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto \tau^{1/2}$	$\propto \tau^{-1}$	$ct_c N \sum_i p_i^2 = 1$
$\gamma = 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto 1/\ln \tau^{-1}$		
$3 < \gamma < 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ {shortest loops scale as $O[\log(N)]$}	$\propto \tau^{1/(\gamma-3)}$		
$\gamma = 3$, $\langle k^2 \rangle_P = \infty$ (finite loops)	$\propto e^{-2T/\langle k \rangle_P}$	$\propto T^{-1}$	$\beta_c \to 0$
$2 < \gamma < 3$, $\langle k^2 \rangle_P = \infty$ (finite loops)	$\propto T^{-1/(3-\gamma)}$		
Ising on the "classical SW networks" (Refs. [14,16])	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
$P(k)$ Poissonian + additional arbitrary links (loops of any length)	$\propto \tau^{1/2}$	$\propto \tau^{-1}$	$ct_c \tilde{\chi}_0(\beta_c J_0; 0) = 1$
Ising on the "heterogeneous SW networks" (Ref. [22] and this paper)	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
$\gamma > 5$, $\langle k^4 \rangle_P < \infty$ (arbitrary loops of any length)	$\propto \tau^{1/2}$	$\propto \tau^{-1}$	$ct_c N \sum_{i,j} \tilde{\chi}_{0;i,j} p_i p_j = 1$
$\gamma = 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ (arbitrary loops of any length)	$\propto 1/\ln \tau^{-1}$		
$3 < \gamma < 5$, $\langle k^4 \rangle_P = \infty$, $\langle k^2 \rangle_P < \infty$ (arbitrary loops of any length)	$\propto \tau^{1/(\gamma-3)}$		
$\gamma = 3$, $\langle k^2 \rangle_P = \infty$ (arbitrary loops of any length)	$\propto e^{-2T/\langle k \rangle_P}$	$\propto T^{-1}$	$\beta_c \to 0$
$2 < \gamma < 3$, $\langle k^2 \rangle_P = \infty$ (arbitrary loops of any length)	$\propto T^{-1/(3-\gamma)}$		
Ising on "static and growing spatial networks" (Ref. [23])	$\bar{m}$	χ	Equation for $t_c = \tanh(\beta_c J)$
Self-similarity $2 < \gamma < 3$, $\langle k^2 \rangle_P = \infty$ (loops in a metric space)	$\propto T^{-1/(3-\gamma)}$	$\propto T^{-1}$	$\beta_c \to 0$

where $p \in [0,1]$, and the $\{p_i\}$ are a set of hidden variables[1] [17–20], each proportional to the average degrees $\{\bar{k}_i\}$ of the graph $\boldsymbol{c}$ of the nodes $i = 1, \ldots, N$ [i.e., the degrees in the absence of the graph $(\mathcal{L}_0, \Gamma_0)$]. Usually, the hidden variables depend on one (or more) continuous parameters $\mu \in \mathcal{I}$, and on N. Given the mean degree $c > 0$ (so that, on average, there are in total $cN/2$ bonds) of the graph $\boldsymbol{c}$, we assume that, for a continuous subset $\mathcal{J} \subset \mathcal{I}$, asymptotically in N, we can write

$$f(p_i, p_j) = cNp_i p_j, \tag{6}$$

where

$$c \stackrel{def}{=} \sum_i \frac{\bar{k}_i}{N}. \tag{7}$$

For the validity of the results that we present in the next section, we require the number of links (i,j) for which Eq. (6) is not true to be less than $O(N^\alpha)$, as long as $\alpha < 1$. We will prove in fact that, in the thermodynamic limit, the free energy of the model (see below) is not affected by the presence of the $O(N^\alpha)$ links for which Eq. (6) is not true if $\alpha < 1$. As a probability, Eq. (6) for $f(p_i, p_j)$ will be manifestly violated in $\mathcal{I} \setminus \mathcal{J}$ whenever $cNp_i p_j > 1$. Note that, if $p_i \neq 0$ for any given N (a requirement that is true for any graph in which there are no isolated nodes), for $N \to \infty$ the terms $cNp_i p_j$ tend either to 0 or to ∞, therefore, the number of links (i,j) for which Eq. (6) is not true for N large approaches

$$\mathcal{N}_N \stackrel{def}{=} \sum_{i<j} \theta(cNp_i p_j - 1), \tag{8}$$

where $\theta(x) = 0$ or 1 if $x < 0$ or $x \geqslant 0$, respectively. In Appendix A, we show that, if in the ensemble $\mathcal{C}$ the probability $p(k)$ to have a vertex with degree k scales, for k large, as a power law $p(k) \sim k^{-\gamma}$, then

$$\mathcal{N}_N < \frac{N^{2-\gamma} c^{1-\gamma}}{2(\gamma - 1)} \log(N), \tag{9}$$

[1]Usually, the hidden variables are represented not with the $\{p_i\}$, but with the set $\{\theta_i\}$, where $\theta_i = \sqrt{cN}\, p_i$.

TABLE II. Connected correlation functions $\tilde{\chi}_l$ (connected correlation of two randomly chosen spins at given distance l) and $\tilde{\chi}_{ij}$ [average connected correlation of two given spins i and j (see Sec. II C)] of the Ising model built on the configuration model; the classical small-world network built by overlapping the classical random graph with additional links associated to an arbitrary graph $(\mathcal{L}_0,\Gamma_0)$ [$\tilde{\chi}_{0;i,j}$ stands for the connected correlation function associated to $(\mathcal{L}_0,\Gamma_0)$]; the heterogeneous small-world networks (this paper) built by overlapping hidden variable models with $(\mathcal{L}_0,\Gamma_0)$. In the heterogeneous small-world networks, the formula for $\tilde{\chi}_{ij}$ is valid when the underlying network $(\mathcal{L}_0,\Gamma_0)$ has dimension $d_0 \leqslant 1$. For the general case, the dependence on i and j is more complicated [see Eq. (75)], but the dependence on N is the same. For what concerns the configuration network model, for $\gamma \leqslant 3$, in Ref. [25] it is speculated that $\tilde{\chi}_l \sim 0$ for l not too small, while for $l \sim O(1)$, one has $\tilde{\chi}_l \sim t^l$, a result that is not in contradiction with our achievement (see Sec. II C). However, in Ref. [25], $\tilde{\chi}_l$ was obtained directly by using the treelike assumption, which, for $\gamma \leqslant 3$, is wrong even in the configuration model (in fact, the formulas of Ref. [25] give $\tilde{\chi}_l = 0$ for any $l \geqslant 2$, which is clearly not exact). See Ref. [7] and footnote 2. Apart from technical details, the main point we stress is that, in previous works, it was not possible to see that (i) finite size effects are always strong in scale-free networks; (ii) for $\gamma > 3$ they decay as slowly as $O(1/N^{(\gamma-3)/(\gamma-1)})$; and (iii) finite size effects persist even in the thermodynamic limit when $\gamma \leqslant 3$ and, as a consequence, correlations of two given spins can be strong (power-law-like) when $\gamma \leqslant 3$.

Ising on the "configuration network" (Ref. [25])	Correlations $\tilde{\chi}_l$ of two spins at distance l; $N \to \infty$
$\gamma > 5$, $\langle k^4\rangle_p < \infty$ {length of shortest loops scale as $O[\log(N)]$}	$\tilde{\chi}_l \sim t^l$, $[t = \tanh(\beta J)]$
$\gamma = 5$, $\langle k^4\rangle_p = \infty$, $\langle k^2\rangle_p < \infty$ {shortest loops scale as $O[\log(N)]$}	$\tilde{\chi}_l \sim t^l$
$3 < \gamma < 5$, $\langle k^4\rangle_p = \infty$, $\langle k^2\rangle_p < \infty$ {shortest loops scale as $O[\log(N)]$}	$\tilde{\chi}_l \sim t^l$
$\gamma = 3$, $\langle k^2\rangle_p = \infty$ (finite loops)	$\tilde{\chi}_l$?
$2 < \gamma < 3$, $\langle k^2\rangle_p = \infty$ (finite loops)	$\tilde{\chi}_l$?
Ising on the "classical SW networks" (Ref. [14])	Correlations $\tilde{\chi}_{ij}$ of two given spins i and j; N finite
$P(k)$ Poissonian + additional arbitrary links (arbitrary loops)	$\tilde{\chi}_{ij} = \tilde{\chi}_{0;i,j} + O\left(\frac{ct}{N}\frac{[\tilde{\chi}_0]^2}{1-ct\tilde{\chi}_0}\right)$
Ising on the "heterogeneous SW networks" (this paper)	Correlations $\tilde{\chi}_{ij}$ of two given spins i and j; N finite
$\gamma > 3$, $\langle k^4\rangle_p < \infty$ (arbitrary loops of any length)	$\tilde{\chi}_{ij} = \tilde{\chi}_{0;i,j} + O\left(\frac{t}{t_c}\frac{1}{1-t/t_c}\frac{(ij)^{-1/(\gamma-1)}}{N^{(\gamma-3)/(\gamma-1)}}\right)$
$3 \geqslant \gamma > 2$, $\langle k^4\rangle_p = \infty$ (arbitrary loops of any length)	$\tilde{\chi}_{ij} = \tilde{\chi}_{0;i,j} + O\left((ij)^{-1/(\gamma-1)}\right)$

so that the requirement $\mathcal{N}_N = O(N^\alpha)$ with $\alpha < 1$ is equivalent to have $\gamma > 2$, and, for N large but finite, the error we make per spin in neglecting these $O(N^\alpha)$ contributions is $O[N^{1-\gamma}\log(N)]$ for $\gamma > 2$.[2] As an example of a scale-free model, we can consider the choice

$$f(p_i,p_j) = 1 - e^{-cNp_ip_j}, \tag{10}$$

$$p_i \overset{def}{=} \frac{i^{-\mu}}{\sum_{j\in\mathcal{L}_0} j^{-\mu}} \simeq \frac{i^{-\mu}(1-\mu)}{N^{1-\mu}}, \tag{11}$$

where $\mu \in [0,1)$. Equations (10) and (11) define the static model introduced in [17]. Note that the so called fermionic constraint that avoids having multiple bonds is automatically satisfied by Eq. (5). As has been shown, this constraint leads to some weak disassortative degree-degree correlations for $\mu > 1/2$ [21]. In the thermodynamic limit $N \to \infty$, for $\mu \in (0,1)$, Eqs. (10) and (11) lead to a number of long-range connections per site distributed according to a power law with mean c and exponent γ given by

$$\gamma = 1 + \frac{1}{\mu}, \tag{12}$$

so that $\gamma \in (2,\infty)$. For $\mu \in (0,1/2)$ $(\gamma > 3)$, Eq. (10) takes the simpler form (6), while for $\mu \in [1/2,1)$ $(2 < \gamma \leqslant 3)$, Eq. (10) can be written as Eq. (6) only when i and j are sufficiently distant $ij \gg N^{2-1/\mu}$, while for lower distances $ij \ll N^{2-1/\mu}$, we have $p_{ij}(c_{ij} = 1) \simeq 1$.

The free energy F and the averages $\overline{\langle\mathcal{O}\rangle^l}$, with $l = 1,2$, are defined in the usual (quenched) way as ($\beta = 1/T$)

$$-\beta F \overset{def}{=} \sum_{\boldsymbol{c}_0,\boldsymbol{c}} P(\boldsymbol{c}_0,\boldsymbol{c}) \int d\mathcal{P}(\boldsymbol{J}_0,\boldsymbol{J}) \log(Z_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}}) \tag{13}$$

and

$$\overline{\langle\mathcal{O}\rangle^l} \overset{def}{=} \sum_{\boldsymbol{c}_0,\boldsymbol{c}} P(\boldsymbol{c}_0,\boldsymbol{c}) \int d\mathcal{P}(\boldsymbol{J}_0,\boldsymbol{J}) \langle\mathcal{O}\rangle^l_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}, \quad l = 1,2 \tag{14}$$

where $Z_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}$ is the partition function of the quenched system

$$Z_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}} = \sum_{\{\sigma_i\}} e^{-\beta H_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}(\{\sigma_i\})}, \tag{15}$$

[2] It should be noted that, as a matter of fact, from the analysis performed on $\mathcal{J}$, one is allowed to make the analytic continuation to get the results in the full set $\mathcal{I}$. Notice the strict analogy with what is usually (tacitly) done in the Ising model defined on the configuration model [24]: One uses the local treelike ansatz to derive the equation for the order parameter in the region $\gamma > 3$, then one extrapolates by analytic continuation the result to the region $3 \geqslant \gamma > 2$, where the tree ansatz is wrong even locally. The reason why the analytic continuation works is the same as ours: The extensive free energy does not depend on the number of contributions for which the treelike ansatz is wrong since this number grows less slowly than $O(N^\alpha)$ with some $\alpha < 1$. It should, however, be recalled that the treelike ansatz used to get directly local quantities loop sensitive, as the spin-spin correlations would lead to a completely wrong result for $3 \geqslant \gamma > 2$. The proper way to get the spin-spin correlation consists in solving the model for $\gamma > 3$ in the presence of a nonuniform external field and then to analytically continue the result to the range $3 \geqslant \gamma > 2$.

$\langle\mathcal{O}\rangle_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}$ the Boltzmann average of the quenched system ($\langle\mathcal{O}\rangle$ depends on the given realization of $\boldsymbol{J}$, $\boldsymbol{J}_0$, $\{\boldsymbol{c}_0\}$ and $\boldsymbol{c}$: $\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{\boldsymbol{c}_0,\boldsymbol{c};\boldsymbol{J}_0,\boldsymbol{J}}$; for simplicity, we later will omit to write these dependences)

$$\langle\mathcal{O}\rangle_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}} \stackrel{def}{=} \frac{\sum_{\{\sigma_i\}} \mathcal{O} e^{-\beta H_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}(\{\sigma_i\})}}{Z_{\boldsymbol{c}_0,\boldsymbol{c},\boldsymbol{J}_0,\boldsymbol{J}}}, \tag{16}$$

and $d\mathcal{P}(\boldsymbol{J}_0,\boldsymbol{J})$ and $P(\boldsymbol{c}_0,\boldsymbol{c})$ are product measures given in terms of arbitrary measures (all normalized to 1) for the short- and long-range couplings $d\mu_0(J_{0;i,j}) \geqslant 0$, $d\mu(J_{i,j}) \geqslant 0$, and in terms of the introduced link probabilities [Eqs. (4) and (5)] $p_0(c_{i,j}) \geqslant 0$, and $p_{ij}(c_{i,j}) \geqslant 0$:

$$d\mathcal{P}(\boldsymbol{J}_0,\boldsymbol{J}) \stackrel{def}{=} \prod_{(i,j),i<j} d\mu(J_{i,j}) \prod_{(i,j)\in\Gamma_0} d\mu_0(J_{0;i,j}), \tag{17}$$

$$P(\boldsymbol{c}_0,\boldsymbol{c}) \stackrel{def}{=} \prod_{(i,j),i<j} p_{ij}(c_{i,j}) \prod_{(i,j)\in\Gamma_0} p_0(c_{0;i,j}). \tag{18}$$

B. A note on small-world networks à la Watts and Strogatz

The class of our small-world scale-free models given by Eqs. (2)–(7) is very general. Note, in particular, that the graph $(\mathcal{L}_0,\Gamma_0)$ is completely arbitrary and can contain closed paths of any length. We stress that the resulting network, the union of the graph $(\mathcal{L}_0,\Gamma_0)$ in which each link is removed with a probability p, with the scale-free random graph $\boldsymbol{c}$ can be seen as a scale-free grand canonical generalization of the original small-world graph of Watts and Strogatz [13], although we do not perform here a true rewiring. Since we let the probability $p \in [0,1]$ and the mean $c \in (0,\infty)$ be arbitrary, our method to build small-world networks is more general even for the non-scale-free case $\mu = 0$ ($\gamma = \infty$). However, we can always restrict our class of small-world networks to those having a total average connectivity, which does not change with p by choosing c such that the total number of links of the graph $\boldsymbol{c}$ is equal to the total number of removed links of $(\mathcal{L}_0,\Gamma_0)$. Up to corrections $O(1/\sqrt{N})$, we can accomplish this for any sample by simply taking $c = c_0 p$, where c_0 is the average connectivity of $(\mathcal{L}_0,\Gamma_0)$. We anticipate, however, that the critical behavior of these models is not affected by any particular choice of p and c, the only condition being $c > 0$. In fact, as we will see soon in Sec. III, the sole role of the parameter p is to give a renormalized effective coupling $J_0(p)$ to be used as though we had the original graph $(\mathcal{L}_0,\Gamma_0)$ with no removed links. Since the class of universality does not depend on J_0 as soon as $c > 0$, it follows that the critical behavior of this class of generalized small-world models is the same for any $p \in [0,1]$ as soon as $c > 0$.[3]

C. Correlations of two given spins and correlations of two spins at given chemical distance

The quantities of major interest are the averages and the quadratic averages of the correlation functions, which for simplicity will be indicated by $C^{(1)}$ and $C^{(2)}$. For example, the following are nonconnected correlation functions of order k:

$$C^{(1)} = \overline{\langle\sigma_{i_1},\ldots,\sigma_{i_k}\rangle}, \tag{19}$$

$$C^{(2)} = \overline{\langle\sigma_{i_1},\ldots,\sigma_{i_k}\rangle^2}, \tag{20}$$

where $k \geqslant 1$ and the indices $i_1,\ldots,i_k$ are supposed all different. For simplicity, we will keep using the symbols $C^{(1)}$ and $C^{(2)}$ also for the connected correlation function since they obey the same rules of transformations. We point out that the set of indices $i_1,\ldots,i_k$ is fixed along the process of the two averages, with respect to the couplings (17) and the graph realizations (18). This implies, in particular, that, given the spin with index i and the spin with index j, once the averages have been performed, their chemical distance remains undefined, while the only meaningful distance between i and j is the distance defined over $\mathcal{L}_0$, which we will indicate as $||i-j||_0$. Interestingly, for the cases in which $(\mathcal{L}_0,\Gamma_0)$ is a regular lattice, $||i-j||_0$ is a Euclidean distance. Therefore, throughout this paper, it must be kept in mind that, for example, $C^{(1)}(||i-j||_0) = \overline{\langle\sigma_i\sigma_j\rangle}$ is very different from the correlation function $G^{(1)}(l)$ of two points at a fixed chemical distance l, i.e., the minimum number of links to join two points among both the links of Γ_0 and the links of the random graph realization $\boldsymbol{c}$. In fact, if, e.g., for the homogeneous case $p_i \equiv 1/N$ with $J_0 = 0$, one considers all the possible realizations of the Poisson graph, and then all the possible distances l between two given points i and j, one has

$$\begin{aligned} C^{(1)}(||i-j||_0) &= \overline{\langle\sigma_i\sigma_j\rangle} - \overline{\langle\sigma_i\rangle\langle\sigma_j\rangle} \\ &= \sum_{l=1}^{N} P_N(l) G^{(1)}(l), \end{aligned} \tag{21}$$

where $P_N(l)$ is the probability that, in the system with N spins, the shortest path between the vertices i and j has length l. If we now use $G^{(1)}(l) \sim [\tanh(\beta J)]^l$ [25] (in the P region holds the exact equality) and the fact that the average of l with respect to $P_N(l)$ is of the order $\log(N)$, we see that the two-point connected correlation function (21) goes to 0 in the thermodynamic limit. Similarly, in the Poissonian graph, all the connected correlation functions defined in this way are zero in the thermodynamic limit. However, as we will see in Sec. III D, this independence of the variables holds only if $J_0 = 0$ and $\gamma > 3$. Furthermore, even for $\gamma > 3$, finite size effects may result in strong correlations in the finite network.

III. AN EFFECTIVE FIELD THEORY

A. The self-consistent equation

Depending on the temperature T, and on the parameters μ and those of the probability distributions $d\mu(\cdot)$ and $d\mu_0(\cdot)$, the small-world model may stably stay either in the paramagnetic (P), in the ferromagnetic (F), or in the spin-glass (SG) phase. In our approach for the F and SG phases, there are two natural order parameters that will be indicated by $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$. Similarly, for any correlation function, quadratic or not, there are two natural quantities indicated by $C^{(\mathrm{F})}$ and $C^{(\mathrm{SG})}$, and that in turn will be calculated in terms of $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$, respectively. To avoid confusion, it should be kept in mind

[3] In the non-scale-free case, the similarity between small-world models obtained by pure rewiring or pure addition of links has already been speculated by several authors but never proved.

that, in our approach, for any observable $\mathcal{O}$, there are, in principle, always two solutions that we label as F and SG, but, as we shall discuss soon, for any temperature, only one of the two solutions is stable and useful in the thermodynamic limit.

In the following, we use the label 0 to specify that we are referring to the pure model with Hamiltonian (2). Note that all the equations presented in this paper have meaning and usefulness also for sufficiently large but finite size N. For simplicity, we shall often omit to write the dependence on N.

Let $m_{0i}(\beta J_0,\{\beta h_j\})$ be the stable magnetization of the spin i in the pure model (2) with coupling J_0 and in the presence of a generic external field $\{h_j\}$ at inverse temperature β. In Appendix B, we prove that the order parameter $m^{(\mathrm{F})}$ or $m^{(\mathrm{SG})}$ of the model defined in Eqs. (3)–(7), with the condition $\alpha < 1$ (equivalent to $\gamma > 2$), satisfies the following self-consistent equation:

$$m^{(\Sigma)} = \sum_i m_{0i}\left(\beta J_0^{(\Sigma)};\{Np_j ct^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) p_i, \quad (22)$$

where the effective fields $t^{(\mathrm{F})}$, $t^{(\mathrm{SG})}$ and couplings $J_0^{(\mathrm{F})}$ and $J_0^{(\mathrm{SG})}$ are given by

$$t^{(\mathrm{F})} = \int d\mu(J)\tanh(\beta J), \quad (23)$$

$$t^{(\mathrm{SG})} = \int d\mu(J)\tanh^2(\beta J), \quad (24)$$

$$\tanh\left(\beta J_0^{(\mathrm{F})}\right) = (1-p)\int d\mu_0(J_0)\tanh(\beta J_0), \quad (25)$$

and

$$\tanh\left(\beta J_0^{(\mathrm{SG})}\right) = (1-p)\int d\mu_0(J_0)\tanh^2(\beta J_0). \quad (26)$$

Note that $|J_0^{(\mathrm{F})}| > J_0^{(\mathrm{SG})}$. For later use, we introduce also the short notations

$$t_0^{(\mathrm{F})} \stackrel{def}{=} \tanh\left(\beta J_0^{(\mathrm{F})}\right) \quad (27)$$

and

$$t_0^{(\mathrm{SG})} \stackrel{def}{=} \tanh\left(\beta J_0^{(\mathrm{SG})}\right). \quad (28)$$

The meaning of the order parameters $m^{(\Sigma)}$ is quite natural being given by

$$(m^{(\Sigma)})^{l_\Sigma} = \sum_i p_i \overline{\langle\sigma_i\rangle^{l_\Sigma}}, \quad (29)$$

where $l_\Sigma = 1,2$ for $\Sigma = \mathrm{F}$ or SG, respectively.

The free energy density $f^{(\Sigma)}$ coming from Eq. (13) involves a generalized Landau free energy density $L^{(\Sigma)}$ from which it differs only for trivial terms independent from $m^{(\Sigma)}$. The complete expression for $f^{(\Sigma)}$ in terms of $L^{(\Sigma)}$ is reported in Appendix C. The term $L^{(\Sigma)}$ reads ($\beta f^{(\Sigma)}$ = trivial terms $+L^{(\Sigma)}/l^{(\Sigma)}$, with $l^{(\Sigma)} = 1,2$ for Σ =F, SG, respectively), and is given by

$$L^{(\Sigma)}(m^{(\Sigma)}) \stackrel{def}{=} \frac{ct^{(\Sigma)}(m^{(\Sigma)})^2}{2} + \beta f_0\left(\beta J_0^{(\Sigma)},\{Np_j ct^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right), \quad (30)$$

with $f_0(\beta J_0,\{\beta h_i\})$ being the free energy density of the pure model (2). For given β, among all the possible solutions of Eqs. (22), in the thermodynamic limit, for both Σ= F and SG, the true solution $\bar{m}^{(\Sigma)}$(or leading solution) is the one that minimizes $L^{(\Sigma)}$.

Finally, let k be the order of a given correlation function $C^{(1)}$ or $C^{(2)}$. The averages and the quadratic averages over the disorder $C^{(1)}$ and $C^{(2)}$ are (see Appendix B for details)

$$C^{(1)} = C^{(\mathrm{F})} \quad \text{in F}, \quad (31)$$

$$C^{(1)} = 0, \quad k \text{ odd} \quad \text{in SG}, \quad (32)$$

$$C^{(1)} = C^{(\mathrm{SG})}, \quad k \text{ even} \quad \text{in SG} \quad (33)$$

and

$$C^{(2)} = (C^{(\mathrm{F})})^2 \quad \text{in F}, \quad (34)$$

$$C^{(2)} = (C^{(\mathrm{SG})})^2 \quad \text{in SG}, \quad (35)$$

where, for sufficiently large N,

$$C^{(\Sigma)} = C_0\left(\beta J_0^{(\Sigma)};\{Np_j ct^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) \times O\left(\frac{1}{N^\delta}\right), \quad (36)$$

where in turn $C_0(\beta J_0,\{\beta h_i\})$ is the correlation function of the pure model (2); and, finally, $\delta \geqslant 1$ only for $k = 1$, while in general $0 \leqslant \delta < 1$ for $k > 1$ and $\delta = 0$ if $\gamma < 3$ (see Sec. III D).

From Eqs. (34) and (35) for $k = 1$, we note that the Edward-Anderson order parameter [26] $C^{(2)} = \overline{\langle\sigma\rangle^2} = q_{\mathrm{EA}}$ is equal to $(C^{(\mathrm{SG})})^2 = (m^{(\mathrm{SG})})^2$ only in the SG phase, whereas in the F phase, we have $q_{\mathrm{EA}} = (m^{(\mathrm{F})})^2$. Therefore, since $m^{(\mathrm{SG})} \neq m^{(\mathrm{F})}$, $m^{(\mathrm{SG})}$ is not equal to $\sqrt{q_{\mathrm{EA}}}$; in our approach, $m^{(\mathrm{SG})}$ represents a sort of spin glass order parameter [27]. In general, our method is able to establish exactly the phase boundary P-F and P-SG, but not the frontiers F-SG when both the order parameters give a nonzero solution. Furthermore, while Eq. (29) for $\Sigma = \mathrm{F}$ can be derived, for $\Sigma = \mathrm{SG}$ it remains only a plausible ansatz (see discussion at the end of the Sec. VII A of Ref. [14]). Note, however, that, at least for lattices $\mathcal{L}_0$ having only loops of even length, the stable P region is always that corresponding to a P-F phase diagram, so that in the P region the correlation functions must be calculated only through Eqs. (31) and (34).

As an immediate consequence of Eq. (22), we get the susceptibility $\tilde{\chi}^{(\Sigma)}$ of the model (throughout the paper we will use only the dimensionless definition of the susceptibility)

$$\tilde{\chi}^{(\Sigma)} \stackrel{def}{=} \frac{\partial m^{(\Sigma)}}{\partial(\beta h)} = \frac{\sum_i p_i \sum_j \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)};\{Np_l ct^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right)}{1 - ct^{(\Sigma)} N \sum_{i,j} \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)};\{Np_l ct^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) p_i p_j}, \quad (37)$$

where $\tilde{\chi}_{0;i,j}$ stands for the two-point connected correlation function of the pure model

$$\tilde{\chi}_{0;i,j} \stackrel{def}{=} \langle\sigma_i\sigma_j\rangle_0 - \langle\sigma_i\rangle_0\langle\sigma_j\rangle_0. \tag{38}$$

Note that, as is evident in all the above equations, even when the pure model in the presence of a uniform external field is translational invariant, for any nonzero value of the order parameter $m^{(\Sigma)}$, the disordered model is no longer translational invariant. Note, in particular, that $\tilde{\chi}^{(\Sigma)}$ refers to the weighted order parameter (29) so that it does not coincide with the usual unweighted sum of the connected correlation functions. In fact, from Eq. (22), it follows that

$$\tilde{\chi}^{(\Sigma)} = \sum_{i,j} p_i[\overline{\langle\sigma_i\sigma_j\rangle^{l_\Sigma} - \langle\sigma_i\rangle^{l_\Sigma}\langle\sigma_j\rangle^{l_\Sigma}}]. \tag{39}$$

B. Critical surface (thermal and percolative)

Note that, for β sufficiently small (see later), Eq. (22) has always the solution $m^{(\Sigma)} = 0$ and, furthermore, if $m^{(\Sigma)}$ is a solution, $-m^{(\Sigma)}$ is a solution as well. From now on, if not explicitly said, we will refer only to the positive (possibly zero) solution, the negative one being understood. A solution $m^{(\Sigma)}$ of Eq. (22) is stable (but in general not unique) if

$$ct^{(\Sigma)}N\sum_{i,j}\tilde{\chi}_{0;i,j} \times\left(\beta J_0^{(\Sigma)};\{Np_l ct^{(\Sigma)}m^{(\Sigma)} + \beta h\}\right)p_i p_j < 1. \tag{40}$$

From Eqs. (22) or (40), we see that, in the thermodynamic limit, the critical surface crossing that the system passes from a P region to a non-P region satisfies

$$ct_c^{(\Sigma)}N\sum_{i,j}\tilde{\chi}_{0;i,j}\left(\beta_c^{(\Sigma)}J_0^{(\Sigma)};0\right)p_i p_j = 1. \tag{41}$$

Equation (41) gives the critical surface of the model in the plane (β,c) as a function of p and the other parameters of the model $(d\mu_0, d\mu, \{p_i\})$.

1. Critical temperature

For a given value of c, Eq. (41) provides the critical temperature. From Eq. (22), it is immediate to recognize that, for $J_0 \neq 0$,

$$\beta_c^{(\Sigma)} < \beta_{c0}^{(\Sigma)}, \tag{42}$$

while $\beta_c^{(\Sigma)} = \beta_{c0}^{(\Sigma)}$ for $J_0 = 0$, where $\beta_{c0}^{(\Sigma)}$ is the critical temperature of the pure model with coupling $J_0^{(\Sigma)}$. It is clear that, when $(\mathcal{L}_0,\Gamma_0)$ is not translational invariant, there exists an optimal choice of labeling the sites $i = 1,\ldots,N$, which gives the lowest $\beta_c^{(\Sigma)}$, and that corresponds to the choice that maximizes the functional $F_\beta(\{p_i\})$, where

$$F_\beta(\{p_i\}) \stackrel{def}{=} cN\sum_{i,j}\tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)};0\right)p_i p_j.$$

We will come back to this interesting issue in Sec. V.

2. Percolation threshold: Clustering versus percolation threshold

The theory can be projected toward the limit $\beta \to \infty$ where, for $\Sigma = \mathrm{F}$, we get an effective percolation theory. Here, the region P corresponds to the region in which (in the thermodynamic limit) the parameters (c,c_0,p) are such that no giant connected component exists ($m^{(F)} = 0$). Note, in particular, that if c_{0c} is the percolation threshold of the initial graph $(\mathcal{L}_0,\Gamma_0)$ (if c_{0c} does not exist, we can set formally $c_{0c} = \infty$) in order to remain in the region P, the connectivity $c_0^{(p)} = c_0(1-p)$ of the graph obtained from the graph $(\mathcal{L}_0,\Gamma_0)$ in which each link has been removed at random with probability p must satisfy $c_0(1-p) \leqslant c_{0c}$, otherwise, a giant connected component already exists [and the stability condition (40) at $\beta \to \infty$ with $m^{(F)} = 0$ is violated]. From Eq. (41), in the thermodynamic limit it follows the equation for the percolation threshold c_c as a function of p:

$$c_c N\sum_{i,j}\tilde{\chi}_{0;i,j}[\tanh^{-1}(1-p);0]p_i p_j = 1 \quad \text{with} \quad c_0(1-p) \leqslant c_{0c}, \tag{43}$$

where we have used the fact that $\lim_{\beta\to\infty}\tanh(\beta J_0^{(\mathrm{F})}) = \tanh(1-p)$. Alternatively, Eq. (43) can be rewritten in terms of only graph elements as

$$c_c N\sum_{i,j}\left(\delta_{i,j} + \mathcal{N}_{0;i,j}^{(p)}\right)p_i p_j = 1, \quad c_0(1-p) \leqslant c_{0c} \tag{44}$$

where $\mathcal{N}_{0;i,j}^{(p)} = 1$ if, in the graph $(\mathcal{L}_0,\Gamma_0)$ from which each link has been removed at random with probability p, between the vertex i and the vertex j there exists at least a path of links and $\mathcal{N}_{0;i,j}^{(p)} = 0$ otherwise.

Given p, if the condition $c_0(1-p) \leqslant c_{0c}$ is not satisfied, then a giant connected component is present and we can set $c_c = 0$. It is interesting to see in more details the case in which we choose $c = c_0 p$ so that, as we vary p, the total connectivity is fixed and equal to c_0 (the "rewired" small world). This study is important since it leads us to understand how the presence of short loops affects diffusion processes on general networks. In particular, a strong interest concerns the following question: *In the presence of short loops, how does the percolation threshold change?* If we set $c = c_0 p$, from Eq. (44) we get the percolation threshold c_{0c} as a function of the rewiring parameter p:

$$c_{0c}^{(p)} pN\sum_{i,j}\left(\delta_{i,j} + \mathcal{N}_{0;i,j}^{(p)}\right)p_i p_j = 1, \quad c_0^{(p)}(1-p) \leqslant c_{0c}. \tag{45}$$

From Eq. (45), we see that p has two effects on $c_{0c}^{(p)}$: the prefactor p in the left-hand side of Eq. (45) tends to decrease $c_{0c}^{(p)}$, while the other tends to decrease $\mathcal{N}_{0;i,j}^{(p)}$ and then to increase $c_{0c}^{(p)}$. However, as we shall see soon, in general, $c_{0c}^{(p)}$ decreases with p due to the general mechanism according to which clustering diminishes the percolation threshold.

A special case is the one in which $p_i \equiv 1/N$; i.e., the classical small world (no heterogeneity). In this case, Eq. (43) simplifies as

$$c_{0c}^{(p)} p\tilde{\chi}_0[\tanh^{-1}(1-p);0] = 1, \quad c_0^{(p)}(1-p) \leqslant c_{0c}. \tag{46}$$

So, for example, if $(\mathcal{L}_0,\Gamma_0)$ is the Erdös-Rényi random graph [15] (in the canonical representation), with mean connectivity c_0, from (valid in the P region)

$$\tilde{\chi}_0(\beta J_0;0)=\frac{1}{1-c_0\tanh(\beta J_0)} \tag{47}$$

to be inserted in Eq. (46), we get back obviously the well known percolation threshold $c_{0c}^{(p)}=1$, independently of the value of p. Depending on the problem, given $c_0<c_{0c}$, in general one can be more interested in reading Eq. (43) either as an equation for p or for c. We can consider, for example, the case in which $(\mathcal{L}_0,\Gamma_0)$ is an ensemble of arbitrary disconnected finite clusters (dimers, triangles, ..., or mixtures of them) for which there is no percolation threshold (or formally $c_{c0}=\infty$). For example, for a set of $N/2$ disconnected dimers ($c_0=1$), $N/3$ disconnected triples ($c_0=1\times 2/3+2\times 1/3$), $N/3$ disconnected triangles ($c_0=2$), $N/4$ disconnected squares ($c_0=2$), and $N/5$ disconnected pentagons ($c_0=2$), we have, respectively,

$$\tilde{\chi}_0(\beta J_0;0)=\frac{2e^{\beta J_0}}{e^{\beta J_0}+2e^{-\beta J_0}} \quad \text{(dimers)}, \tag{48}$$

$$\tilde{\chi}_0(\beta J_0;0)=\frac{1}{3}\frac{9e^{2\beta J_0}+2+2e^{-2\beta J_0}}{e^{2\beta J_0}+2+e^{-2\beta J_0}} \quad \text{(triples)}, \tag{49}$$

$$\tilde{\chi}_0(\beta J_0;0)=\frac{3e^{3\beta J_0}+e^{-\beta J_0}}{e^{3\beta J_0}+3e^{-\beta J_0}} \quad \text{(triangles)}, \tag{50}$$

$$\tilde{\chi}_0(\beta J_0;0)=\frac{4e^{4\beta J_0}+4}{e^{4\beta J_0}+7} \quad \text{(squares)}, \tag{51}$$

$$\tilde{\chi}_0(\beta J_0;0)=\frac{5e^{5\beta J_0}+11e^{\beta J_0}}{e^{5\beta J_0}+15e^{\beta J_0}} \quad \text{(pentagons)}. \tag{52}$$

If we consider the case with no heterogeneity $p_i\equiv 1/N$, from Eqs. (48)–(52) plugged in Eq. (43) for $p=0$, we get, respectively, the following percolation thresholds c_c:

$$c_c=1/2 \quad \text{(dimers)}, \tag{53}$$

$$c_c=1/3 \quad \text{(triples)}, \tag{54}$$

$$c_c=1/3 \quad \text{(triangles)}, \tag{55}$$

$$c_c=1/4 \quad \text{(squares)}, \tag{56}$$

$$c_c=1/5 \quad \text{(pentagons)}, \tag{57}$$

and, in general, for polygons of $m\geqslant 3$ sides ($c_0=2$),

$$c_c=1/m \quad \text{(polygons of } m \text{ sides)}. \tag{58}$$

Notice that the clustering coefficient for dimers and triples is zero, and for closed polygons of m sides decreases with m. Of course, one recovers that $c_c=1/m\to 0$ for $m\to\infty$ since an ensemble of N/m disconnected polygons of length m for $m=N\to\infty$ becomes equivalent to a closed chain for which we already know that $c_c=0$. Eqs. (48)–(52) can be used in general also for $p>0$. So, for example, from Eqs. (46) and (50), by using the replacement $\beta J_0\to\tanh^{-1}(1-p)$, we get the equation for the percolation threshold p_c of an ensemble of disconnected triangles ($c_0=2$) from which each link has been removed with probability p ($c_0^{(p)}=2p$) and "rewired" as a "long-range" link:

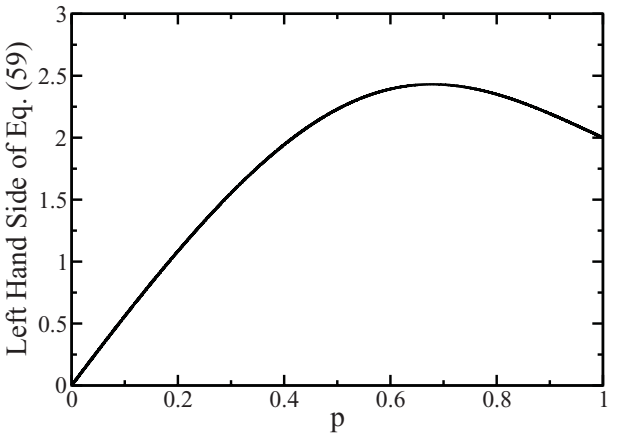

FIG. 1. Plot of the left-hand side of Eq. (59) [$(\mathcal{L}_0,\Gamma_0)$ is a set of disjoint triangles] as a function of the dilution probability p.

$$2p\frac{3e^{3\tanh^{-1}(1-p)}+e^{-\tanh^{-1}(1-p)}}{e^{3\tanh^{-1}(1-p)}+3e^{-\tanh^{-1}(1-p)}}=1. \tag{59}$$

In Fig. 1, we plot the left-hand side of Eq. (59) as a function of p. Equation (59) is solved for $p_c=0.183\,406$.

Let us come back now to the general heterogeneous case. From Eq. (45), we see that, given two regular graphs $(\mathcal{L}_0,\Gamma_0)$ and $(\mathcal{L}_0,\Gamma_0')$, both having the same average connectivity $c_0=c_0'$ (so that $|\Gamma_0'|=|\Gamma_0|$), between $(\mathcal{L}_0,\Gamma_0)$ and $(\mathcal{L}_0,\Gamma_0')$, the sum in the left-hand side of Eq. (45) will be greater for the graph having the smaller clustering coefficient, which in turn will result in a lower value for $c_{0c}^{(p)}$. In fact, given a vertex i and its local connectivity $c_0(i)$, the smaller the clustering coefficient around the vertex i is, the larger the number of different vertices j connected to i (while when the clustering coefficient is large, a same vertex j will be reached from the vertex i by many different paths) will be, so that $\sum_j \mathcal{N}_{0;i,j}$ will be greater, which in turn will give rise, via Eq. (45), to a smaller clustering coefficient, and similarly for $\mathcal{N}_{0;i,j}^{(p)}$ for any given p. In conclusion, as already discussed in [28–30], clustering increases the percolation threshold. Equation (58) for the polygons represents a clear example of this mechanism for the particular choice $c_0=2$, $p=0$, and $p_i\equiv 1/N$.

We conclude this section with a remark on the recent methods used by Newman [8] and Gleeson [12] by which families of clustered networks are introduced and analytically exactly solved by generating function techniques. Although these networks have a finite clustering coefficient, they can still be mapped to effective treelike graphs. So, for example, for the ensemble of disconnected finite clusters as the ones we have analyzed in Eqs. (48)–(58), we could also use the method [8] to solve the percolation problem,[4] but not, for example, the case in which $(\mathcal{L}_0,\Gamma_0)$ is a d_0 dimensional lattice. In fact, the main condition that allows the methods [8] or [12] to be applied is

[4]For example, for the ensemble of disconnected triangles, by using the same formalism of [8], it is easy to see that by choosing $p_{s,t}=p_s\delta_{t,1}$, p_s being Poissonian with mean $\mu=c$, we reach Eq. (55).

the absence of overlaps among the module elements (links or triangles or any kind of finite cluster) while, for example, in a two-dimensional lattice, we always have overlap among the square "module elements."

C. Critical behavior

In this section, we prove that the critical behavior of an arbitrary heterogeneous graph as defined through Eq. (5), which includes, in particular, scale-free graphs, is robust with respect to the addition of any graph $(\mathcal{L}_0,\Gamma_0)$ provided $(\mathcal{L}_0,\Gamma_0)$ is not in turn a heterogeneous graph. In [14], we have shown this result for the homogeneous small-world model corresponding to the case $p_i \equiv 1/N$. More precisely, the critical behavior for $p_i \equiv 1/N$ and $p = 0$ has been shown to be classical mean field for $t_0^{(\mathrm{F})} \geqslant 0$, while for $t_0^{(\mathrm{F})} < 0$, first-order phase transitions are also possible (see also [31]). Here, we will restrict the analysis only to the case $t_0^{(\mathrm{F})} \geqslant 0$. First of all, from Eqs. (37) and (41) we observe immediately that the critical exponents for the susceptibility, above and below the critical temperature, are both equal to 1. Note in particular that, above the critical temperature and zero external field, the susceptibility can be written in the simpler form

$$\tilde{\chi}^{(\Sigma)} = \frac{\sum_i p_i \sum_j \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)};\{0\}\right)}{1 - t^{(\Sigma)}/t_c^{(\Sigma)}}. \tag{60}$$

Let us now turn to the analysis of the order parameter near the critical point. For $J_0 = 0$, i.e., for the pure static model, we have

$$m_{0i}\left(\beta J_0^{(\Sigma)};\{\beta h_j\}\right) = \tanh(\beta h_i) \tag{61}$$

so that the self-consistent equation (22) strongly simplifies in

$$m^{(\Sigma)} = g(m^{(\Sigma)}), \tag{62}$$

where

$$g(m^{(\Sigma)}) \stackrel{def}{=} \sum_i \tanh(N p_i c t^{(\Sigma)} m^{(\Sigma)}) p_i. \tag{63}$$

The critical behavior of the pure static model, i.e., with $J_0 = 0$ for the scale-free choice [Eqs. (10) and (11)], has been studied in Ref. [32]. Let us focus on the P-F transition. For $\Sigma = \mathrm{F}$, Eq. (62) is equal to Eq. (21) of [32]. We recall that, due to the power-law character of the distribution $\{p_j\}$, we can not derive the correct critical behavior by simply expanding in the sum in $g(m^{(\mathrm{F})})$ term by term for small $m^{(\mathrm{F})}$. As shown in [32], it is necessary to keep track of all the terms of the sum present in $g(m^{(\mathrm{F})})$. This is done by evaluating the sum with the corresponding integral that gives rise to a singular term proportional to $(m^{(\mathrm{F})})^{\gamma-2}$ plus regular terms proportional to $(m^{(\mathrm{F})})$, $(m^{(\mathrm{F})})^3$, and so on. As a consequence, when we solve the self-consistent equation to leading order in $m^{(\mathrm{F})}$, if T and τ indicate the temperature and the reduced temperature, respectively, we get the well known anomalous mean-field behavior $m^{(\mathrm{F})} \sim O(\tau^{1/2})$ (i.e., classical mean field) for $\gamma > 5$, $m^{(\mathrm{F})} \sim O(\tau^{1/(\gamma-3)})$ for $3 < \gamma < 5$, and $m^{(\mathrm{F})} \sim O(T^{-(\gamma-2)/(3-\gamma)})$ for $2 < \gamma < 3$. Note that the critical behavior of the order parameter $m^{(\mathrm{F})} = \sum_i \overline{\langle\sigma_i\rangle} p_i$ is different from the unweighted one defined as $\overline{m} = \sum_i \overline{\langle\sigma_i\rangle}/N$ when $2 < \gamma < 3$. In such a case, from Eq. (36) one can use $\overline{m} \sim t^{(\mathrm{F})} m^{(\mathrm{F})}$ from which it follows that $\overline{m} \sim O(T^{-1/(3-\gamma)})$ for $2 < \gamma < 3$.

Let now $J_0 \neq 0$. It is clear that if the graph $(\mathcal{L}_0,\Gamma_0)$ is in turn a pure scale-free graph with exponent γ', then the joined network will have an anomalous critical behavior characterized by the minimum between γ and γ'. Less obvious is to understand what happens if $(\mathcal{L}_0,\Gamma_0)$ has a finite-dimensional structure or some special topology with short loops. In particular, we can pose the following question: Does the critical behavior change by adding, via short loops, many paths between far spins, or may the critical exponent for m depend on J_0? Let us consider the self-consistent equation (22) in general. The exact expression of $m_{0i}(\beta J_0^{(\mathrm{F})};\{\beta h_j\})$ for a generic nonhomogeneous external field $\{h_j\}$ represents a formidable task. Note that, as mentioned above, to analyze the critical behavior, we can not expand for small fields $\{\beta h_j\}$. We can, however, perform an expansion to the lowest order in $t_0^{(\mathrm{F})} = \tanh(\beta J_0^{(\mathrm{F})})$. It is then easy to see that, for $\{h_j\} \neq 0$, at the order $O(t_0^{(\mathrm{F})})$ we have

$$m_{0i}(\beta J_0;\{\beta h_j\}) = \tanh(\beta h_i) + t_0^{(\mathrm{F})}[1 - \tanh^2(\beta h_i)] \times \sum_{j\in\mathcal{N}_0(i)} \tanh(\beta h_j), \tag{64}$$

where $\mathcal{N}_0(i)$ is the set of the first neighbors of the vertex i in the graph $(\mathcal{L}_0,\Gamma_0)$. It must be said that without the condition $\{h_j\} \neq 0$, Eq. (64) might be wrong since the lowest nonzero terms in $t_0^{(\mathrm{F})}$ would involve closed paths of at least length 3, while Eq. (64) contains only paths of length 1. More precisely, near the critical point, due to the fact that the fields $\{h_j\}$ are infinitesimal but not zero, we can neglect higher order corrections in $t_0^{(\mathrm{F})}$. By plugging Eq. (64) into (22) for $\Sigma = \mathrm{F}$, we have

$$m^{(\mathrm{F})} = g(m^{(\mathrm{F})}) + \Delta_1(m^{(\mathrm{F})}) + \Delta_2(m^{(\mathrm{F})}), \tag{65}$$

where we have introduced

$$\Delta_1(m^{(\mathrm{F})}) \stackrel{def}{=} t_0^{(\mathrm{F})} \sum_i p_i \sum_{j\in\mathcal{N}_0(i)} \tanh(N p_j c t^{(\mathrm{F})} m^{(\mathrm{F})}), \tag{66}$$

$$\Delta_2(m^{(\mathrm{F})}) \stackrel{def}{=} t_0^{(\mathrm{F})} \sum_i p_i \tanh^2(N p_i c t^{(\mathrm{F})} m^{(\mathrm{F})}) \times \sum_{j\in\mathcal{N}_0(i)} \tanh(N p_j c t^{(\mathrm{F})} m^{(\mathrm{F})}), \tag{67}$$

both to be compared with the J_0 independent term $g(m^{(\mathrm{F})})$. Let us analyze the bigger contribution $\Delta_1(m^{(\mathrm{F})})$ and focus on the simpler cases in which the graph $(\mathcal{L}_0,\Gamma_0)$ has a fixed connectivity $|\mathcal{N}_0(i)| \equiv c_0$, where $|\mathcal{N}_0(i)|$ stands for the cardinality of the set $\mathcal{N}_0(i)$. Let us suppose first that $c_0 = 1$ [i.e., $(\mathcal{L}_0,\Gamma_0)$ is an ensemble of dimers]. In general, given any normalized distribution $p_i \geqslant 0$ different from the homogeneous one, and any function $f(x) \geqslant 0$ increasing with x, the following property holds:

$$\sum_i p_i f(p_{j_0(i)}) < \sum_i p_i f(p_i), \tag{68}$$

where $j_0(i)$ stands for the first single neighbor of i in $(\mathcal{L}_0,\Gamma_0)$, while

$$\sum_i p_i f(p_{j_0(i)}) = \sum_i p_i f(p_i) \tag{69}$$

only for $p_i \equiv 1/N$. Note that, by definition, $i \to j_0(i)$ is a bijection on $\mathcal{L}_0$ and that $j_0(i) \neq i$. We can, however, formally enlarge the definition of $j_0(i) \neq i$ to include also the case $j_0(i) = i$ (a self-link). The inequality (68) tells us that when we choose $j_0(i) = i$, we get an optimal overlap between the distribution $\{p_i\}$ and the function $f(\cdot)$. For the general case $|\mathcal{N}_0(i)| \equiv c_0 \geqslant 1$, given a vertex i, we can enumerate the c_0 neighbors of i as $j_0^{(1)}(i), \ldots, j_0^{(c_0)}(i)$. Each upper index $l = 1, \ldots, c_0$ represents an oriented axis so that, for each $l = 1, \ldots, c_0$, the function $j_0^{(l)}(i)$ is a bijection on $\mathcal{L}_0$. For example, if $(\mathcal{L}_0,\Gamma_0)$ is the one dimensional ring, we have the two bijections $j_0^{(1)}(i) = i + 1$ and $j_0^{(2)}(i) = i - 1$. By applying Eq. (68) to each oriented axis, we then get

$$\sum_i p_i \sum_{j \in \mathcal{N}_0(i)} f(p_j) = \sum_{l=1}^{c_0} \sum_i p_i f(p_{j_0^{(l)}(i)}) < c_0 \sum_i p_i f(p_i). \tag{70}$$

By using Eq. (70) to our case with $f(x) = \tanh(x)$ and for $t_0^{(\mathrm{F})} > 0$, we see that, for $m^{(\mathrm{F})} > 0$, we have always

$$0 < \Delta_1(m^{(\mathrm{F})}) < t_0^{(\mathrm{F})} c_0 g(m^{(\mathrm{F})}). \tag{71}$$

In turn, $0 < \Delta_2(m^{(\mathrm{F})}) < \Delta_1(m^{(\mathrm{F})})$ and, furthermore, as already mentioned, near the critical point, higher order corrections in $t_0^{(\mathrm{F})}$ will be all lower than the first term proportional to $t_0^{(\mathrm{F})}$. In conclusion, for $t_0^{(\mathrm{F})} > 0$, from Eq. (22) and the above inequalities, we get

$$g(m^{(\mathrm{F})}) < m^{(\mathrm{F})} < \left(1 + t_0^{(\mathrm{F})} c_0\right) g(m^{(\mathrm{F})}) \; O\left(t_0^2\right) g(m) + o[g(m^{(\mathrm{F})})], \tag{72}$$

where $o[g(m^{(\mathrm{F})})]$ stands for corrections smaller than $g(m^{(\mathrm{F})})$. In general, for t_0 finite, it is possible to prove that

$$g(m) < m < \tilde{\chi}_0\left(\beta J_0^{(\mathrm{F})}; 0\right) g(m) + o[g(m)], \tag{73}$$

where $\tilde{\chi}_0(\beta J_0^{(\mathrm{F})}; 0)$ is the susceptibility of the pure model (2) with coupling $J_0^{(\mathrm{F})}$ and $h_i \equiv 0$. The proof is given in Appendix D. Since near the critical point in the region $\beta_{c0}^{(F)} > \beta > \beta_c$, it is $\tilde{\chi}_0(\beta J_0^{(\mathrm{F})}; 0) < \infty$, we see that Eq. (73) implies that the critical behavior of Eq. (22) remains always as that corresponding to the term $g(m)$, i.e., as if it were $J_0 = 0$. We can finally consider the case in which $(\mathcal{L}_0,\Gamma_0)$ is a Poissonian graph (Erdös-Rényi random graph in the canonical representation) with mean connectivity c_0. To this aim, we can start from the fully connected graph and remove from it randomly each of its $N(N-1)/2$ links with a probability $p = 1 - c_0/N$; the resulting graph will be our Poissonian graph with mean connectivity c_0. Since we have already proved that when $(\mathcal{L}_0,\Gamma_0)$ is the fully connected graph with a coupling $O(1/N)$, the critical behavior remains equal to that of the model with $J_0 = 0$ [the effective couplings in this case being given by $\tan(\beta J_0^{(\mathrm{F})}) = \tanh(\beta J_0) c_0/N$], we conclude that, also for a Poissonian graph, the critical behavior of the small-world model remains the same as if it were $J_0 = 0$.

For the P-SG transition of the model with $J_0 = 0$, we can evaluate the weighted Edward-Anderson order parameter as $q_{\mathrm{EA}} = (m^{(\mathrm{SG})})^2$. Since the critical behavior of $m^{(\mathrm{SG})}$ is identical to that of $m^{(\mathrm{F})}$, we get that the critical exponent for q_{EA} is simply given by twice the γ dependent critical exponent for $m^{(\mathrm{F})}$ that we have discussed before. This result is in contrast with that of Ref. [32] for the region $4 < \gamma < 5$. The source of such a contrast might be related to the already mentioned fact that Eq. (29) for $\Sigma = \mathrm{SG}$ remains only a plausible ansatz. We do not discuss here this issue further, but we stress that, whatever the critical P-SG behavior of the model with $J_0 = 0$ may be, by applying the same procedure as done above for the case P-F, we arrive at the conclusion that also the critical P-SG behavior is infinitely robust with respect to the addition of any nonheterogeneous graph $(\mathcal{L}_0,\Gamma_0)$. The same conclusions also hold, of course, for the critical exponent of the order parameter of the percolation problem.

D. Correlation functions

Another remarkable consequence of our theory comes from Eq. (36). We see in fact that, in the thermodynamic limit, any correlation function of the model, at least for $\gamma > 3$, fits with the correlation function of the pure model but immersed in an effective field that is exactly zero in the P region and zero external field ($\{h = 0\}$). In other words, in terms of correlation functions, in the P region, the small-world model and the pure model are indistinguishable (modulo the transformation $J_0 \to J_0^{(\mathrm{SG})}$ for $\Sigma = \mathrm{SG}$). Note, however, that this assertion holds only for a given correlation function calculated in the thermodynamic limit. In fact, the corrective $O(1/N^\delta)$ term appearing in the right-hand side of Eq. (36) can not be neglected when we sum the correlation functions over all the sites $i \in \mathcal{L}_0$, as to calculate the susceptibility; yet, it is just this corrective $O(1/N^\delta)$ term that gives rise to the singularities of the model. More precisely, for the two-point connected correlation function defined as

$$\tilde{\chi}_{i,j}^{(\Sigma)} \stackrel{def}{=} \overline{\langle \sigma_i \sigma_j \rangle^{l_\Sigma} - \langle \sigma_i \rangle^{l_\Sigma} \langle \sigma_j \rangle^{l_\Sigma}}, \tag{74}$$

where $l_\Sigma = 1,2$ for $\Sigma = \mathrm{F}, \mathrm{SG}$, respectively, we have

$$\tilde{\chi}_{i,j}^{(\Sigma)} = \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}; \{N p_j c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) + c t^{(\Sigma)} N \frac{\sum_l \tilde{\chi}_{0;i,l}\left(\beta J_0^{(\Sigma)}; \{N p_q c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) p_l \sum_n p_n \tilde{\chi}_{0;n,j}\left(\beta J_0^{(\Sigma)}; \{N p_q c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right)}{1 - c t^{(\Sigma)} N \sum_{l,n} \tilde{\chi}_{0;l,n}\left(\beta J_0^{(\Sigma)}; \{N p_q c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) p_l p_n}, \tag{75}$$

where the dependence on N in $\tilde{\chi}^{(\Sigma)}_{i,j}$ and $\tilde{\chi}_{0;i,j}$ is understood. In the homogeneous case $p_i \equiv 1/N$, Eq. (75) becomes

$$\tilde{\chi}^{(\Sigma)}_{i,j} = \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}; Nct^{(\Sigma)}m^{(\Sigma)} + \beta h\right) + \frac{ct^{(\Sigma)}}{N} \frac{\sum_l \tilde{\chi}_{0;i,l}\left(\beta J_0^{(\Sigma)}; ct^{(\Sigma)}m^{(\Sigma)} + \beta h\right) \sum_n \tilde{\chi}_{0;n,j}\left(\beta J_0^{(\Sigma)}; ct^{(\Sigma)}m^{(\Sigma)} + \beta h\right)}{1 - ct^{(\Sigma)}\tilde{\chi}_0\left(\beta J_0^{(\Sigma)}; ct^{(\Sigma)}m^{(\Sigma)} + \beta h\right)}, \tag{76}$$

which, when $(\mathcal{L}_0, \Gamma_0)$ is in turn homogeneous, reduces to [14]

$$\tilde{\chi}^{(\Sigma)}_{i,j} = \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}; Nct^{(\Sigma)}m^{(\Sigma)} + \beta h\right) + \frac{ct^{(\Sigma)}}{N} \frac{\left[\tilde{\chi}_0\left(\beta J_0^{(\Sigma)}; ct^{(\Sigma)}m^{(\Sigma)} + \beta h\right)\right]^2}{1 - ct^{(\Sigma)}\tilde{\chi}_0\left(\beta J_0^{(\Sigma)}; ct^{(\Sigma)}m^{(\Sigma)} + \beta h\right)}. \tag{77}$$

Equation (75) is easily obtained by derivation of the mean-field equation (22) generalized to the case of an arbitrary external field $\{h_i\}$ (see Appendix E). Equation (75) clarifies the structure of the correlation functions in general small-world models. In the right-hand side, we have two terms: the former is a distance-dependent [the distance, if any, defined in the graph $(\mathcal{L}_0, \Gamma_0)$] short-range term whose finite correlation length for $T \neq T_{c0}^{(\Sigma)}$ makes it summable [over all the nodes (i, j)], and the latter is instead a term that takes into account the heterogeneity of the system, possibly power-law-like, also in the P phase, which turns out to be summable thanks to a global $1/N^{\delta}$ factor, where δ is the exponent appearing in Eq. (36). For the two-point connected correlation function, at least for the case in which $(\mathcal{L}_0, \Gamma_0)$ is a regular lattice, as will be clear in the next section, δ takes the value

$$\delta = \begin{cases} \frac{\gamma-3}{\gamma-1} & \text{for } \gamma > 3, \\ 0 & \text{for } 2 < \gamma \leqslant 3. \end{cases} \tag{78}$$

Once we perform the weighted sums with the distribution $\{p_i\}$, both the terms in the right-hand side of Eq. (75) give a finite contribution to the susceptibility. It is, in fact, immediate to verify that, by inserting Eq. (75) in (39), we get back Eq. (37).

We see here a novel fact: In scale-free models in finite but large systems, correlations between two given spins can be power-law-like even above the critical surface. Furthermore, we see from Eq. (78) that such phenomena become persistent even in the thermodynamic limit when $\gamma \leqslant 3$. At this point, it is worth comparing these scenarios with the scenarios one has in other systems. By focusing only on the second term of the right-hand side of Eq. (75), we find the following. In finite d-dimensional models, according to the Ornstein-Zernike form [33], at any T but the critical one T_c, one has exponentially small correlations, while at T_c, the correlation function decays as a power law with the distance with an exponent $d - 2 + \eta$, where η is the critical exponent of the correlation length. Roughly speaking, this implies that, in finite-dimensional models, there are essentially two possible correlations for near [i.e., at a distance $O(1)$ at $T \neq T_c$] and far [i.e., at a distance $O(N)$ at $T = T_c$] spins, with values $O(1)$ and $O(1/N^{d-2+\eta})$, respectively, and the total number of such couples of spins are $O(N)$ and $O(N^2)$, respectively. In the fully connected model with a coupling $O(1/N)$, or in classical random graphs, or in homogeneous small-world models, at any T the correlation function decays instead as $1/N$ for any couple of the $N(N-1)/2$ spins, with no spatial dependence (the correlation length goes to infinity). As will be clear in the next section, in heterogeneous small-world networks with a power-law degree distribution $k^{-\gamma}$, at any T we can instead distinguish three families of correlations: Given two spins that are both far from a hub, they have correlations $O(1/N)$, and the total number of such couples of spins is of the order $O(aN^2)$ with $a < 1$; given two spins, one of which is a hub and the other not, they have correlations $O(1/N^{(\gamma-2)/(\gamma-1)})$, and the number of such couples of spins is $O(bN)$, where b is a decreasing function of $1/\gamma$; finally, given two spins, which are both a hub, they have correlations $O(1/N^{(\gamma-3)/(\gamma-1)})$, and the number of such couples of spins is $O(1)$.

IV. EXAMPLES

In Sec. III B 2, we have seen some simple applications to the homogeneous case ($p_i \equiv 1/N$). Here, we discuss some examples where we can apply, analytically, the general results of the previous section to the heterogeneous case. Since we have already solved the issues for the critical behavior, we will focus only on the critical surface and on the correlation functions.

A. Viana-Bray on the scale-free graph

In the case $J_0 = 0$ so that there is no additional graph, for historical reasons we refer to this as the Viana-Bray model [34] on the scale-free graph. This model was solved in [32] and, for the network version called "configuration model" (which is a network realization slightly different from the hidden variables network), the Ising model was already extensively studied in [24] almost one decade ago. Since $J_0 = 0$ for m_0, we can use Eq. (61) from which in particular it follows that, for $\beta < \beta_{c0}$, we have $\tilde{\chi}_{0;i,j} = \delta_{i,j}$. By inserting this in Eq. (41), we get the critical surface $t_c^{(\Sigma)}$

$$ct_c^{(\Sigma)} N \sum_i p_i^2 = 1, \tag{79}$$

which, for large N under the choices (10) and (11), gives

$$ct_c^{(\Sigma)} \frac{(1-\mu)^2}{(1-2\mu)}(1 - N^{2\mu-1}) = 1. \tag{80}$$

The critical surface given by Eq. (79) coincides with the one found in [32]. Note that $N\sum_i p_i^2$ is related to the second and first moments of the degree distribution $P(k)$, $\langle k \rangle_P = c$ and $\langle k^2 \rangle_P$, in terms of which Eq. (79) becomes identical to the critical surface valid for the pure scale-free graph obtained by using the configuration model [24] when $\langle k^2 \rangle_P < \infty$ [note,

however, that Eq. (80) is valid in general also when $1 < 2\mu < 2$ where $\langle k^2 \rangle_P = \infty$]:

$$t_c^{(\Sigma)} \frac{\langle k^2 \rangle_P - \langle k \rangle_P}{\langle k \rangle_P} = 1. \tag{81}$$

As anticipated before, we find out instead a quite uncommon behavior for the correlation functions which was not observed previously. Let us consider only the case $\Sigma = \mathrm{F}$ in the P region and let us consider the choices (10) and (11). From Eq. (80) we see that, when N grows, $t_c^{(\mathrm{F})}$ and $t^{(\mathrm{F})}$ in the P region remain finite for $2\mu < 1$, while they go to 0 for $2\mu > 1$ (logarithmically for $2\mu = 1$); therefore, in the latter case, for finite N, we can evaluate the correlation function at a temperature scaling with the critical one. In conclusion, from Eqs. (80) and (75) applied with $\tilde{\chi}_{0;i,j}|_{\{h_l=0\}} = \delta_{i,j}$ in the finite network for $2\mu < 1$ and at any temperature above the critical one, we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} = \delta_{i,j} + \frac{t^{(\mathrm{F})}}{t_c^{(\mathrm{F})}} \frac{(1-2\mu)}{\left(1 - t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)} \frac{(ij)^{-\mu}}{N^{1-2\mu}}, \tag{82}$$

whereas for $2\mu > 1$ at any temperature scaling with the critical one and in the P region ($\beta < \beta_c$), we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} = \delta_{i,j} + \frac{\beta}{\beta_c^{(\mathrm{F})}} \frac{(2\mu-1)}{\left(1 - \beta/\beta_c^{(\mathrm{F})}\right)} (ij)^{-\mu}, \tag{83}$$

where we have made use of the fact that, up to negligible terms for N large, $t^{(\mathrm{F})}/t_c^{(\mathrm{F})} = \beta/\beta_c^{(\mathrm{F})}$. By using $p_i \simeq \langle k_i \rangle / \sum_j \langle k_j \rangle$, $\langle k_i \rangle$ being the average degree of the vertex i, we can express approximately Eqs. (82) and (83) in terms of the vertex degree as

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} \simeq \delta_{i,j} + \frac{t^{(\mathrm{F})}}{t_c^{(\mathrm{F})}} \frac{(1-2\mu)}{(1-\mu)^2\left(1 - t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)} \frac{\langle k_i \rangle \langle k_j \rangle}{\langle k \rangle_P^2 N} \tag{84}$$

for $2\mu < 1$, and

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} \simeq \delta_{i,j} + \frac{\beta}{\beta_c^{(\mathrm{F})}} \frac{(2\mu-1)}{(1-\mu)^2\left(1 - \beta/\beta_c^{(\mathrm{F})}\right)} \frac{\langle k_i \rangle \langle k_j \rangle}{\langle k \rangle_P^2 N^{2\mu}} \tag{85}$$

for $2\mu > 1$. However, by using only Eqs. (75) and (79), we can get the correlation function in a form that is completely independent of the form for the p_i's,

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} \simeq \delta_{i,j} + \frac{t^{(\mathrm{F})}}{\left(1 - t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)} \frac{\langle k_i \rangle \langle k_j \rangle}{\langle k \rangle_P N}. \tag{86}$$

Comparison of Eq. (86) with Eqs. (82) and (83) shows that the strongest correlations involve the nodes i's with the highest degree with $\langle k_i \rangle \sim O(N^\mu)$ for $\mu < 1/2$ ($\gamma > 3$) and $\langle k_i \rangle \sim O(N^{1/2})$ for $\mu \geqslant 1/2$ ($\gamma \leqslant 3$).

B. "Gas" of dimers in a scale-free network

Here, we consider the case in which $(\mathcal{L}_0, \Gamma_0)$ is a set of N disconnected dimers (so that there are $2N$ sites). This case represents the simplest example with $J_0 \neq 0$ in which $m_{0i}(\beta J_0^{(\Sigma)}; \{\beta h_j\})$ can be exactly calculated. We have

$$m_{0i}\left(\beta J_0^{(\Sigma)}; \{\beta h_j\}\right) = \frac{\tanh(\beta h_i) + t_0^{(\Sigma)} \tanh(\beta h_{j_0(i)})}{1 + t_0 \tanh(\beta h_i) \tanh(\beta h_{j_0(i)})}, \tag{87}$$

where $t_0^{(\Sigma)} = \tanh(\beta J_0^{(\Sigma)})$ and $j_0(i)$ stands for the first neighbor of i. By derivation, we get the correlation function of the pure model $\tilde{\chi}_{0;i,j}$ which, in the P region, takes the form

$$\tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}; 0\right) = \begin{cases} 1, & j = i \\ t_0^{(\Sigma)}, & j = j_0(i). \end{cases} \tag{88}$$

Therefore, for the critical surface, we have

$$c t_c^{(\Sigma)} N \left[\sum_i p_i^2 + t_0^{(\Sigma)} \sum_i p_i p_{j_0(i)} \right] = 1. \tag{89}$$

With respect to the critical surface of the model with $J_0 = 0$ (the above Viana-Bray case), we see in Eq. (89) the presence of a term proportional to $t_0^{(\Sigma)}$. How much this term affects $t_c^{(\Sigma)}$ depends on how the dimers are placed, i.e., on how we choose the first neighbors $\{j_0(i)\}$. Since by definition the dimers are not connected, in general, for $j_0(i)$ we can take $j_0(i) = i + k$, mod N where k is a constant integer in the range $[1, N]$. The exact evaluation of $t_c^{(\Sigma)}$ for N large remains simple only if k does not depend on N or $k = O(N)$. Under the choices (10) and (11), for the former case we get

$$c t_c^{(\Sigma)} \frac{\left(1 + t_{c0}^{(\Sigma)}\right)(1-\mu)^2}{(1-2\mu)} (1 - N^{2\mu-1}) = 1, \tag{90}$$

whereas for the latter case, the critical surface remains not affected by $t_0^{(\Sigma)}$ as in Eq. (80). When k does not grow with N, for the correlation function for the $\Sigma = \mathrm{F}$ case in the P region for $2\mu < 1$, we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} = \tilde{\chi}_{0;i,j} + \frac{t^{(\mathrm{F})}}{t_c^{(\mathrm{F})}} \frac{(1-2\mu)}{\left(1 + t_0^{(\mathrm{F})}\right)\left(1 - t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)} \times \frac{\left\{i^{-\mu} + t_0^{(\mathrm{F})}[j_0(i)]^{-\mu}\right\}\left\{j^{-\mu} + t_0^{(\mathrm{F})}[j_0(j)]^{-\mu}\right\}}{N^{1-2\mu}}, \tag{91}$$

whereas for $2\mu > 1$ at a temperature scaling with the critical one, we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} = \tilde{\chi}_{0;i,j} + \frac{\beta}{\beta_c^{(\mathrm{F})}} \frac{(2\mu-1)}{\left(1 + t_0^{(\mathrm{F})}\right)\left(1 - \beta/\beta_c^{(\mathrm{F})}\right)} \times \left\{i^{-\mu} + t_0^{(\mathrm{F})}[j_0(i)]^{-\mu}\right\}\left\{j^{-\mu} + t_0^{(\mathrm{F})}[j_0(j)]^{-\mu}\right\}. \tag{92}$$

Similar expressions hold for the correlation in the case in which $k = O(N)$, the only difference being the absence of the prefactor $1/(1 + t_0^{(\mathrm{F})})$. More, in general, independently of the form for the p_i's, in terms of the average degrees we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})} \simeq \delta_{i,j} + \frac{t^{(\mathrm{F})}}{\left(1 - t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)} \times \frac{\left[\langle k_i \rangle + t_0^{(\mathrm{F})} \langle k_{j_0(i)} \rangle\right]\left[\langle k_j \rangle + t_0^{(\mathrm{F})} \langle k_{j_0(j)} \rangle\right]}{\langle k \rangle_P N}. \tag{93}$$

C. A one dimensional chain through the scale-free network

Here, we consider the case in which $(\mathcal{L}_0, \Gamma_0)$ is a a one dimensional chain with periodic boundary conditions and such that the first site of the chain corresponds to the site $i = 1$ of the static network, the second site of the chain corresponds to the site $i = 2$ of the static network, and so on. As we have

learned in Sec. III C, this or any other choice will not alter the critical behavior of the whole system that remains the same as in the absence of the chain. In the P region, the correlation function of the pure model is given by

$$\tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)};0\right)=\left[t_0^{(\Sigma)}\right]^{|j-i|}, \tag{94}$$

from which [by using Eq. (41)] we get the critical surface

$$ct_c^{(\Sigma)}N\sum_{i,j}p_ip_j\left[t_0^{(\Sigma)}\right]^{|j-i|}=1. \tag{95}$$

Let us consider the choices (10) and (11). For $|t_0^{(\Sigma)}|\ll 1$, we need to keep track only of the term $O(t_0^{(\Sigma)})$ and, for N large, we get

$$ct_c^{(\Sigma)}\frac{\left(1+2t_{c0}^{(\Sigma)}\right)(1-\mu)^2}{(1-2\mu)}(1-N^{2\mu-1})=1. \tag{96}$$

In general, Eq. (96) is exact only in the region $2\mu\geqslant 1$ so that $t_{c0}^{(\Sigma)}\to 0$ for $N\to\infty$. Notice the difference with respect to the gas of dimers case in Eq. (90) for the presence of a factor 2 in front of the term proportional to $t_{c0}^{(\Sigma)}$. When $|t_0^{(\Sigma)}|\ll 1$, for the correlation function for the $\Sigma=\mathrm{F}$ case in the P region with $2\mu<1$, we have

$$\tilde{\chi}_{i,j}^{(\mathrm{F})}\simeq\left[t_0^{(\mathrm{F})}\right]^{|j-i|}+\frac{t^{(\mathrm{F})}}{t_c^{(\mathrm{F})}}\frac{(1-2\mu)}{\left(1-t^{(\mathrm{F})}/t_c^{(\mathrm{F})}\right)}\times\frac{\left(1+2t_0^{(\mathrm{F})}\right)(ij)^{-\mu}}{N^{1-2\mu}}, \tag{97}$$

whereas for $2\mu>1$ at a temperature scaling with the critical one, we have (with a better approximation)

$$\tilde{\chi}_{i,j}^{(\mathrm{F})}\simeq\left[t_0^{(\mathrm{F})}\right]^{|j-i|}+\frac{\beta}{\beta_c^{(\mathrm{F})}}\frac{(2\mu-1)}{\left(1-\beta/\beta_c^{(\mathrm{F})}\right)}\times\left(1+2t_0^{(\mathrm{F})}\right)(ij)^{-\mu}, \tag{98}$$

where we have approximated $i+1\simeq i-1\simeq i$. The analytical evaluation of the left-hand side of Eq. (95) for $t_0^{(\Sigma)}$ finite remains a difficult task and we have to resort to a numerical evaluation at a sufficiently large value of N such that finite size effects become negligible. As we have just learned, finite size effects can have a very slow relaxation rate in scale-free graphs; in evaluating the correlation functions, when $\mu<1/2$ ($\gamma>3$), they decay as slowly as $1/N^{1-2\mu}$, while they persist even in the thermodynamic limit when $\mu>1/2$ ($\gamma\leqslant 3$). As we have seen above, however, we can easily handle the latter case since $t_0^{(\Sigma)}$ is always small. We see then that the most difficult numerical task in the evaluation of the left-hand side of Eq. (95), as well as in general formulas involving sums of correlation functions, occurs in the case of a distribution with $\mu\to 1/2^-$ ($\gamma\to 3^-$).

In Fig. 2, we plot simulations for the susceptibility χ and for the Binder cumulant U [35], respectively, as a function of the temperature T for several system sizes N and compare the location of the maximums with the theoretical T_c evaluated at a very large value of N where we observe stationarity within the statistical errors. Finally, in Fig. 3, for growing but finite sizes N, we plot the position of the "finite size $T_c(N)$," defined as the position of the maximum of the susceptibility χ with respect to the temperature. We evaluate such quantities for

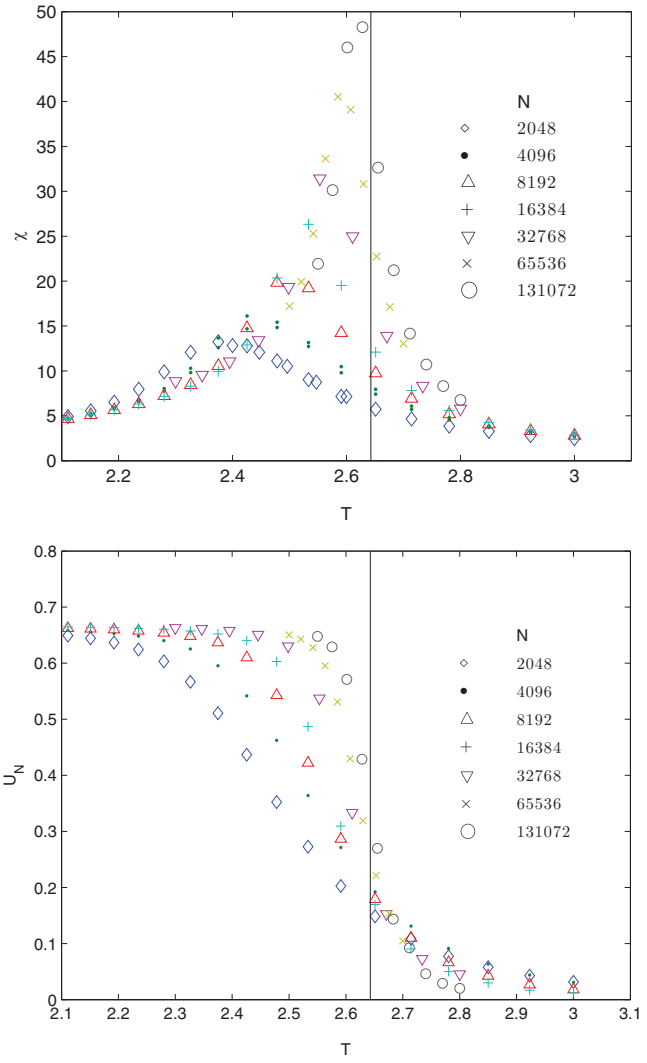

FIG. 2. (Color online) Plots of the susceptibility χ (top panel) and of the Binder cumulant U (bottom panel) as a function of the temperature T for the random models (3)–(5) in which $(\mathcal{L}_0,\Gamma_0)$ is a one dimensional chain and the random network is generated via the choices (10) and (11) with $\mu=1/3$ (corresponding to $\gamma=4$). The other parameters of the model are $c=1$, $J=J_0=1$, and $p=0$. The vertical line comes from the solution of Eq. (95) with $N=131\,072$.

both simulations and theoretical data of the same system as a function of $N^{1/2}$. Note that the latter evaluation coincides simply with the solution coming from Eq. (95). From Fig. 3, we find confirmation of two facts: (i) Eq. (95) (as well as all the effective field theory in general) has a clear meaning also at finite sizes; (ii) since for a mean-field universality class it is expected to be at criticality $\tilde{\chi}_c(N)\sim N^{1/2}$ [36], as also confirmed in [31], and since, on the other hand, from Eq. (60) for finite N, we have $\tilde{\chi}_c(N)\sim O(1)/[T_c(N)-T_c]$, we get $T_c(N)\sim T_c+O(1)/N^{1/2}$, in accordance with Fig. 3.

V. APPLICATION TO NETWORK DESIGN

As we have seen in Sec. III B 2, if $(\mathcal{L}_0,\Gamma_0)$ is a homogeneous graph, i.e., its adjacency matrix $c_{0;i,j}$ has some periodicity,

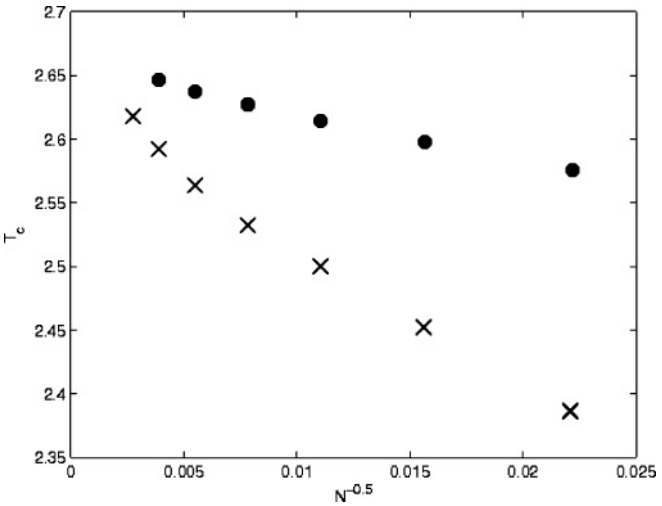

FIG. 3. Plots of the "finite size critical temperature" $T_c(N)$ as a function of the system size N for theoretical (dots) and for simulation data (crosses) of the same system of Fig. 2. We stress that, although the accordance between the theoretical and the simulation data is poor for relatively small values of N, as explained at the end of Sec. IV C, the two trends for large N must fit with a $N^{1/2}$ behavior and approach the same value in the limit $N \to \infty$. The figure confirms our analysis.

clustering increases the percolation threshold. A different question arises instead if between the $\{p_i\}$ and the $\{c_{0;i,j}\}$ there is some correlation. Given the desired degree sequence and then the weights $\{p_i\}$, we see that, if $(\mathcal{L}_0,\Gamma_0)$ is not translational invariant, we can optimize the percolation by labeling the sites in such a way that the functional $F_\infty(\{p_i\})$ is maximized, where [as before here p is the probability by which each link of the pure graph $(\mathcal{L}_0,\Gamma_0)$ is removed]

$$F_\infty(\{p_i\}) \stackrel{def}{=} cN \sum_{i,j} \tilde{\chi}_{0;i,j}[\tanh^{-1}(1-p);0]p_i p_j. \quad (99)$$

Alternatively, Eq. (99) can be rewritten in terms of the graph elements of $(\mathcal{L}_0,\Gamma_0)$ as

$$F_\infty(\{p_i\}) = cN \sum_{i,j} \left(\delta_{i,j} + \mathcal{N}^{(p)}_{0;i,j}\right) p_i p_j, \quad (100)$$

where $\mathcal{N}^{(p)}_{0;i,j} = 1$ if, in the graph $(\mathcal{L}_0,\Gamma_0)$ from which each link has been removed at random with probability p, between the vertex i and the vertex j there exists at least one path of links, and $\mathcal{N}^{(p)}_{0;i,j} = 0$ otherwise.

Once the $\{p_i\}$ that optimizes $F_\infty(\{p_i\})$ has been found through a suitable labeling, the corresponding network will have, in general, a percolation threshold c_c given by the equation $F_\infty(\{p_i\}) = 1$ that is a minima with respect to all the possible $N!$ labelings. From Eq. (99), we see that a simple approximate heuristic receipt to approach this optimum consists in choosing a labeling of the weights $\{p_i\}$, $i_1, i_2, \ldots, i_N$, such that

$$p_i \geqslant p_j, \quad \Leftrightarrow \quad \tilde{\chi}_{0;i} \geqslant \tilde{\chi}_{0;j}, \quad \forall i,j \in \mathcal{L}_0, \quad (101)$$

where $\tilde{\chi}_{0;i}[\tanh^{-1}(1-p);0]$ stands for the total correlation of the graph $(\mathcal{L}_0,\Gamma_0)$ at zero temperature at the point i:

$$\tilde{\chi}_{0;i} \stackrel{def}{=} \sum_j \tilde{\chi}_{0;i,j}[\tanh^{-1}(1-p);0] \quad (102)$$

or, alternatively,

$$\tilde{\chi}_{0;i} = 1 + \sum_j \mathcal{N}^{(p)}_{0;i,j}, \quad (103)$$

i.e., $\tilde{\chi}_{0;i}$ can be seen as the average total number of nodes connected to i (including the node i itself) in the graph $(\mathcal{L}_0,\Gamma_0)$ from which each link has been removed at random with probability p.

Our optimal design problem can be precisely formulated as follows. Given a graph $(\mathcal{L}_0,\Gamma_0)$ in which each link is removed with probability p, and given a desired additional degree sequence $\{\bar{k}_i\}$ (proportional to the weights $\{p_i\}$) having an average connectivity c, we have to place the $L = cN/2$ additional "long-range" links on $(\mathcal{L}_0,\Gamma_0)$ in such a way that the resulting graph has a maximal percolating cluster. Within our effective field theory, this task amounts to say that $F_\infty(\{p_i\})$ is maximized. As we have explained in [37], however, a network, at least in the absence of congestion [38], benefits from optimal communication features at the percolation point. Adding further links after this point makes the network less sensitive to signals. On the other hand, if, for a given value of c, we have found an optimal labeling that maximizes $F_\infty(\{p_i\})$, from Eq. (99) we see that changing only c will leave still the choice of the labeling as an extremal choice for $F_\infty(\{p_i\})$. Therefore, we can speak of label optimization for $F_\infty(\{p_i\})$ regardless of the value of c. In particular, after finding the optimal labeling, we will be free to choose for c a value such that $F_\infty(\{p_i\}) = 1$, so that we will be in the percolation threshold. Among all the other possible labelings, the network will have a minimal percolation threshold. This implies that, for the found optimal labeling, the graph will benefit from optimal communication features but with minimal cost (if the cost is given by c). Our design strategy is therefore in the same philosophy of [39], where the optimality was defined with respect to synchronization. Although, in general, these two different criteria of design may give rise to different networks, we argue that, in the absence of congestion, some general properties about efficient communication are shared.[5]

A particularly interesting case is the one in which the graph $(\mathcal{L}_0,\Gamma_0)$ is split into disjoint subsets, which we could then call isolated communities. In [37], we had considered the problem of percolation for a generic set of n communities, isolated or not, in which the additional "long-range" links were defined through an additional $n \times n$ matrix $\boldsymbol{c}$ of intraconnectivities (inside the community) and interconnectivities (among the communities). In that problem, the unknown quantity to be found was the critical matrix $\boldsymbol{c}$ at which percolation sets in,

[5]We think that if our optimal-percolation criterion is equipped with further constraints on the $\{p_i\}$, the two keys of design would share more and more properties.

and it was easy to find that the critical $\boldsymbol{c}$ must satisfy the equation

$$\det(\mathbf{1} - \tilde{\boldsymbol{\chi}}_{\mathbf{0}} \cdot \boldsymbol{c}) = 0, \tag{104}$$

$\tilde{\boldsymbol{\chi}}_{\mathbf{0}}$ being the matrix of the relative intrasusceptibilities and intersusceptibilities among the communities. In particular, Eq. (104) for the case in which $(\mathcal{L}_0,\Gamma_0)$ is simply a disjoint set of nodes becomes

$$\det(\boldsymbol{c}) = 1, \tag{105}$$

which constitutes a clean generalization of the well known percolation threshold $c = 1$ of the case $n = 1$ corresponding to the Erdös-Rényi random graph [15]. However, Eq. (104) is a single equation in the $n \times n$ unknown matrix elements of $\boldsymbol{c}$, $c^{(l,m)}$, therefore, there are infinite solutions for $n > 1$. Given the matrix $\tilde{\boldsymbol{\chi}}_{\mathbf{0}}$, now, the analogous optimization problem that we have defined before amounts to look for the matrix $\boldsymbol{c}$ that gives the maximum value of the largest eigenvalue of the matrix $\tilde{\boldsymbol{\chi}}_{\mathbf{0}} \cdot \boldsymbol{c}$ under the constraint that the total cost is fixed: $\sum_{l,m} c^{(l,m)}/n = c$ [40]. In general, in this kind of problem, one can find several solutions that represent the local maximum, but the asymmetric ones, if any, are those that guarantee better communication performance [37]. This fact is reminiscent of the so called "starlike" configuration, which is known to provide the best communication and searchability performance in the absence of congestion [38]. In the problem considered in [37], however, the additional "long-range" links were uniformly distributed among the communities, although with the use of $n \times n$ different average connectivities $c^{(l,m)}$. Here we face instead the problem in which we have one single additional connectivity c, but the "long-range" links follow a generic desired degree sequence $\{p_i\}$, which, in particular, can be scale free. We observe that, in the case in which the graph $(\mathcal{L}_0,\Gamma_0)$ is split into n disjoint communities $(\mathcal{L}_0^{(l)},\Gamma_0^{(l)})$, $l = 1,\ldots,n$, the heuristic solution (101) provides the exact minima for $F_\infty(\{p_i\})$ when all the communities have equal size and are internally homogeneous (i.e., $\tilde{\chi}_{0;i,j}$ is the same for any $i,j \in \mathcal{L}_0^{(l)}$).

Let us consider, for example, the case in which we have n disjoint communities of size $N^{(l)}$, $l = 1,\ldots,N$, such that $N = \sum_l N^{(l)}$. Let us suppose that each community $(\mathcal{L}_0^{(l)},\Gamma_0^{(l)})$ consists in a homogeneous random graph with average connectivity $c_0^{(l)} < 1$ [we want to consider only situations in which the set $(\mathcal{L}_0,\Gamma_0)$ is not already a percolating cluster]. From Eq. (77), we have simply

$$\tilde{\chi}_{0;i,j} = \delta_{i,j} + \frac{c_0^{(l)}}{N^{(l)}\left[1 - c_0^{(l)}\right]}, \quad i,j \in \mathcal{L}_0^{(l)}, \quad l = 1,\ldots,n, \tag{106}$$

from which we get the following total correlator [Eq. (102) with the choice $p = 0$], which depends only on the community index l:

$$\tilde{\chi}_0^{(l)} \overset{def}{=} \tilde{\chi}_{0;i} = 1 + \frac{c_0^{(l)}}{1 - c_0^{(l)}}, \quad i \in \mathcal{L}_0^{(l)}, \quad l = 1,\ldots,n. \tag{107}$$

Taking into account that the n communities of $(\mathcal{L}_0,\Gamma_0)$ do not interact (there is no interlink), by plugging Eq. (106) in Eq. (99), we have

$$F_\infty(\{p_i\}) = cN\sum_i p_i^2 + \sum_{l=1}^{n} \frac{cc_0^{(l)}}{\alpha^{(l)}\left[1 - c_0^{(l)}\right]}\left[\left(\sum_{i\in\mathcal{L}_0^{(l)}} p_i\right)^2 - \sum_{i\in\mathcal{L}_0^{(l)}} p_i^2\right], \tag{108}$$

where $\alpha^{(l)} \overset{def}{=} N^{(l)}/N$. For N large, Eq. (108) becomes

$$F_\infty(\{p_i\}) = cN\sum_i p_i^2 + c\sum_{l=1}^{n} \frac{c_0^{(l)}}{\alpha^{(l)}\left[1 - c_0^{(l)}\right]}\left(\sum_{i\in\mathcal{L}_0^{(l)}} p_i\right)^2, \tag{109}$$

which, in turn, can be rewritten in terms of the $\tilde{\chi}_0^{(l)}$ as

$$F_\infty(\{p_i\}) = cN\sum_i p_i^2 + c\sum_{l=1}^{n} \frac{\tilde{\chi}_0^{(l)} - 1}{\alpha^{(l)}}\left(\sum_{i\in\mathcal{L}_0^{(l)}} p_i\right)^2. \tag{110}$$

From Eq. (110), we see that the heuristic solution (101) provides manifestly the global maximum for $F_\infty(\{p_i\})$ only when all the communities have the same size $\alpha^{(l)} = 1/n$, otherwise, the exact global maximum will be provided by the labelings of the p_i's such that

$$p_i \geqslant p_j, \quad \Leftrightarrow \quad \frac{\tilde{\chi}_0^{(l)} - 1}{\alpha^{(l)}} \geqslant \frac{\tilde{\chi}_0^{(m)} - 1}{\alpha^{(m)}}, \quad \forall i \in \mathcal{L}_0^{(l)}, \forall j \in \mathcal{L}_0^{(m)}. \tag{111}$$

Of course, due to the homogeneity of the communities, we have at least as many equivalent global maxima as $\prod_{l=1}^{n} N^{(l)}!$ (if the ratios $[\tilde{\chi}_0^{(l)} - 1]/\alpha^{(l)}$ are not all different, the number of equivalent global maxima is greater).

In the above example, we had three important simplifications: (i) the communities were not interacting; (ii) each community was homogeneous; and (iii) we were able to calculate analytically the terms $\tilde{\chi}_{0;i,i}$. In the most general case, none of the above conditions are satisfied. In particular, when condition (iii) is not satisfied, to calculate the total correlator $\tilde{\chi}_0^{(l)}$ of the lth community, defined as

$$\tilde{\chi}_0^{(l)} \overset{def}{=} \sum_{i,j\in\mathcal{L}_0^{(l)}} \tilde{\chi}_{0;i,j}, \quad l = 1,\ldots,n \tag{112}$$

we have to resort to a Monte Carlo strategy either by an Ising model at low temperature (simulated annealing) [in view of Eq. (99)] or by a direct graph analysis [in view of Eq. (100)]. However, if conditions (i) and (ii) are satisfied, Eq. (110) is still exact and the global maximum will be given by Eq. (111) with the total correlator defined by Eq. (112). It is interesting to note that if, as usually happens, the communities are hierarchically organized in nested communities at deeper and deeper levels,

and conditions (i) and (ii) are still satisfied at each level of the hierarchy, we can iterate the above procedure through a natural generalization of Eqs. (110)–(112) at each level of the hierarchy. For example, if the communities are organized into two levels, i.e., $(\mathcal{L}_0,\Gamma_0)$ is split in n_1 communities $l_1 = 1,\ldots,n_1$, each one in turn split in n_{l_1} communities as

$$\begin{aligned}(\mathcal{L}_0,\Gamma_0) &= \cup_{l_1=1}^{n_1}\left(\mathcal{L}_0^{(l_1)},\Gamma_0^{(l_1)}\right)\\ &= \cup_{l_1=1}^{n_1}\cup_{l_2=1}^{n_{l_1}}\left(\mathcal{L}_0^{(l_1,l_2)},\Gamma_0^{(l_1,l_2)}\right),\end{aligned} \tag{113}$$

it is then easy to see that Eq. (110) generalizes to

$$\begin{aligned}F_\infty(\{p_i\}) &= cN\sum_i p_i^2\\ &+ c\sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_{l_1}}\frac{\tilde{\chi}_0^{(l_1,l_2)}-1}{\alpha^{(l_1,l_2)}}\left(\sum_{i\in\mathcal{L}_0^{(l_1,l_2)}} p_i\right)^2,\end{aligned} \tag{114}$$

which has a global maximum in correspondence of the following labeling, natural generalization of Eq. (111):

$$\begin{aligned}p_i \geqslant p_j, \quad \Leftrightarrow \quad \frac{\tilde{\chi}_0^{(l_1,l_2)}-1}{\alpha^{(l_1,l_2)}} \geqslant \frac{\tilde{\chi}_0^{(m_1,m_2)}-1}{\alpha^{(m_1,m_2)}},\\ \forall i \in \mathcal{L}_0^{(l_1,l_2)},\ \forall j \in \mathcal{L}_0^{(m_1,m_2)},\end{aligned} \tag{115}$$

where now the total correlators $\tilde{\chi}_0^{(l_1,l_2)}$ and the coefficients $\alpha^{(l_1,l_2)}$ are defined as

$$\tilde{\chi}_0^{(l_1,l_2)} \stackrel{def}{=} \sum_{i,j\in\mathcal{L}_0^{(l_1,l_2)}} \tilde{\chi}_{0;i,j}, \tag{116}$$

$$\alpha^{(l_1,l_2)} \stackrel{def}{=} \frac{N}{N^{(l_1,l_2)}}, \tag{117}$$

with $N^{(l_1,l_2)} \stackrel{def}{=} |\mathcal{L}_0^{(l_1,l_2)}|$.

Whatever the graph $(\mathcal{L}_0,\Gamma_0)$ may be, the task to compute via a Monte Carlo method the total correlators usually requires a computational cost, which grows only polynomially in the system size N. A serious problem comes, however, when conditions (i) or (ii) are not satisfied. In this case, in fact, the heuristic solution (111) (or its generalization to the hierarchical case) in general will not provide the global maximum for $F_\infty(\{p_i\})$. As an intermediate situation, it may happen that condition (i) is not exactly satisfied, but the interaction among different communities is weak so that the heuristic solution (111) (or its generalizations), via the evaluation and comparison of the total correlators, is still a good starting point for the numerical search of the exact global maximum of $F_\infty(\{p_i\})$, especially when also condition (ii) is almost satisfied. However, when the communities are well connected to each other, or there is no community structure at all, $F_\infty(\{p_i\})$ in general presents an exponential number of local maximum and, in fact, the computational complexity of the search for the global maximum of $F_\infty(\{p_i\})$ becomes equivalent to the traveling salesman problem, which is an NP-hard problem [41]. In this case, in the graph $(\mathcal{L}_0,\Gamma_0)$ there is a high degree of frustration and the heuristic ansatz (101) might be very far, not only from the exact global solution, but in general also from the local solutions. We conclude, however, by stressing that, despite this worst case scenario for the most general optimization problem in which one is forced to check for almost all the possible $N!$ labelings of the p_i's, the optimization of $F_\infty(\{p_i\})$ remains still exponentially advantageous with respect to a direct inspection (in which c is supposed to be given) of all the possible graphs that one can build up by adding $L = cN/2$ long-range links over the graph $(\mathcal{L}_0,\Gamma_0)$. In fact, given c, if we evaluate the number of ways $\mathcal{N}_c$ to lie $L = cN/2$ long-range links among N nodes, for c finite and N large, we get

$$\begin{aligned}\mathcal{N}_c &= \binom{\frac{N(N-1)}{2}}{\frac{cN}{2}}\\ &\sim \exp\left[\frac{N(N-1)}{2} + \left(\frac{c}{2}-1\right)N\log(N)\right] \gg N!.\end{aligned} \tag{118}$$

VI. CONCLUSIONS

In this paper, we have considered in detail, and in a more general framework, the heterogeneous small-world model that was briefly presented in Ref. [22], providing now all the complete proofs and new applications. By using an effective field theory, we prove in particular that the critical behavior is never affected by the presence of short loops (see Table I). We then apply the general result to the study of percolation, correlation functions, and network design.

By studying the percolation, we have shown, by considering several analytically solvable examples, the role played by short loops in modifying the percolation threshold in networks. In particular, we have seen how the presence of short loops increases the percolation point [see Eq. (58)].

By studying the correlation functions, we have found that, for a scale-free network with or without short loops, finite size effects can be very strong [see Table II and Eqs. (36) and (78)]. Moreover, when γ, the exponent of the degree distribution, is as small as $\gamma \leqslant 3$, the finite size effects become persistent even in the thermodynamic limit, with the strongest correlations being those among hubs. We stress that this is true even in the paramagnetic region and with or without short-range couplings, contradicting then the common opinion that correlations in purely mean-field models always disappear in the thermodynamic limit.[6]

Finally, we have seen that the formula for the percolation threshold suggests a natural way to optimize the communication features among communities even if they interact. We propose and discuss the efficiency of a heuristic solution [see Eqs. (101)–(103)] at several levels: isolated and homogeneous communities, weakly interacting communities, and ill defined communities. The worst case scenario, in which there is no evident community structure, is an NP-hard problem equivalent to the traveling salesman problem, nevertheless, the use of the formula is still exponentially convenient with

[6]This scenario, however, is compatible with the fact that, when $\gamma \leqslant 3$, a network can be ultra-small-world with an average distance between nodes, which can be of the order $\log[\log(N)]$, or even finite in the thermodynamic limit [2].

respect to a direct inspection of the network. We think that, at least in the absence of load congestion, our algorithm can find important real-world applications.

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APPENDIX A: BOUNDING $\mathcal{N}_N$

By using, as in [21], the approximation $\bar{k}_i / \sum_j \bar{k}_j \sim p_i$, where $\bar{k}_i$ is the average degree of the vertex i, from Eq. (8) we have

$$\mathcal{N}_N = \sum_{i<j} \theta\left(\frac{\bar{k}_i \bar{k}_j}{Nc} - 1\right), \tag{A1}$$

which can be rewritten as

$$\mathcal{N}_N = \frac{N}{2} \sum_i \mathcal{P}\left(k > \frac{Nc}{k_i} | k_i\right) p(k_i), \tag{A2}$$

where $p(k_i)$ is the probability that vertex i has degree k_i, and $\mathcal{P}(k > \frac{Nc}{k_i}|k_i)$ is the conditional probability that, given that the vertex i has degree k_i, a randomly chosen vertex different from i has degree greater than Nc/k_i. Due to the weak degree-degree correlation of the network, from $\mathcal{P}(k > \frac{Nc}{k_i}|k_i) \cong \mathcal{P}(k > \frac{Nc}{k_i})$, from Eq. (A2) we have

$$\mathcal{N}_N = \frac{N}{2} \sum_i p(k_i) \int_{\frac{Nc}{k_i}}^{k_M(N)} dk\, p(k), \tag{A3}$$

where $p(k)$ is the probability that a randomly chosen vertex has degree k and $k_M(N)$ is the maximum allowed degree in the network. Of course, it is always $k_M(N) \leqslant N$. By using now the hypothesis that for k large $p(k) \sim k^{-\gamma}$, we arrive at

$$\begin{aligned} \mathcal{N}_N &< \frac{N}{2(\gamma-1)} \int_1^{k_M(N)} dk\, p(k) k^{\gamma-1} (Nc)^{1-\gamma} \\ &< \frac{N^{2-\gamma} c^{1-\gamma}}{2(\gamma-1)} \log(N), \end{aligned} \tag{A4}$$

where we have used $k_M(N) \leqslant N$.

APPENDIX B: DERIVATION OF THE SELF-CONSISTENT EQUATION

In this appendix, we derive Eqs. (22)–(26). Sometimes, to indicate a link we will use the symbol (i,j), or more simply ij. Let us rewrite explicitly the adimensional Hamiltonian (3) as follows:

$$\begin{aligned} \beta H_{\boldsymbol{c}_0, \boldsymbol{c}} = &- \sum_{(i,j)\in\Gamma_0} (c_{0;ij} \beta J_{0;ij} + c_{ij} \beta J_{ij}) \sigma_i \sigma_j \\ &- \sum_{i<j,(i,j)\notin\Gamma_0} c_{ij} \beta J_{ij} \sigma_i \sigma_j - \beta h \sum_i \sigma_i. \end{aligned} \tag{B1}$$

In [42], we have introduced the following mapping. Given a lattice $\mathcal{L}$ with $N = |\mathcal{L}|$ spins, and a generic quenched Hamiltonian $H_{\tilde{\boldsymbol{J}}}$,

$$\beta H_{\tilde{\boldsymbol{J}}} = - \sum_{i<j} \beta \tilde{J}_{ij} \sigma_i \sigma_j - \beta h \sum_i \sigma_i, \tag{B2}$$

where the couplings $\{\tilde{J}_{ij}\}$ are distributed according to a given distribution $\{d\tilde{\mu}_{ij}\}$, let us consider the two following related nonrandom Ising Hamiltonians with labels $\Sigma = \mathrm{F}$ and SG:

$$\beta H^{(\Sigma)} = - \sum_{(i,j)} \beta \tilde{J}_{ij}^{(\Sigma)} \sigma_i \sigma_j - \beta h \sum_i \sigma_i, \tag{B3}$$

where the effective couplings $\beta \tilde{J}_{ij}^{(\Sigma)}$ are given by

$$\tanh(\beta \tilde{J}_{ij}^{(\Sigma)}) = \int d\tilde{\mu}_{ij}(\tilde{J}_{ij}) \tanh^{l_\Sigma}(\beta \tilde{J}_{ij}), \tag{B4}$$

with $l_\Sigma = 1,2$ for $\Sigma = \mathrm{F}$ or SG, respectively. In [42], we have shown that, if the effective couplings $\beta \tilde{J}_{ij}^{(\mathrm{F})}$ or $\beta \tilde{J}_{ij}^{(\mathrm{SG})}$ are at least $O(1/N)$ on the fully connected graph (also called complete graph) $(\mathcal{L}, \Gamma_f)$, then, in the paramagnetic (P) region, the pure model with the effective Hamiltonian $H^{(\Sigma)}$, with $\Sigma =$ F or SG, gives rise to the same nontrivial part of the free energy (see Appendix C) and the same correlation functions of the original Hamiltonian $H_{\tilde{\boldsymbol{J}}}$, the stable phase between F and SG being determined by the minimum of the corresponding associated free energies $f^{(\mathrm{F})}$ or $f^{(\mathrm{SG})}$. This, in particular, gives us the exact critical surfaces paramagnetic-ferro (P-F) and paramagnetic spin-glass (P-SG) and, by a simple analytic continuation, approximations also out of the P region, which allow us to get the critical behavior. The above condition on the effective couplings can be expressed as an infinite dimensionality of the model. Let us apply the mapping to our small-world scale-free case. The quenched Hamiltonian (B1) can be rewritten in the form (B2) where

$$\tilde{J}_{ij} \overset{def}{=} \begin{cases} c_{0;ij} J_{0;ij} + c_{ij} J_{ij}, & (i,j) \in \Gamma_0 \\ c_{ij} J_{ij}, & (i,j) \notin \Gamma_0. \end{cases} \tag{B5}$$

By applying Eq. (B4) to our case with the independent measures $p_0(c_{0;i,j})$ and $p_{ij}(c_{i,j})$ defined by Eqs. (4) and (5) and with $d\mu_{0ij} = d\mu_0$ and $d\mu_{ij} = d\mu$ being two arbitrary independent measures, we arrive at the following effective couplings:

$$\tanh(\beta \tilde{J}_{ij}^{(\Sigma)}) = \begin{cases} (1-p) \int d\mu_0(J_0) \tanh^{l_\Sigma}(\beta J_0) + O\left(\frac{1}{N}\right), & (i,j) \in \Gamma_0 \\ f(p_i, p_j) \int d\mu(J) \tanh^{l_\Sigma}(\beta J), & (i,j) \notin \Gamma_0. \end{cases} \tag{B6}$$

In particular, for large N, in the region $\mathcal{J}$ where the factorization (6) $f(p_i,p_j)=cNp_ip_j$ takes place, from Eqs. (B6) we get

$$\tilde{J}_{ij}^{(\Sigma)}=\begin{cases}\tanh^{-1}\left[t_0^{(\Sigma)}\right], & (i,j)\in\Gamma_0\\ cNp_ip_jt^{(\Sigma)}, & (i,j)\notin\Gamma_0\end{cases} \tag{B7}$$

where $t_0^{(\Sigma)}$ and $t^{(\Sigma)}$ are defined as in Eqs. (23)–(26).

We have to evaluate the partition function

$$Z^{(\Sigma)}=\sum_{\{\sigma_i\}}e^{-\beta H_0^{(\Sigma)}+t^{(\Sigma)}\sum_{i<j}f(p_i,p_j)\sigma_i\sigma_j+\beta h\sum_i\sigma_i},$$

where

$$H_0^{(\Sigma)}=-\beta J_0^{(\Sigma)}\sum_{(i,j)\in\Gamma_0}\sigma_i\sigma_j.$$

By using Eq. (B7), we rewrite $Z^{(\Sigma)}$ as

$$Z^{(\Sigma)}=\sum_{\{\sigma_i\}}e^{-\beta H_0^{(\Sigma)}+t^{(\Sigma)}cN\sum_{i<j}p_ip_j\sigma_i\sigma_j}\times e^{\beta h\sum_i\sigma_i+O[N^{2-\gamma}\log(N)]}, \tag{B8}$$

where $O[N^\alpha\log(N)]$ stands for the contributions coming from the links (i,j) for which the factorization in the second line of Eq. (B7) is not true and we have used Eq. (A4). For N large but finite, the corrective term $O[N^\alpha\log(N)]$ can always be neglected, the error per spin being of order $O[N^{1-\gamma}\log(N)]$.

In the following, we will suppose that $t^{(\Sigma)}$ is positive. The derivation for $t^{(\Sigma)}$ negative differs from the other derivation just for a rotation of $\pi/2$ in the complex m plane, and leads to the same result that one can obtain by analytically continuing the equations derived for $t^{(\Sigma)}>0$ to the region $t^{(\Sigma)}<0$. By using the Gaussian transformation, we can rewrite $Z^{(\Sigma)}$ as

$$Z^{(\Sigma)}=c_N\sum_{\{\sigma_i\}}e^{-\beta H_0^{(\Sigma)}}\int_{-\infty}^{\infty}dm\,e^{-\frac{1}{2}t^{(\Sigma)}cNm^2}\times e^{\sum_i\left(t^{(\Sigma)}cNmp_i+\beta h\right)\sigma_i}, \tag{B9}$$

where c_N is a normalization constant

$$c_N=\sqrt{\frac{t^{(\Sigma)}cN}{2\pi}},$$

and, in the exponent of Eq. (B9), we have again neglected terms of order $O(1)$. For finite N, we can exchange the integral and the sum over the σ's. By using the definition of the pure model with Hamiltonian H_0, Eq. (2), whose free energy density, for a given coupling βJ_0 and for an arbitrary (inhomogeneous) external field $\{\beta h_i\}$, is indicated by $f_0(\beta J_0,\{\beta h_i\})$, we arrive at

$$Z^{(\Sigma)}=c_N\int_{-\infty}^{\infty}dm\,e^{-NL^{(\Sigma)}(m)}, \tag{B10}$$

where we have introduced the function

$$L^{(\Sigma)}(m)=\frac{1}{2}ct^{(\Sigma)}m^2+\beta f_0\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNmp_j+\beta h\}\right). \tag{B11}$$

By using $\partial_{\beta h_i}N\beta f_0(\beta J_0,\{\beta h_j\})=-m_{0i}(\beta J_0,\{\beta h_j\})$ and $\partial_{\beta h_j};m_0(\beta J_0,\{\beta h_l\})=\tilde{\chi}_{0;i,j}(\beta J_0,\{\beta h_l\})$, where $\tilde{\chi}_{0;i,j}\overset{def}{=}\langle\sigma_i\sigma_j\rangle_0-\langle\sigma_i\rangle_0\langle\sigma_j\rangle_0$, we get

$$L'^{(\Sigma)}(m)=t^{(\Sigma)}c\times\left[m-\sum_i m_{0i}\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNmp_j+\beta h\}\right)p_i\right], \tag{B12}$$

$$L''^{(\Sigma)}(m)=t^{(\Sigma)}c\left[1-t^{(\Sigma)}cN\times\sum_{i,j}\tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNmp_l+\beta h\}\right)p_ip_j\right]. \tag{B13}$$

If the integral in Eq. (B10) converges for any N, by performing saddle point integration, we see that the saddle point $m^{(\Sigma)}$ is solution of the equation

$$m^{(\Sigma)}=\sum_i m_{0i}\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNm^{(\Sigma)}p_j+\beta h\}\right)p_i, \tag{B14}$$

so that, if the stability condition

$$t^{(\Sigma)}cN\sum_{i,j}\tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNm^{(\Sigma)}p_l+\beta h\}\right)p_ip_j<1$$

is satisfied, in the thermodynamic limit we arrive at the following expression for the free energy density $f^{(\Sigma)}$ of the related Ising model:

$$\beta f^{(\Sigma)}=\left[\frac{t^{(\Sigma)}}{2}cm^2+\beta f_0\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNmp_j+\beta h\}\right)\right]_{m=m^{(\Sigma)}}. \tag{B15}$$

Similarly, in the thermodynamic limit,[7] any correlation function $C^{(\Sigma)}$ of the related Ising model is given in terms of the correlation function C_0 of the pure model by the following relation:

$$C^{(\Sigma)}=C_0\left(\beta J_0^{(\Sigma)},\{t^{(\Sigma)}cNmp_j+\beta h\}\right)\big|_{m=m^{(\Sigma)}}. \tag{B16}$$

The saddle point solution $m^{(\mathrm{F})}$ represents the weighted magnetization (29) of the related Ising model, as can be checked directly by deriving Eq. (B15) with respect to βh and by using Eq. (B14). For $\Sigma=\mathrm{SG}$, Eq. (29) remains an ansatz.

If the saddle point equation (B14) has more stable solutions, the "true" free energy and the "true" observable of the related Ising model will be given by Eqs. (B15) and (B16), respectively, calculated at the saddle point solution that minimizes Eq. (B15) itself and that we will indicate with $m^{(\Sigma)}$.

Let us call $\beta_{c0}^{(\Sigma)}$ the inverse critical temperature of the pure model with coupling $J_0^{(\Sigma)}$ and zero external field, possibly

[7]Note, however, that finite size effects are responsible for the critical behavior of the system and, furthermore, as we show in Sec. III D, the case for $2\mu>1$ must be carefully calculated since the correction terms responsible for the critical behavior, for suitable choices of the spin indices i and j, may take values up to $O(1)$.

with $\beta_{c0}^{(\Sigma)} = \infty$ if no phase transition exists in the pure model. As stressed in Sec. III B, for the pure model we use the expression "critical temperature" for any temperature where the magnetization m_0 at zero external field passes from 0 to a nonzero value, continuously or not. Note that, as a consequence, if $J_0^{(\Sigma)} < 0$, we have formally $\beta_{c0}^{(\Sigma)} = \infty$, independently from the fact that some antiferromagnetic order may be not zero in the pure model. Let us start to make the obvious observation that a necessary condition for the related Ising model to have a phase transition at $h = 0$ and for a finite temperature is the existence of some paramagnetic region $\mathrm{P}^{(\Sigma)}$, where $m^{(\Sigma)} = 0$. By expanding for small $m^{(\Sigma)} = 0$, we see from the saddle point equation (B14) that, for $h = 0$, a necessary condition for $m^{(\Sigma)} = 0$ to be a solution is that be $\beta \leqslant \beta_{c0}^{(\Sigma)}$ for any β in $\mathrm{P}^{(\Sigma)}$. In a few lines we will see, however, that the inequality must be strict if $\beta_{c0}^{(\Sigma)}$ is finite, which, in particular, excludes the case $J_0 < 0$ (for which the inequality to be proved is trivial). Let us suppose for the moment that be $\beta_c^{(\Sigma)} < \beta_{c0}^{(\Sigma)}$. For $\beta < \beta_{c0}^{(\Sigma)}$ and $h = 0$, the saddle point equation (B14) has always the trivial solution $m^{(\Sigma)} = 0$, which, according to the stability condition, is also a stable solution if

$$t^{(\Sigma)} cN \sum_{i,j} \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}, \{0\}\right) p_i p_j < 1. \tag{B17}$$

The solution $m^{(\Sigma)} = 0$ starts to be unstable when

$$t^{(\Sigma)} cN \sum_{i,j} \tilde{\chi}_{0;i,j}\left(\beta J_0^{(\Sigma)}, \{0\}\right) p_i p_j = 1. \tag{B18}$$

Equation (B18), together with the constraint $\beta_c^{(\Sigma)} \leqslant \beta_{c0}^{(\Sigma)}$, gives the critical temperature of the related Ising model $\beta_c^{(\Sigma)}$. In the region of temperatures where Eq. (B17) is violated, Eq. (B14) gives two symmetrical stable solutions $\pm m^{(\Sigma)} \neq 0$. Furthermore, from Eqs. (B14) and (B18), we see also that, if we make the very plausible assumption that the number of vertices i for which $p_i \geqslant 1/N$ grows with N as aN [see Eq. (11)], with a asymptotically constant for N large, due to the fact that the pure model has a divergent susceptibility at $\beta_{c0}^{(\Sigma)}$, the case $\beta_c^{(\Sigma)} = \beta_{c0}^{(\Sigma)}$ is impossible unless be $t^{(\Sigma)} = 0$. We have therefore proved that $\beta_c^{(\Sigma)} < \beta_{c0}^{(\Sigma)}$. Note that, for $J_0^{(\Sigma)} \geqslant 0$ and $\beta < \beta_{c0}^{(\Sigma)}$, Eq. (B17) is violated only for $\beta > \beta_c^{(\Sigma)}$, whereas for $J_0^{(\Sigma)} < 0$, Eq. (B17) in general may be violated also in finite regions of the β axis.

APPENDIX C: FREE ENERGY

Concerning the full expression of the free energy density, we proceed as follows. If $\varphi^{(\Sigma)}$ is the high temperature part of the free energy density $f^{(\Sigma)}$ of the related Ising model that we have solved in Appendix B, then

$$\begin{aligned} -\beta f^{(\Sigma)} = & \lim_{N\to\infty} \frac{1}{N} \sum_{(i,j)\in\Gamma_0} \log\left[\cosh\left(\beta J_0^{(\Sigma)}\right)\right] \\ & + \lim_{N\to\infty} \frac{1}{N} \sum_{i<j} \log\left[\cosh\left(ct^{(\Sigma)} N p_i p_j\right)\right] \\ & + \log\left[2\cosh(\beta h)\right] + \varphi^{(\Sigma)}. \end{aligned} \tag{C1}$$

On the other hand, the free energy of the model obeys

$$\begin{aligned} -\beta f = & \log\left[2\cosh(\beta h)\right] + \varphi \\ & + \lim_{N\to\infty} \frac{1}{N} \sum_{(i,j)\in\Gamma_0} \int d\mu_0(J_0)(1-p) \log\left[\cosh(\beta J_0)\right] \\ & + \lim_{N\to\infty} \frac{1}{N} \sum_{i<j} \int d\mu(J) \log\left[\cosh(\beta J)\right] cN p_i p_j. \end{aligned} \tag{C2}$$

Therefore, by using the mapping $\varphi = \varphi^{(\Sigma)}/l_\Sigma$ and $\beta f^{(\Sigma)} = L^{(\Sigma)}(m^{(\Sigma)})$, where $L^{(\Sigma)}$ is given by Eq. (B11), and $m^{(\Sigma)}$ is the solution of the self-consistent Eq. (22), and by choosing Σ according to which is minimum between $L^{(\mathrm{F})}(m^{(\mathrm{F})})$ and $L^{(\mathrm{SG})}(m^{(\mathrm{SG})})$, comparing Eq. (C1) with (C2) we get the total free energy βf. It is clear, however (as already anticipated), that the only part of the free energy that depends on the order parameter and that is therefore responsible for the critical behavior of the system and the correlation functions is φ. The rest of the free energy is important only to calculate the total specific heat.

APPENDIX D: PROOF OF EQ. (73)

Let us start to express the partition function of the pure model in the high temperature expansion. In general, for the partition function Z of an Ising model having a set of links $b \in \Gamma$ taking the couplings $\{J_b\}$ and in the presence of arbitrary external fields $\{h_i\}$, we have

$$\begin{aligned} Z(\{J_b\};\{h_i\}) = & \prod_{b\in\Gamma} \cosh(\beta J_b) \prod_{i=1}^{N} \cosh(\beta h_i) \\ & \times \sum_{\{\sigma_i\}} \prod_{b\in\Gamma} \left[1 + \sigma_{i_b}\sigma_{j_b} \tanh(\beta J_b)\right] \\ & \times \prod_{i=1}^{N} \left[1 + \sigma_i \tanh(\beta h_i)\right], \end{aligned} \tag{D1}$$

where i_b and j_b are the two sites linked by the link b. It is not difficult to recognize that Z can be rewritten as a sum over paths as follows:

$$\begin{aligned} Z(\{J_b\};\{h_i\}) = & \prod_{b\in\Gamma} \cosh(\beta J_b) \prod_{i=1}^{N} \cosh(\beta h_i) \\ & \times \sum_{\gamma\in\mathcal{T}} \prod_{b\in\gamma} t_b \prod_{i\in\partial\gamma} t_i, \end{aligned} \tag{D2}$$

where $\mathcal{T}$ is the set of all possible multipaths on Γ, including then all the possible combinations of closed and open paths; $\partial\gamma$ stands for the subset of vertices that belong to the border of the multipath γ (if it has at least one open path component); and we have introduced the short notations $t_b = \tanh(\beta J_b)$ and $t_i = \tanh(\beta h_i)$. Note that the cardinality $|\partial\gamma|$ is always an even number. We now want to calculate the average magnetization $\langle\sigma_i\rangle$. By derivating Eq. (D2) with respect to βh_i, we get

$$\langle\sigma_i\rangle = t_i + \left(1 - t_i^2\right) \frac{\sum_{\gamma\in\mathcal{T}^{(i)}} \prod_{b\in\gamma} t_b \prod_{j\in\partial\gamma\setminus i} t_j}{\sum_{\gamma\in\mathcal{T}} \prod_{b\in\gamma} t_b \prod_{j\in\partial\gamma} t_j}, \tag{D3}$$

where $\mathcal{T}^{(i)}$ stands for the subset of $\mathcal{T}$ having at least one open path component that passes through the vertex i. As done in

Sec. III D, we can not expand the terms t_i for small h_i, but we can neglect $(t_i)^2$ terms. In other words, we expand $\langle\sigma_i\rangle$ at the least nonzero order (not in the $\{h_j\}$, and nor in the $\{t_b\}$) but in the $\{t_j\}$. Within this approximation, Eq. (D3) becomes

$$\langle\sigma_i\rangle = t_i + \frac{\sum_{\gamma\in\mathcal{T}_1^{(i)}}\prod_{b\in\gamma} t_b t_{j^{(i)}}}{\sum_{\gamma\in\mathcal{C}}\prod_{b\in\gamma} t_b} + O\left(\{t_j^2\}\right), \qquad \text{(D4)}$$

where now $\mathcal{T}_1^{(i)}$ stands for the subset of $\mathcal{T}$ having one and only one open path component that passes through the vertex i, $j^{(i)}$ is second end of this path component passing through i, and $\mathcal{C}$ stands for the set of all the closed multipaths on Γ. Notice that, as anticipated, this latter definition makes the calculation exact with respect to the presence of loops of any length and taking any coupling. Similarly, for the connected correlation function, by deriving once more with respect to βh_j we get

$$\tilde{\chi}_{ij} = \delta_{i,j} + \frac{\sum_{\gamma\in\mathcal{T}_1^{(i,j)}}\prod_{b\in\gamma} t_b}{\sum_{\gamma\in\mathcal{C}}\prod_{b\in\gamma} t_b} + O\left(\{t_j^2\}\right), \qquad \text{(D5)}$$

where $\mathcal{T}_1^{(i,j)}$ stands for the subset of $\mathcal{T}$ having one and only one open path component that passes through both the vertices i and j. From Eq. (D5), one can obtain the susceptibility $\tilde{\chi}$ up to $O(\{t_j^2\})$ terms by summing over i and j and dividing by N. In particular, for a regular lattice, we have

$$\tilde{\chi} = 1 + \frac{\sum_{\gamma\in\mathcal{T}_1^{(i_0)}}\prod_{b\in\gamma} t_b}{\sum_{\gamma\in\mathcal{C}}\prod_{b\in\gamma} t_b} + O\left(\{t_j^2\}\right), \qquad \text{(D6)}$$

where i_0 is an arbitrary vertex chosen as reference. If we now, for a regular lattice, plug in Eq. (D4) in the self-consistent equation (22), use the definition (63) and the property (68), we get the bound

$$m < g(m) + \frac{\sum_{\gamma\in\mathcal{T}_1^{(i_0)}}\prod_{b\in\gamma} t_b g(m)}{\sum_{\gamma\in\mathcal{C}}\prod_{b\in\gamma} t_b} + O\left(\{t_j^2\}\right), \qquad \text{(D7)}$$

which, by using (D5), leads immediately to Eq. (73).

APPENDIX E: DERIVATION OF EQ. (75)

Equations (B10)–(B14) are already in a form able to take into account the presence of an arbitrary inhomogeneous external field $\{h_j\}$; in these equations, we simply have to substitute everywhere in their arguments $\{h\}$ with $\{h_j\}$. Then, by deriving $L^{(\Sigma)}(m)$ with respect to βh_i and by using, as in Appendix B, $\partial_{\beta h_i} N\beta f_0(\beta J_0,\{\beta h_j\}) = -m_{0i}(\beta J_0,\{\beta h_j\})$ and $\partial_{\beta h_j} m_0(\beta J_0,\{\beta h_l\}) = \tilde{\chi}_{0;i,j}(\beta J_0,\{\beta h_l\})$, and the self-consistent equation for the order parameter $m^{(\Sigma)}$, we get immediately

$$m_i^{(\Sigma)} = m_{0i}\left(\beta J_0^{(\Sigma)}, \left\{t^{(\Sigma)} cN m^{(\Sigma)} p_j + \beta h_j\right\}\right), \qquad \text{(E1)}$$

which confirms Eq. (75) for the correlation functions of order $k = 1$. Then, by deriving in turn $m_i^{(\Sigma)}$ with respect to βh_j, and by using

$$\frac{\partial m^{(\Sigma)}}{\partial(\beta h_j)} = \frac{\sum_n p_n \tilde{\chi}_{0;n,j}\left(\beta J_0^{(\Sigma)}; \{N p_q c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right)}{1 - ct^{(\Sigma)} N \sum_{l,n} \tilde{\chi}_{0;l,n}\left(\beta J_0^{(\Sigma)}; \{N p_q c t^{(\Sigma)} m^{(\Sigma)} + \beta h\}\right) p_l p_n}, \qquad \text{(E2)}$$

we reach immediately Eq. (75).

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