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Zeroth law compatibility of nonadditive thermodynamics

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Nonextensive thermodynamics is criticized by the statement that the zeroth law cannot be satisfied with nonadditive composition rules. In this paper we determine the general functional form of those nonadditive composition rules that are compatible with the zeroth law of thermodynamics. We find that this general form is additive for the formal logarithms of the original quantities and the familiar relations of thermodynamics apply to these. Our result offers a possible solution to the long-standing questions about equilibrium between extensive and nonextensive systems or systems with different nonextensivity parameters.

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I. INTRODUCTION

Since Gibbs's famous expression

$$S = \sum_i p_i \ln \frac{1}{p_i} \tag{1}$$

was formulated, several other expressions have been proposed and are in use for the entropy. Many applications stem from the field of theoretical informatics [1,2]. A common motivation in the hunt for further formulas is for consideration of entanglement or correlation between elementary events or states of the system under study. A fairly well known suggestion is due to Rényi; in his formula, a parameter, nowadays denoted by q, occurs [3].

A more recent endeavor was initiated by Tsallis [4,5], namely, to build thermodynamics and statistical physics on a mathematical formulation accounting for the interdependence between alternative states of a complex system, but possibly remaining almost as simple and powerful as the classical theory. Based on the nonadditive nature of the entropy formula promoted by Tsallis [4,6], almost all familiar properties of classical thermodynamics have been questioned, reiterated and are partially still open. One of the open questions relates the maximum entropy principle to the zeroth law of thermodynamics: How can the transitivity property of thermal equilibrium and the definition of the empirical temperature be preserved in this more complex approach? There was, in particular, a sharply formulated critique against using anything other than the classical Boltzmann formula, stating that temperature, heat exchange, and probability factorizability were not clearly conceptualized, and in particular that the parameter q could not be interpreted in a way compatible with the zeroth law [7].

Quite a few attempts have been devoted to the clarification of this issue [8–14]. The generalization of thermostatistics by considering nonadditive entropy functions was proposed long ago [4], but its relation to alternative suggestions [15–18] is not yet completely understood [5]. One challenge to thermodynamics using nonadditive entropy functions is its surmised noncompatibility with the zeroth law. On the other hand, it was shown recently that one can achieve some consequences of these theories—most prominently power-law-tailed equilibrium distributions of the energy in canonical statistical systems—by assuming that the entropy is additive, but other fundamental thermodynamic variables, e.g., the energy, are not [19,20]. Moreover, there are indications that both entropy and energy may be nonadditive [13,21].

In this paper we approach the problem of nonextensive equilibrium in a more general setting: given—as a rule nonadditive—composition rules for the classical extensives, originally treated as additive quantities, what constraints can be derived from the zeroth law for their functional form? Furthermore, what consequences can be formulated for the usefulness of one or another entropy formula, and in particular what is the role of any q-like parameter in the thermodynamical equilibrium state?

II. THE ZEROTH LAW: FACTORIZABILITY

In order to arrive at answers to the questions raised in the Introduction, we briefly repeat the main steps of the classical construction of the zeroth law and then we generalize it to nonadditive composition rules of entropy and energy. We treat these composition rules as independent, including in this way the purely entropic and purely energetic nonadditivity as particular cases. At the end of this process we aim at having clear constraints on the interplay between different nonextensivity parameters in a heterogeneous thermal equilibrium.

In carrying out this program, we regard the problem as a *purely thermodynamic question*, and elaborate the answer without relying on statistical arguments. Instead of additivity we consider general relations for fundamental thermodynamic quantities of composed systems. We refer to these relations as *composition rules*. We accept the maximum entropy principle and investigate the most general form of the composition rules that is compatible with the factorizability of the equilibrium condition.

The basic variables of classical thermodynamics are the so-called extensive physical quantities (X), like the energy (E), particle number of chemical components (N_i), and entropy (S). The adjective "extensive" denotes two different characteristics of these fundamental physical quantities. It is customary to assume that these quantities are added if we put two thermodynamic bodies together:

$$X_{12} = X_1 + X_2. \tag{2}$$

This special property of composition is called *additivity*. On the other hand, traditionally one assumes that these quantities characterize the systems down to the smallest meaningful scale when operating with finite densities of the extensive quantities [22]:

$$\rho_X = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X_i < \infty. \tag{3}$$

Here we divided the body into N different parts and X_i belongs to the ith of them. This property is called *extensivity*. The two properties are related, but not equivalent. If a quantity is additive, then it is extensive, but there are extensive and yet nonadditive quantities, too [5,20].

Another important fundamental group of thermodynamic quantities are called intensives, and their definition is related to thermodynamic equilibrium and to the zeroth and the second laws of thermodynamics. As a first step we do not analyze the relation of these concepts in their utmost generality, but rather start with a simple approach. We rely on the principle that the entropy is a function of the fundamental quantities and it is maximal in equilibrium. In this way we implicitly assume that the entropy is meaningful also (slightly) out of equilibrium. Let us consider the simplest example, where there are only two thermodynamic bodies, both characterized solely by their respective energies E_1 and E_2. By assuming that the energy is additive, we prescribe that the total energy of the composed system of the two bodies E_{12} is given by the formula

$$E_{12}(E_1,E_2) = E_1 + E_2. \tag{4}$$

The bodies are characterized by their respective entropies and we assume that the entropy is additive:

$$S_{12}(E_1,E_2) = S_1(E_1) + S_2(E_2) \tag{5}$$

with $S_i(E_i)$ being the respective equations of state. Maximizing the entropy, while assuming that the total energy of the two bodies is conserved ($dE_{12} = 0$), although an exchange of energy between the bodies is possible, one easily derives that

$$dS_{12}(E_1,E_2) = \frac{\partial S_1}{\partial E_1} dE_1 + \frac{\partial S_2}{\partial E_2} dE_2 = (S_1' - S_2')\, dE_1 = 0, \tag{6}$$

where the prime denotes the derivative of a single variable function. From this, one concludes that equilibrium requires the equality of the derivatives:

$$S_1'(E_1) = S_2'(E_2). \tag{7}$$

The zeroth law law of thermodynamics is formulated as the requirement of *transitivity* of the thermodynamic equilibrium state. This transitivity property implies the existence of an empirical temperature. One can easily see that transitivity is satisfied the conditions above and the derivatives of the single body entropies are suitable empirical temperatures, because they are characteristics of the respective bodies and *do not depend on the properties of the partner body or the interaction*. Furthermore any monotonic function of the single body entropy derivative is an equally usable empirical temperature.

In this classical train of thought the maximum entropy condition written with additive composition rules leads to equation (7) that is *factorizable*: the two sides depend on the quantities of the respective bodies. In the above case this property is trivial, and one realizes that the additivity properties of the energy and the entropy are important ingredients. In this way additivity is a sufficient condition for the zeroth law, but the question is whether it is also necessary.

We investigate now the problem of factorizability of the zeroth law when both the entropy and energy follow nonadditive composition rules. The maximum entropy principle with fixed combined energy leads to vanishing total differentials:

$$\begin{aligned} dS_{12} &= \frac{\partial S_{12}}{\partial S_1} S_1' dE_1 + \frac{\partial S_{12}}{\partial S_2} S_2' dE_2 = 0, \\ dE_{12} &= \frac{\partial E_{12}}{\partial E_1} dE_1 + \frac{\partial E_{12}}{\partial E_2} dE_2 = 0. \end{aligned} \tag{8}$$

Here we assumed that the composition laws for entropy and energy (and other further extensives) are independent, i.e., the following natural assumption about the functional form of the composition rules has been made: $S_{12} = S_{12}(S_1,S_2)$ and $E_{12} = E_{12}(E_1,E_2)$ are functions of the corresponding variables of the subsystems for all possible equations of state $S_1(E_1)$ and $S_2(E_2)$.

A nontrivial solution for the energy changes dE_1 and dE_2 exists if the determinant vanishes. This occurs if

$$\frac{\partial E_{12}}{\partial E_2} \frac{\partial S_{12}}{\partial S_1} S_1' = \frac{\partial E_{12}}{\partial E_1} \frac{\partial S_{12}}{\partial S_2} S_2'. \tag{9}$$

The most general form of the partial derivatives now may include different two-variable functions (indexed with 1 and 2):

$$\begin{aligned} \frac{\partial S_{12}}{\partial S_1} &= F_1(S_1) G_2(S_2) H_1(S_1,S_2), \\ \frac{\partial S_{12}}{\partial S_2} &= F_2(S_2) G_1(S_1) H_2(S_1,S_2), \\ \frac{\partial E_{12}}{\partial E_1} &= A_1(E_1) B_2(E_2) C_1(E_1,E_2), \\ \frac{\partial E_{12}}{\partial E_2} &= A_2(E_2) B_1(E_1) C_2(E_1,E_2). \end{aligned} \tag{10}$$

The vanishing determinant condition (9) requires

$$A_2 B_1 C_2 \times F_1 G_2 H_1 S_1' = A_1 B_2 C_1 \times F_2 G_1 H_2 S_2'. \tag{11}$$

This equation factorizes to (E_1,S_1)- and (E_2,S_2)-dependent terms only if

$$\frac{C_2(E_1,E_2)}{C_1(E_1,E_2)} = \frac{H_2(S_1,S_2)}{H_1(S_1,S_2)}. \tag{12}$$

If one considers that the factorizability of the maximum entropy principle should not depend on any particular form of the equation of state $S(E)$, then the above ratio can only be a constant.[1] Its value can easily be absorbed into one

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[1] This is analogous to the separable wave function ansatz in the quantum mechanical description of the H atom.

of the factorizing component functions, so as an immediate consequence

$$\begin{aligned} C_1(E_1,E_2) &= C_2(E_1,E_2), \\ H_1(S_1,S_2) &= H_2(S_1,S_2). \end{aligned} \tag{13}$$

These equalities are the basis for considering formal logarithms for the entropy and energy separately. The factorized form of the zeroth law in this case,

$$\frac{B_1 F_1}{A_1 G_1} S_1' = \frac{B_2 F_2}{A_2 G_2} S_2', \tag{14}$$

selects out the following choice for the thermodynamic temperature:

$$\frac{1}{T} = \frac{B(E)F(S)}{A(E)G(S)} S'(E). \tag{15}$$

Finally, using the definitions

$$\begin{aligned} \hat{L}(S) &:= \int \frac{F(S)}{G(S)} dS, \\ L(E) &:= \int \frac{A(E)}{B(E)} dE, \end{aligned} \tag{16}$$

we arrive at

$$\frac{1}{T} = \frac{\partial \hat{L}(S)}{\partial L(E)}. \tag{17}$$

The zeroth law requires that this common value be introduced as the reciprocal temperature.

The functions of the original thermodynamical variables $\hat{L}(S)$ and $L(E)$ defined in Eq. (16) can be used to map the original composition rules to a simple addition. That is, due to Eqs. (10) and (13) one has

$$\begin{aligned} C_1 &= \frac{1}{A_1 B_2} \frac{\partial E_{12}}{\partial E_1} = \frac{1}{A_2 B_1} \frac{\partial E_{12}}{\partial E_2} = C_2, \\ H_1 &= \frac{1}{F_1 G_2} \frac{\partial S_{12}}{\partial S_1} = \frac{1}{F_2 G_1} \frac{\partial S_{12}}{\partial S_2} = H_2, \end{aligned} \tag{18}$$

which can be rearranged into the following form:

$$\begin{aligned} \frac{B_1}{A_1} \frac{\partial E_{12}}{\partial E_1} &= \frac{B_2}{A_2} \frac{\partial E_{12}}{\partial E_2}, \\ \frac{G_1}{F_1} \frac{\partial S_{12}}{\partial S_1} &= \frac{G_2}{F_2} \frac{\partial S_{12}}{\partial S_2}. \end{aligned} \tag{19}$$

Utilizing now the definitions (16) for both E_1, E_2 and S_1, S_2 separately, the partial derivatives simplify:

$$\begin{aligned} \frac{\partial E_{12}}{\partial L_1} &= \frac{\partial E_{12}}{\partial L_2}, \\ \frac{\partial S_{12}}{\partial \hat{L}_1} &= \frac{\partial S_{12}}{\partial \hat{L}_2}. \end{aligned} \tag{20}$$

The general solution of such partial differential equations is an arbitrary function of the sum of variables:

$$\begin{aligned} E_{12} &= \Phi(L_1 + L_2), \\ S_{12} &= \Psi(\hat{L}_1 + \hat{L}_2). \end{aligned} \tag{21}$$

In the still quite general case when the Φ and Ψ functions are strictly monotonic, they are invertible. This inverse can be indexed by the composite system, so we arrive at

$$\begin{aligned} L_{12}(E_{12}) &= L_1(E_1) + L_2(E_2), \\ \hat{L}_{12}(S_{12}) &= \hat{L}_1(S_1) + \hat{L}_2(S_2). \end{aligned} \tag{22}$$

Since the $L_i(X_i)$ functions map nonadditive quantities to the addition, they can justifiably be called *formal logarithms*. Since $\hat{L}(S)$ and $L(E)$ are additive quantities, they are also extensive. Therefore Eq. (17) defines $1/T$ as a truly intensive quantity since it is a derivative of an extensive quantity with respect to another extensive quantity.

The extensively studied nonextensive entropy with additive energy composition rule and its reverse, i.e., consideration of a nonadditive energy with additive entropy, are particular cases of the above result. When the composition rule is the addition itself, the corresponding formal logarithms become the respective identity functions, $L(E) = E$ and $\hat{L}(S) = S$. The classical result is recovered when both quantities are composed additively. For the entropic compositon rule Abe has derived the most general functional form based on the factorization of the zeroth law in homogeneous equilibrium [14]. His result (see Eq. (34) in [14]) conforms to ours (22) after the following mapping of the pseudoadditivity rule to the addition:

$$\hat{L}(S) = \frac{1}{\lambda} \ln\left[1 + \lambda H_\lambda(S)\right]. \tag{23}$$

Physically, regarding different pieces of the same material, all $L(E)$ and $\hat{L}(S)$ functions are the same—this is the case of composition rules with a formal logarithm, already proved to emerge in the limit of infinite repetitions of an arbitrary rule [20]. The statistics in the generalized thermodynamical limit of repeated compositions therefore ensures the fulfillment of the zeroth law. In a general setting, however, these functions—or "only" their parameters—may differ. Moreover, the mapping function for the interacting composed system also may differ from both subsystems' corresponding functions.

On the one hand, this leads to a variety of equilibria depending on the types of the subsystems and composite systems (extensive or nonextensive). On the other hand, simple composition laws for homogeneous cases—when all formal logarithm functions are the same with the same value of all parameters—may become more involved when considering a zeroth-law-compatible heterogeneous equilibrium between different systems. Our results above, Eqs. (22) and (17), also answer the long-debated question about equilibrium between extensive and nonextensive systems.

III. THE ZEROTH LAW: TRANSITIVITY

The classical formulation of the zeroth law in thermodynamics emphasizes the transitivity of the equilibrium condition without introducing the concept of entropy [23]: when the subsystems 1 and 2 are in thermal equilibrium and independently the subsystems 2 and 3, then it follows that also the subsystems 1 and 3 are. This is a universal principle.

Our condition above, derived from factorization of the constrained maximum entropy principle, automatically satisfies this transitivity. We demonstrate this with the use of the nonadditive energy composition rule; other cases can be derived analogously. The key observation is that we have established additivity of composite functions of the energies of the respective subsystems, $L_i(E_i)$. Therefore it is natural to assume that these functions are characteristic of the subsystems and only the double-indexed formal logarithms $L_{ij}(E_{ij})$ are characteristic of the interaction between subsystems in equilibrium. In this way all subsystems develop the same individual formal logarithm irrespective of which other system they equilibrate with.

Assumption of the opposite, i.e., a partner-dependent individual formal logarithm, would violate the transitivity. Let us consider the three possible pairings of three subsystems. The composite energies satisfy

$$\begin{aligned} E_{12} &= \Phi_{12}[L_1(E_1) + L_2(E_2)], \\ E_{23} &= \Phi_{23}[\tilde{L}_2(E_2) + L_3(E_3)], \\ E_{13} &= \Phi_{13} - [\tilde{L}_1(E_1) + \tilde{L}_3(E_3)]. \end{aligned} \tag{24}$$

If $\tilde{L} \neq L$, then the equilibrium condition is not automatically transitive. In the special case of additive entropy but nonadditive energy composition considered here, Eq. (9) reads pairwise as

$$\begin{aligned} S_1'(E_1)\frac{\partial E_{12}}{\partial E_2} &= S_2'(E_2)\frac{\partial E_{12}}{\partial E_1}, \\ S_3'(E_3)\frac{\partial E_{23}}{\partial E_2} &= S_2'(E_2)\frac{\partial E_{23}}{\partial E_3}, \\ S_1'(E_1)\frac{\partial E_{13}}{\partial E_3} &= S_3'(E_3)\frac{\partial E_{13}}{\partial E_1}. \end{aligned} \tag{25}$$

From here the ratios of the respective $S'(E)$ factors can be expressed:

$$\begin{aligned} \frac{S_1'(E_1)}{S_2'(E_2)} &= \frac{\frac{\partial E_{12}}{\partial E_1}}{\frac{\partial E_{12}}{\partial E_2}}, \\ \frac{S_2'(E_2)}{S_3'(E_3)} &= \frac{\frac{\partial E_{23}}{\partial E_2}}{\frac{\partial E_{23}}{\partial E_3}}, \\ \frac{S_1'(E_1)}{S_3'(E_3)} &= \frac{\frac{\partial E_{13}}{\partial E_1}}{\frac{\partial E_{13}}{\partial E_3}}, \end{aligned} \tag{26}$$

leading to the condition

$$\frac{\frac{\partial E_{12}}{\partial E_1}}{\frac{\partial E_{12}}{\partial E_2}} \, \frac{\frac{\partial E_{23}}{\partial E_2}}{\frac{\partial E_{23}}{\partial E_3}} = \frac{\frac{\partial E_{13}}{\partial E_1}}{\frac{\partial E_{13}}{\partial E_3}}. \tag{27}$$

This compared with the form (24) reveals the following consistency requirement:

$$\frac{\Phi_{12}' L_1'}{\Phi_{12}' L_2'} \, \frac{\Phi_{23}' \tilde{L}_2'}{\Phi_{23}' L_3'} = \frac{\Phi_{13}' \tilde{L}_1'}{\Phi_{13}' \tilde{L}_3'}. \tag{28}$$

This condition can be reduced easily to obtain

$$\frac{L_1'}{\tilde{L}_1'} = \frac{L_2'}{\tilde{L}_2'} \, \frac{L_3'}{\tilde{L}_3'}. \tag{29}$$

Since this equality is required for any permuted arrangement of the indices 1, 2, and 3, one concludes that the transitivity of thermal equilibrium can only be satisfied if the $\tilde{L}'$ and L' functions are identical. This is a necessary and sufficient condition. Adding the observation that a physically sensible composition rule satisfies the triviality condition, i.e., a composition with zero does not change the value, in general $L(0) = 0$ is required. In this case also the $\tilde{L}$ and L functions themselves are identical. Requiring furthermore that for small energies—when nonadditive effects are, as a rule, relatively reduced—the addition reemerges, $L'(0) = 1$ is also set. In the following discussion we assume that these properties are fulfilled.

IV. THE EXAMPLE OF TSALLIS'S ENTROPY COMPOSITION FORMULA

A nonadditive (and for factorizing probabilities also nonextensive) entropy formula, promoted by Tsallis [4,5], satisfies the following composition rule:

$$S_{12} = S_1 + S_2 + \hat{a} S_1 S_2. \tag{30}$$

Here for brevity we used the notation $\hat{a} = 1 - q$. The additive entropy systems have $\hat{a} = 0$; nonextensive systems another value of this parameter. The question of whether two such systems with different $\hat{a}$ parameters can come into thermal equilibrium compatible with the zeroth law has been raised on quite a few occasions [7,21].

According to our main result in this paper the answer is affirmative for a zeroth-law-compatible equilibrium, provided that the general case is treated via the formal logarithm. The formal logarithm for the above rule is easy to derive by observing that

$$\begin{aligned} 1 + \hat{a} S_{12} &= 1 + \hat{a} S_1 + \hat{a} S_2 + \hat{a}^2 S_1 S_2 \\ &= (1 + \hat{a} S_1)(1 + \hat{a} S_2). \end{aligned} \tag{31}$$

The product is mapped to the addition by the logarithm and scaled to satisfy $\hat{L}'(0) = 1$:

$$\hat{L}(S) = \frac{1}{\hat{a}} \ln(1 + \hat{a} S). \tag{32}$$

This trivial property has been observed several times; see, e.g., Ref. [24]. For a general thermal equilibrium between two subsystems with different values of the nonadditivity parameters $\hat{a}_1$ and $\hat{a}_2$, a zeroth-law-compatible equilibrium emerges—according to Eq. (19)—if

$$\frac{1}{\hat{a}_{12}} \ln(1 + \hat{a}_{12} S_{12}) = \frac{1}{\hat{a}_1} \ln(1 + \hat{a}_1 S_1) + \frac{1}{\hat{a}_2} \ln(1 + \hat{a}_2 S_2). \tag{33}$$

Here one observes that a further nonextensivity parameter, $\hat{a}_{12}$, was also introduced. It is necessary for accounting for nonadditivity in composite systems between elements (atoms, particles) of the different subsystems. See Refs. [25,26] for a parton-cascade-based numerical simulation with a nonadditive energy composition rule. The above relation (33) leads to the direct expression

$$S_{12} = \frac{1}{\hat{a}_{12}}[(1 + \hat{a}_1 S_1)^{\hat{a}_{12}/\hat{a}_1} (1 + \hat{a}_2 S_2)^{\hat{a}_{12}/\hat{a}_2} - 1]. \tag{34}$$

In the case of homogeneously nonadditive subsystems and composed systems one considers $\hat{a}_{12} = \hat{a}_1 = \hat{a}_2 = \hat{a}$ and obtains Eq. (30). We note here that the composition formula (34) does not fulfill the naively expected property when a finite entropy and a zero entropy subsystem are composed, i.e., $S_{12}(S_1,0) \neq S_1$ and $S_{12}(0,S_2) \neq S_2$. This property is, however, trivially satisfied for $\hat{L}_{12}(S_{12})$ when one of the subsystems' formal logarithms, $\hat{L}_1(S_1)$ or $\hat{L}_2(S_2)$, vanishes.

For a composition of two nonadditive subsystems in general one obtains

$$S_{12} = \Psi_{12}[\hat{L}(\hat{a}_1,S_1) + \hat{L}(\hat{a}_2,S_2)]. \tag{35}$$

In the case of $\hat{a}_1 = 0$, i.e., considering thermal equilibrium between an additive and a nonadditive system, one achieves

$$S_{12} = \Psi_{12}[S_1 + \hat{L}(\hat{a}_2,S_2)]. \tag{36}$$

From the viewpoint of zeroth law compatibility it is strongly advised to use the additive formal logarithm of any nonextensive entropy formula. In the case of the Tsallis entropy,

$$S_T = \frac{1}{\hat{a}}\left(\sum_i p_i^{1-\hat{a}} - p_i\right) \tag{37}$$

with $\hat{a} = 1 - q$ and the normalization $\sum_i p_i = 1$, its formal logarithm turns out to be the well known Rényi entropy [3]:

$$S_R = \hat{L}(\hat{a},S_T) = \frac{1}{\hat{a}}\ln(1+\hat{a}S_T) = \frac{1}{1-q}\ln\sum_i p_i^q. \tag{38}$$

V. EQUILIBRIUM DISTRIBUTION FUNCTION

Based on the above arguments, the general thermal equilibrium state satisfying $\beta = 1/T = \partial\hat{L}(S)/\partial L(E)$ motivates us to maximize $\hat{L}(S) - \beta L(E)$ when looking for canonical energy distributions [27]. Since the formal logarithms are additive—even if the original quantities to be composed were not—a distribution can be obtained from

$$\hat{L}(S)[p_i] - \beta\sum_i p_i L(E_i) - \alpha\sum_i p_i = \max. \tag{39}$$

For example, for leading order nonadditivity according to a composition rule of type (34) the Rényi entropy $\hat{L}(S) = \frac{1}{\hat{a}}\ln\sum_i p_i^{1-\hat{a}}$ is to be maximized. Denoting the formal logarithm parameter for the entropy composition by $\hat{a} = 1 - q$ and that for the analogous energy composition by a, one considers

$$\frac{1}{\hat{a}}\ln\sum_i p_i^{1-\hat{a}} - \beta\sum_i p_i\frac{1}{a}\ln(1+aE_i) - \alpha\sum_i p_i = \max. \tag{40}$$

The extremal condition results in

$$p_i = A\,[b(\alpha+\beta L_i)]^{-1/\hat{a}}\,. \tag{41}$$

Here we have introduced the notation

$$A = e^{-\hat{L}(S)}, \quad L_i = \frac{1}{a}\ln(1+aE_i), \quad b = \frac{\hat{a}}{1-\hat{a}}. \tag{42}$$

Then the normalization, the average, and the definition of the entropy read as follows:

$$1 = A\sum_i [b(\alpha+\beta L_i)]^{-1/\hat{a}}\,, \tag{43}$$

$$\langle L\rangle = A\sum_i L_i\,[b(\alpha+\beta L_i)]^{-1/\hat{a}}\,, \tag{44}$$

$$A^{-\hat{a}} = A^{1-\hat{a}}\sum_i [b(\alpha+\beta L_i)]^{1-1/\hat{a}}\,. \tag{45}$$

From this set of equations one obtains the condition

$$1 = b\alpha + b\beta\langle L\rangle. \tag{46}$$

Therefore the equilibrium distribution simplifies to

$$\begin{aligned} p_i &= A\,[1+b\beta(L_i-\langle L\rangle)]^{-1/\hat{a}} = \frac{1}{Z}(1+\hat{a}\hat{\beta}L_i)^{-1/\hat{a}} \\ &= \frac{1}{Z}\left(1+\hat{a}\hat{\beta}\frac{1}{a}\ln(1+aE_i)\right)^{-1/\hat{a}}. \end{aligned} \tag{47}$$

Here we have introduced the shorthand notations

$$Z = \frac{1}{A}(1-b\beta\langle L\rangle)^{1/\hat{a}}, \quad \hat{\beta} = \frac{\beta}{1-\hat{a}(1+\beta\langle L\rangle)}. \tag{48}$$

We should keep in mind that the reciprocal temperature, distinguished by the zeroth law, is the Lagrange multiplier β. This is reflected well by the whole formalism, because the usual thermodynamic relations are valid. A distribution similar to (47)—but with negative values of the energy nonadditivity parameter a—has been derived for particle energy spectra inside jets, observed in electron-electron collision experiments, by taking into account multiplicity fluctuations [28].

It is enlightening to consider now cases where one or the other quantity is composed by additive rules. In the limit of additive entropy but nonadditive energy composition rules ($\hat{a} \to 0$) the canonical distribution approaches

$$p_i = \frac{1}{Z_0}(1+aE_i)^{-\beta/a} \quad \text{where} \quad \ln Z_0 = S_{BG} - \beta\langle E\rangle, \tag{49}$$

and S_{BG} is the Boltzmann-Gibbs entropy. For nonadditive entropy and additive energy, on the other hand, a differently parametrized power-law-tailed distribution emerges:

$$p_i = \frac{1}{Z}(1+\hat{\beta}\hat{a}E_i)^{-1/\hat{a}}. \tag{50}$$

How can one distinguish between these two approaches in a physical situation? In some cases the entropic nonadditivity, in some other cases the energetic nonadditivity, may be preferred. To mention an example, the ideal gas of massless particles can be considered. The average energy of a single particle, $\langle E\rangle$, known to be connected to the temperature, leads to different formulas in these two cases. For the entropic nonadditivity one obtains the familiar result,

$$T = \frac{1}{f}\langle E\rangle, \tag{51}$$

with $f = D$ for massless particles in D spatial dimensions and with $f = D/2$ for the nonrelativistic energy relation $E = \vec{p}^2/2m$. This result is independent of the value of the entropic nonadditivity parameter $\hat{a}$. For the general case using

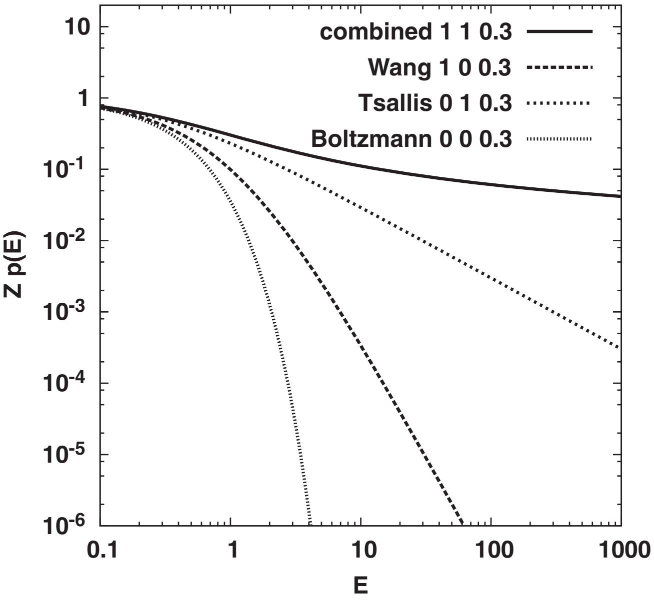

FIG. 1. The stationary energy distributions for different pairings of additive and nonadditive systems with relativistic dispersion relation and obtained by use of the formal logarithms $L(E) = \ln(1 + aE)/a$ and $\hat{L}(S) = \ln(1 + \hat{a}S)/\hat{a}$. In the legend the parameters are a, $\hat{a}$, and $\hat{T} = 1/\hat{\beta}$, in this order.

$E = c\sqrt{\vec{p}^2 + (mc)^2}$, however, the result will depend on the nonadditivity parameter [29,30].

On the other hand, for the energetic nonadditivity, applied under the above conditions, one gets

$$T = \frac{\langle E \rangle}{f + (f+1)a\langle E \rangle}. \tag{52}$$

In this case for $a > 0$ a maximal (so-called *limiting*) temperature arises: $T \leqslant 1/(f+1)a$. We note in passing that if $\langle L(a,E) \rangle$ is fixed instead, the limiting temperature is $T \leqslant 1/fa$ [19]. We would like to emphasize again that this T is the thermodynamical temperature—intensive and satisfying the zeroth law—as defined in Eq. (17).

In Fig. 1 stationary energy distributions are depicted for the equilibrium between additive systems, between energetically or entropically nonadditive systems, and for the case when both nonadditivities are considered. They are labeled according to the nonadditivities as “Boltzmann”, “Tsallis”, “Wang”, and “combined”, respectively. The parameters given in the legend are a, $\hat{a}$, and $\hat{T} = 1/\hat{\beta}$ in this order, in the corresponding energy units.

VI. SUMMARY AND CONCLUSIONS

In this paper we have reversed the usual approach to the zeroth law of thermodynamics in nonextensive thermostatistics. Instead of assuming some particular nonadditive entropy form or composition rule for either the entropy or other basic, traditionally extensive, quantities, we have regarded the zeroth law as more fundamental and derived consequent requirements for the general composition rules. In our analysis we have considered nonadditive composition rules for both the energy and the entropy while composing two subsystems. In principle this method can be extended to all extensive thermodynamical variables.

We have proved that the zeroth law together with the maximum entropy principle strictly restricts the possible functional form of the composition rule, and this restriction can be resolved in a simple manner by using formal logarithms for both the energy and the entropy functions. *The zeroth law essentially enforces the additivity of these formal logarithms of the basic extensive quantities, but not the additivity of the quantities themselves.* In this way it is true that the zeroth law enforces additivity, but it is also true that not only the Boltzmann formula is compatible with the basic principles of thermodynamics.

The temperature, defining thermal equilibrium among systems by following such generalized composition rules instead of simple addition, is the inverse of the Lagrange multiplier associated with formal logarithms of the energy and entropy in the maximum entropy principle.

Finally, we briefly outlined some important properties of the canonical equilibrium distributions arising from this treatment. In the general case, compatible with the zeroth law, these are not at all restricted to a Gibbs exponential of the individual energy.

In our approach several (at first sight seemingly paradoxical) aspects of nonextensive thermostatistics can be explained. We list the most important of them.

(1) The general form of the entropy composition rule (34), derived from the Tsallis-like composition (30), reveals several q parameters. The use of the corresponding formal logarithm with all the same q parameters and functional forms describes the special case of *homogeneous* equilibrium. According to our analysis of the factorizable form of the zeroth law, however, formal logarithms with different functional forms and parameters also can be used to satisfy all requirements. This describes a *heterogeneous* equilibrium; among others that between extensive and nonextensive systems. A practical example is given by a Monte Carlo simulation of nonextensive systems consisting of “red” and “blue” particles. The energetic nonadditivity parameters a_1 for the red-red, a_2 for the blue-blue, and a_{12} for the red-blue interaction can all differ.

(2) Heterogeneous composition formulas can be derived using any suggested entropy formula by determining the corresponding formal logarithm.

(3) It is possible to introduce zeroth-law-compatible energy or entropy nonadditivity either separately or simultaneously. Contrary to previous expectations [13,21], the two different nonadditivities do not imply each other.

(4) Either certain energy or entropy nonadditivities may lead to power-law-tailed equilibrium distributions, but other properties, like the equipartition law, clearly distinguish these two cases. For example, a limiting temperature is the consequence of a repulsive energetic nonadditivity of type $E_{12} = E_1 + E_2 + aE_1E_2$, but an entropic nonadditivity of similar form does not result in such a property.

(5) As emphasized in the Introduction and in [5], additivity is a sufficient but not a necessary condition for extensivity [cf. Eq. (3)]. Some nonadditive rules, when repeated many times on rescaled systems, may lead to addition asymptotically

[20]. It is interesting to note that requiring zeroth law compatibility for general composition rules leads to the conclusion that these rules should also be *associative*. For example, for the energy composition associativity requires

$$E_{12,3}(E_{12}(E_1,E_2),E_3) = E_{1,23}(E_1,E_{23}(E_2,E_3)). \tag{53}$$

Utilizing the representation of the composition rule with correspondingly indexed formal logarithms, one easily derives that

$$L_{12,3}(E_{12,3}) = L_1(E_1) + L_2(E_2) + L_3(E_3) = L_{1,23}(E_{1,23}), \tag{54}$$

i.e., the associativity of the formal logarithms of the energies of different levels of composed subsystems has to be satisfied.

(6) Our approach to extensivity and additivity is also different from a recent classification and use of different nonadditive entropies and related concepts ([5], pp. 91–106). All our results are not bound to a particular composition rule, like (30), or to purely entropic nonadditivity. Any of the traditionally extensive quantities may be composed by almost arbitrary nonadditive rules; all these versions of thermodynamics are zeroth law compatible if the maximum entropy principle is formulated by using all the corresponding formal logarithms. In particular, our conclusion is that in thermal equilibrium the Rényi entropy, as the formal logarithm of the Tsallis entropy, is to be maximized with the traditional normalization of probabilities.

Finally, we note that our analysis and statements are restricted to thermal equilibrium. Far from the equilibrium state there may be other properties, like the convexity, which could indicate a preference for using nonadditive formulas (see the comparison between Tsallis and Rényi entropy in Ref. [5]).

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[1] C. Arndt, *Information Measures (Information and Its Description in Science and Engineering)* (Springer, Berlin, 2001).

[2] I. J. Taneja, *Generalized Information Measures and Their Applications* [http://www.mtm.ufsc.br/~taneja/book/book.html].

[3] A. Rényi, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1960*, edited by J. Neyman, Vol. I (University of California Press, Berkeley, 1961), pp. 547–561.

[4] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[5] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, Berlin, 2009).

[6] M. E. Havrda and S. Charvát, Kybernetika **3**, 30 (1967).

[7] M. Nauenberg, Phys. Rev. E **67**, 036114 (2003).

[8] S. Abe, Physica A **269**, 403 (1999).

[9] S. Abe, S. Martínez, F. Pennini, and A. Plastino, Phys. Lett. A **281**, 126 (2001).

[10] S. Martínez, F. Pennini, and A. Plastino, Physica A **295**, 416 (2001).

[11] C. Ou and J. Chen, Physica A **370**, 525 (2006).

[12] R. Toral, Physica A **317**, 209 (2003).

[13] A. M. Scarfone, Phys. Lett. A **374**, 2701 (2010).

[14] S. Abe, Phys. Rev. E **63**, 061105 (2001).

[15] P. T. Landsberg and V. Vedral, Phys. Lett. A **247**, 211 (1998).

[16] J. Aczél and Z. Daróczy, *On Measures of Information* (Academic Press, New York, 1975).

[17] B. D. Sharma and I. J. Taneja, Metrika **22**, 205 (1975).

[18] G. Kaniadakis, Physica A **296**, 405 (2001).

[19] T. S. Biró and A. Peshier, Phys. Lett. B **632**, 247 (2006).

[20] T. S. Biró, Europhys. Lett. **84**, 56003 (2008).

[21] Q. A. Wang, Eur. Phys. J. B **26**, 357 (2002).

[22] M. C. Mackey, *Time's Arrow: The Origins of Thermodynamic Behaviour* (Springer, New York, 1992).

[23] J. C. Maxwell was the first to formulate the zeroth law in this form: J. C. Maxwell, *Theory of Heat* (Longmans Green, London, 1902). A representative summary of the history is given at [http://www.eoht.info/page/Zeroth+law+of+thermodynamics].

[24] S. Abe, Physica A **300**, 417 (2001).

[25] T. S. Biró, G. Purcsel, and K. Ürmössy, Eur. Phys. J. A **40**, 325 (2009).

[26] T. S. Biró and G. Purcsel, Phys. Lett. A **372**, 1174 (2008).

[27] T. S. Biró, *Is There a Temperature?* (Springer, New York, 2011).

[28] K. Ürmössy, G. Barnaföldi, and T. S. Biró, e-print arXiv:1101.3023.

[29] A. Lavagno, Phys. Lett. A **301**, 13 (2002).

[30] W. M. Alberico and A. Lavagno, Eur. Phys. J. A **40**, 313 (2009).