Perturbation theory for fractional Brownian motion in presence of absorbing boundaries

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Fractional Brownian motion is a Gaussian process $x(t)$ with zero mean and two-time correlations $\langle x(t_1)x(t_2)\rangle = D\left(t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}\right)$, where H, with $0 < H < 1$, is called the Hurst exponent. For $H = 1/2$, $x(t)$ is a Brownian motion, while for $H \neq 1/2$, $x(t)$ is a non-Markovian process. Here we study $x(t)$ in presence of an absorbing boundary at the origin and focus on the probability density $P_+(x,t)$ for the process to arrive at x at time t, starting near the origin at time 0, given that it has never crossed the origin. It has a scaling form $P_+(x,t) \sim t^{-H} R_+(x/t^H)$. Our objective is to compute the scaling function $R_+(y)$, which up to now was only known for the Markov case $H = 1/2$. We develop a systematic perturbation theory around this limit, setting $H = 1/2 + \epsilon$, to calculate the scaling function $R_+(y)$ to first order in ϵ. We find that $R_+(y)$ behaves as $R_+(y) \sim y^{\phi}$ as $y \to 0$ (near the absorbing boundary), while $R_+(y) \sim y^{\gamma} \exp(-y^2/2)$ as $y \to \infty$, with $\phi = 1 - 4\epsilon + O(\epsilon^2)$ and $\gamma = 1 - 2\epsilon + O(\epsilon^2)$. Our ϵ-expansion result confirms the scaling relation $\phi = (1 - H)/H$ proposed in Zoia, Rosso, and Majumdar [Phys. Rev. Lett. **102**, 120602 (2009)]. We verify our findings via numerical simulations for $H = 2/3$. The tools developed here are versatile, powerful, and adaptable to different situations.

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I. INTRODUCTION

Survival of a species of bacteria, translocation of DNA through a nanopore, and diffusion in presence of an absorbing boundary are only a few of many situations where the central question is the survival, or persistence, of the underlying stochastic process. More precisely, persistence, or survival probability $S(t)$, of a process is the probability that the process, starting from an initial positive position, stays positive over a time interval $[0,t]$. For many stochastic processes arising in nonequilibrium systems, persistence decays as a power law $S(t) \sim t^{-\theta}$, where θ is called the persistence exponent [1]. For a simple Markov process such as one-dimensional Brownian motion, $\theta = 1/2$ [2]. On the other hand, the exponent θ is nontrivial whenever the process is non-Markovian, that is, has a memory. In addition to theoretical studies (for a brief review, see [3]), the exponent θ has been measured in a number of experiments [4–10]. Even for Gaussian non-Markovian processes, θ is nontrivial [11]. For the latter processes that are *close* to a Markov process (i.e., whose correlators are close to that of a Gaussian Markov process) the exponent θ was computed perturbatively [12,13]. This perturbation theory has been used for various out-of-equilibrium systems, as the global persistence at the critical point of the Ising model in $d = 4 - \epsilon$ dimensions [14], in simple diffusion close to dimension 0 [15], and in fluctuating fields such as interfaces [16–18].

A quantity that contains more spatial information than persistence $S(t)$ is the probability density $P_+(x,t)$ of the particle at position x and at time t, given that it has survived (stayed positive) up to time t. To investigate $P_+(x,t)$, one can equivalently think of a process on the positive semi-infinite line $[0,\infty]$ with *absorbing* boundary condition at the origin $x = 0$ (see Fig. 1). The question is as follows: How does $P_+(x,t)$ depend on x? In other words, how does presence of an absorbing boundary at the origin change the spatial dependence of the probability density of the particle at time t? In particular, it is clear that $P_+(x,t)$ must vanish as $x \to 0$ and $x \to \infty$. However, how do they vanish there? One of the main messages of our paper is that for generic non-Markovian processes, $P_+(x,t)$ vanishes near its boundaries at $x = 0$ and $x \to \infty$ in a nontrivial way, characterized by nontrivial exponents.

As the persistence $S(t)$, the probability $P_+(x,t)$ can be computed exactly for a Gaussian Markov process, as, for example, a one-dimensional Brownian motion. For non-Markovian processes, even if they are Gaussian, $P_+(x,t)$ was not known. In this work, we consider $P_+(x,t)$ for a class of one-dimensional Gaussian processes known as fractional Brownian motion (fBm), which are parametrized by their Hurst exponent H, with $0 < H < 1$. The case $H = 1/2$ corresponds to ordinary Brownian motion, which is a Markov process, while for $H \neq 1/2$ the process is non-Markovian. The purpose of this paper is to develop a systematic perturbation theory to compute $P_+(x,t)$ for non-Markovian fBm's with $H = 1/2 + \epsilon$, where ϵ is the expansion parameter for the perturbation theory. Here we present the result for $P_+(x,t)$ to $O(\epsilon)$. It can be written as a combination of special functions, that is, error and hypergeometric functions [see Eq. (10)].

Before detailing our results, let us position them into a broader context: Fractional Brownian motion with $H \neq 1/2$ is relevant for polymer translocation through a nanopore. Consider a polymer chain composed of N monomers passing through a pore (translocation) from left to right, as drawn in Fig. 2. The dynamics of this translocation process has been investigated intensively due to its central role in understanding, for example, viral injection of DNA into a host, or RNA transport through nanopores, and mastering such applications as fast DNA or RNA sequencing through engineered channels [19–22]. The translocation coordinate $s(t)$, namely, the label of the monomer crossing the pore at time t, is key to quantitatively describing the translocation process [23–26], which begins when $s = 1$ and ends when $s = N$, that is, when the first and the last monomer of the chain enter the pore, respectively [see Fig. 2]. For large N, when the translocation is not yet complete, one can view $s(t)$ as a stochastic process on the

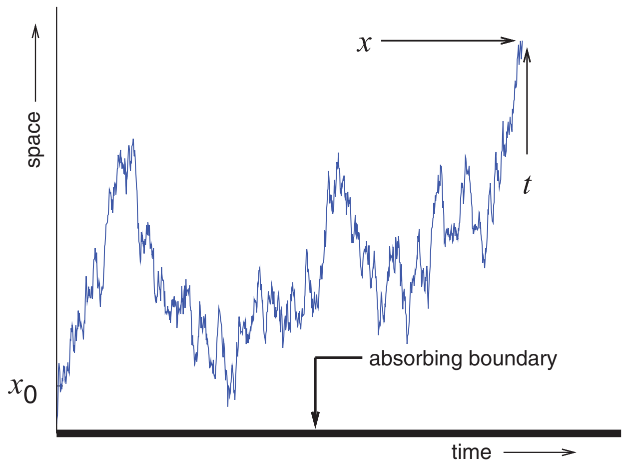

FIG. 1. (Color online) The fractional Brownian motion discussed in the main text.

semi-infinite line with absorbing boundary conditions at $s = 0$. The absorbing boundary at $s = 0$ models that if the chain falls back to the left, that is, on the starting side, it will diffuse away and not try again. The quantity $P_+[s(t) = x,t]$ then represents the probability that x monomers have translocated to the right at time t. To model the process $s(t)$, one observes the following facts: (i) Scaling arguments and numerical simulations show that $s(t)$ is subdiffusive [27]; (ii) in absence of boundaries, numerical simulations indicate that $s(t)$ is a Gaussian process [28]. Based on these observations it was proposed in Ref. [29] that a good candidate for $s(t)$ is a fBm with $H = 1/(1 + 2\nu)$, where the exponent ν describes the growth of the radius of gyration with the number of monomers ($R_g \sim N^\nu$) [30]. Thus, for $\nu \neq 1/2$, $H < 1/2$ and, hence, $s(t)$ is generically a non-Markovian process, with absorbing boundary conditions at $s = 0$ and at $s = N$. Here we consider the limit of $N \to \infty$. Thus, our results for $P_+(x,t)$ of a fBm with $H \neq 1/2$ are directly relevant for polymer translocation.

Directions for further applications are numerous: Recently, a relation was established between the statistics of avalanches associated with the motion of a driven particle in a disordered potential and persistence properties of the latter [31]. Higher-dimensional generalizations are avalanches of extended elastic objects, for which systematic field-theoretic

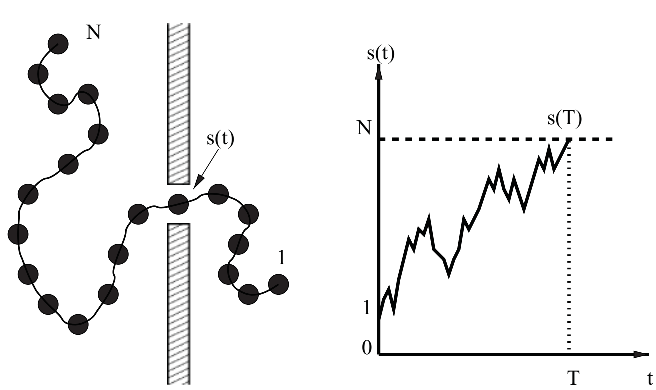

FIG. 2. (Left) Translocation of a polymer chain through a pore. (Right) The translocation coordinate $s(t)$ denotes the number of the monomer that is crossing the pore at time t.

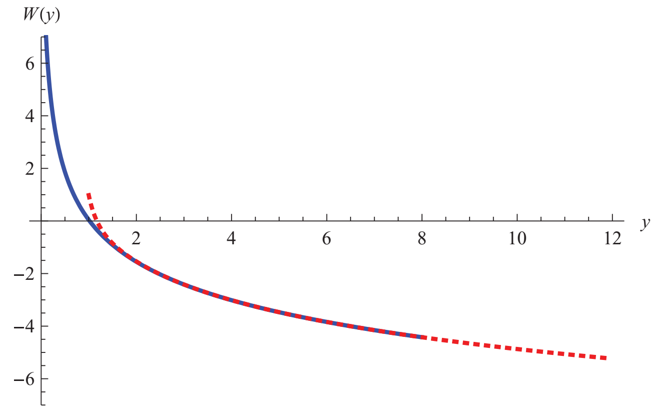

FIG. 3. (Color online) The function $W(y)$ defined in Eq. (10). The solid blue line is the result of the series expansion (11), while the dashed red line is the result of the asymptotic expansion (12). We note that $W(1.016\,94) = 0$.

treatments exist [32–35]. In few cases, no-hitting probabilities can be calculated for extended (nondirected) objects as self-avoiding random walks avoiding extended objects [36]. Other approaches use real-space renormalization [37,38].

This article is organized as follows: Since some of the computations are rather technical, we first provide in Sec. II a brief summary of the main definitions and our principal results. In Sec. III, we introduce basic notations and reproduce the known results for $H = \frac{1}{2}$. Section IV explains the basic ideas of our perturbative approach, sketches the calculation, and discusses some of the subtle points. Our predictions are compared to numerical simulations in Sec. V. Conclusions are presented in Sec. VI. More technical points are relegated to two appendixes: In Appendix A the correction to the action is derived; Appendix B contains the explicit calculation of the perturbation theory. Finally, Appendix C reviews the arguments for the scaling law $\phi = (1 - H)/H$.

II. SUMMARY OF DEFINITIONS AND MAIN FINDINGS

Consider a particle, located at time $t = 0$ at the origin $x = 0$ and free to propagate on the real axis. For Gaussian processes, the probability to find the particle inside the interval $(x,x + \mathrm{d}x)$ at time t is given by

$$P(x,t)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi\,\langle x^2(t)\rangle}}e^{-\frac{x^2}{2\langle x^2(t)\rangle}}\,\mathrm{d}x, \tag{1}$$

where $\langle x^2(t)\rangle$ is the particle's mean square displacement. A natural scaling variable is

$$y = \frac{x}{\sqrt{\langle x^2(t)\rangle}}, \tag{2}$$

and most of the properties of the process are a function of this single variable. For example, the distribution probability in Eq. (1) becomes

$$P(x,t)\,\mathrm{d}x = R(y)\,\mathrm{d}y, \tag{3}$$

$$R(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}. \tag{4}$$

In many problems the motion is confined to an interval, finite or semi-infinite. In presence of *absorbing* boundaries, the probability distribution of the particle position, subject to the condition that the particle has survived, has no longer a simple Gaussian form since it has to vanish at the boundaries. However, one can still express it as a function of the sole scaling variable y defined in Eq. (2), where $\langle x^2(t)\rangle$ is the particle's mean square displacement in the *unconstrained (without boundaries) process over the full real line*. In particular, here we discuss the case where the particle can move on the positive semiaxis and is absorbed whenever $x(t) < 0$. We call $P_+(x,t)$ and $R_+(y)$ with y given in Eq. (2) the *normalized* probability distribution and the scaling function of the problem in presence of an absorbing boundary at the origin,

$$P_+(x,t)\,\mathrm{d}x = R_+(y)\,\mathrm{d}y. \tag{5}$$

In contrast to the free case, the functional form of $R_+(y)$ is not the same for all Gaussian processes but depends on the precise nature of the latter. Here we study a particular class of processes, the fBm, for which the autocorrelation function in absence of boundaries is

$$\langle x(t_1)x(t_2)\rangle = D\left(t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}\right), \tag{6}$$

where H with $0 < H < 1$ is the Hurst exponent. For $H = 1/2$, the fBm identifies with Brownian motion

$$\langle x(t_1)x(t_2)\rangle = 2D\min(t_1,t_2), \tag{7}$$

where D is the diffusion constant. Note that only for $H = 1/2$ is the Gaussian process $x(t)$ Markovian. For other values of H, the process is non-Markovian.

For Brownian motion ($H = 1/2$), the form of $R_+(y)$ can be obtained using the method of images (see Sec. III),

$$R_+^{(0)}(y) = y e^{-\frac{y^2}{2}}. \tag{8}$$

The superscript (0) identifies the case $H = 1/2$. For other values of H, due to the non-Markovian nature of the process, the method of images no longer works and the computation of $R_+(y)$ becomes a challenging problem. In this paper we compute this function, using a perturbative approach for $H = 1/2 + \epsilon$, to first order in ϵ. The final result is

$$R_+(y) = R_+^{(0)}(y)[1 + \epsilon W(y) + O(\epsilon^2)], \tag{9}$$

$$\begin{aligned} W(y) = {} & \frac{1}{6}y^4\,{}_2F_2\left(1,1;\frac{5}{2},3;\frac{y^2}{2}\right) \\ & + \pi(1-y^2)\,\mathrm{erfi}\left(\frac{y}{\sqrt{2}}\right) + \sqrt{2\pi}\,e^{\frac{y^2}{2}}\,y \\ & + (y^2-2)[\ln(2y^2) + \gamma_{\mathrm{E}}] - 3y^2, \end{aligned} \tag{10}$$

where γ_{E} is Euler's constant, ${}_2F_2$ a hypergeometric function, and erfi the imaginary error-function. We can write a *convergent* series expansion,

$$\begin{aligned} W(y) = {} & 4y^4 \sum_{n=0}^{\infty} \frac{2^n n!\, y^{2n}}{(2n+4)!} \\ & - \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\, 2^{\frac{3}{2}-n} y^{2n+1}}{(2n-1)(2n+1)n!} \\ & + (y^2-2)[\ln(2y^2) + \gamma_{\mathrm{E}}] - 3y^2, \end{aligned} \tag{11}$$

where each line is equivalent to the corresponding line in Eq. (10) (see Fig. 3). Both sums converge for all y, but problems of numerical precision appear for $y > 7$. In that region, one can use the asymptotic expansion

$$W(y) = 1 - \gamma_{\mathrm{E}} - \ln(2y^2) + \frac{1}{2y^2} - \frac{1}{2y^4} + \frac{5}{4y^6} + O(y^{-8}). \tag{12}$$

At $y = 7$, the difference between (11) and (12) is smaller than 10^{-6}.

We obtain, at first order in ϵ, the asymptotic expansions of $R_+(y)$,

$$\begin{aligned} R_+(y) & \xrightarrow{y\to 0} y\,[1 - 4\epsilon\ln y - 2\epsilon(\gamma_{\mathrm{E}} + \ln 2) + \cdots], \\ R_+(y) & \xrightarrow{y\to\infty} y e^{-y^2/2}\,[1 - 2\epsilon\ln y + \epsilon(1 - \ln 2 - \gamma_{\mathrm{E}})] \\ & \qquad + \cdots. \end{aligned} \tag{13}$$

These asymptotics can be recast into

$$\begin{aligned} R_+(y) &\sim y^{\phi} \quad \text{for} \quad y \to 0, \\ R_+(y) &\sim y^{\gamma} e^{-\frac{y^2}{2}} \quad \text{for} \quad y \to \infty, \end{aligned} \tag{14}$$

where the two exponents ϕ and γ are at first order in ϵ given by

$$\phi = 1 - 4\epsilon + O(\epsilon^2), \quad \gamma = 1 - 2\epsilon + O(\epsilon^2). \tag{15}$$

In a recent publication [29] (reviewed in Appendix C), a general scaling relation, valid for arbitrary self-affine processes with stationary increments, was proposed between the exponent ϕ, the persistence exponent θ, and the Hurst exponent H,

$$\phi = \frac{\theta}{H}. \tag{16}$$

For fBm, it is known that $\theta = 1 - H$ [16]. This result predicts that for fBm,

$$\phi = \frac{1-H}{H}. \tag{17}$$

One of the objectives of this paper was to verify this scaling relation up to $O(\epsilon)$ in a perturbation theory around $H = 1/2$. Using $H = 1/2 + \epsilon$, one expects $\phi = (1-H)/H = 1 - 4\epsilon + O(\epsilon^2)$ for fBm. This is in agreement with our result (15), putting the scaling arguments on a firmer footing.

It is interesting to note that the scaling function $R_+(y)$ given in Eq. (4) has, at least to $O(\epsilon)$, the same leading large-y behavior $\sim e^{-y^2/2}$ as in the unconstrained case (4). This behavior can be understood by a simple heuristic argument: Far from the boundary the process is not "aware" of the latter. Our calculation reveals that the process nevertheless knows about the boundary, and $R_+(y)$ has a subleading power-law prefactor y^{γ} where γ is a new (independent) exponent, whose result to order ϵ is given in Eq. (15).

Our analytical results are then verified via numerical simulations for $H = 2/3$.

III. PRELIMINARIES: BROWNIAN CASE ($H = 1/2$)

To simplify notations, we set $D = 1$ in the following. The final result (9), expressed in the variable y, is, of course, independent of this choice.

The spreading of a Brownian particle is given by the Fokker-Planck equation

$$\partial_t Z_+^{(0)}(x_0,x,t) = \partial_x^2 Z_+^{(0)}(x_0,x,t), \tag{18}$$

$$Z_+^{(0)}(x_0,x,t=0) = \delta(x-x_0). \tag{19}$$

The propagator $Z_+^{(0)}(x_0,x,t)$ times dx gives the probability to find the Brownian particle inside the interval $(x,x+\mathrm{d}x)$ at time t, knowing that the particle was at x_0 at time $t = 0$. With absorbing boundary conditions at the origin we have, using the method of images,

$$Z_+^{(0)}(x_0,x,t) = \frac{1}{\sqrt{4\pi t}}[e^{-(x-x_0)^2/4t} - e^{-(x+x_0)^2/4t}]. \tag{20}$$

This propagator is not a probability distribution because it is not normalized. Its normalization, the so-called survival probability,

$$S(x_0,t) = \int_0^\infty \mathrm{d}x\, Z_+^{(0)}(x_0,x,t) = \mathrm{erf}\left(\frac{x_0}{2\sqrt{t}}\right), \tag{21}$$

gives the probability that the particle is not yet absorbed by the boundary at $x = 0$. The survival probability vanishes when $x_0 \to 0$; however, in that limit, the probability distribution for the nonabsorbed particles remains well-defined:

$$P_+^{(0)}(x,t) = \lim_{x_0\to 0} \frac{Z_+^{(0)}(x_0,x,t)}{\int_0^\infty \mathrm{d}x\, Z_+^{(0)}(x_0,x,t)}. \tag{22}$$

Another quantity with a finite limit for $x_0 = 0$ is

$$Z_+^{(0)}(x,t) = \lim_{x_0\to 0} \frac{1}{x_0} Z_+^{(0)}(x_0,x,t) = \frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{\pi} t^{3/2}}. \tag{23}$$

This makes it possible to write the probability $P_+^{(0)}(x,t)$ as

$$P_+^{(0)}(x,t) = \frac{Z_+^{(0)}(x,t)}{\int_0^\infty \mathrm{d}x\, Z_+^{(0)}(x,t)} = \frac{x}{2t} e^{-\frac{x^2}{4t}}. \tag{24}$$

Using in Eq. (24) the scaling variable defined in (2), $y = x/\sqrt{2t}$, we recover (8). Equation (24) is simpler than Eq. (22) because the x_0 dependence is discarded from the beginning. We use this definition to compute $Z_+(x,t)$ for $H = 1/2 + \epsilon$.

IV. PERTURBATION THEORY ($H \neq 1/2$)

The process $x(t)$ is Gaussian for all values of H, but it is Markovian only for $H = 1/2$. For all other values of H, the process is non-Markovian and this makes the problem difficult to solve. Our idea is to expand around $H = 1/2$. In a first step, we construct an action, which calculates expectation values of the Gaussian process $x(t)$, with bulk expectation values (6). In a second step, we obtain the propagator with absorbing boundary conditions at $x = 0$. In a third step we calculate the probability $P_+(x,t)$ perturbatively, using the action constructed in step 1. In the fourth step, we put together all pieces and interpret our result.

A. Step 1: The action

For all H, $x(t)$ is a Gaussian process; therefore, the statistical weight of a path $x(t')$ without any boundary is proportional to $\exp(-\mathcal{S}[x])$, where the action $\mathcal{S}[x]$ is quadratic in x and given by

$$\mathcal{S}[x] = \int_0^t dt_1 \int_0^t dt_2\, \frac{1}{2} x(t_1) G(t_1,t_2) x(t_2). \tag{25}$$

Note that we use standard field-theoretic notation, noting $f(x)$ a function of the variable x, and $\mathcal{S}[x]$ a functional, depending on the function $x(t')$, with $0 < t' < t$.

The kernel $G(t_1,t_2)$ of the action is related to the autocorrelation function of the process via

$$G^{-1}(t_1,t_2) = \langle x(t_1)x(t_2)\rangle. \tag{26}$$

For $H = 1/2$, the action is simple. In this case, setting $D = 1$,

$$[G^{(0)}]^{-1}(t_1,t_2) = \langle x(t_1)x(t_2)\rangle = 2\min(t_1,t_2). \tag{27}$$

Using the result (A7) in Eq. (25), we recover the standard Brownian action

$$\mathcal{S}^{(0)}[x] = \frac{1}{4}\int_0^t dt'\, (\partial_{t'} x)^2 . \tag{28}$$

For a generic value of H the kernel $G(t_1,t_2)$ becomes nonlocal. For $H = 1/2 + \epsilon$ one can write

$$\mathcal{S}[x] = \mathcal{S}^{(0)}[x] + \epsilon\, \mathcal{S}^{(1)}[x] + \cdots, \tag{29}$$

where $\mathcal{S}^{(0)}[x]$ is the action (28) and $\mathcal{S}^{(1)}[x]$ has been computed in Appendix A:

$$\begin{aligned}\mathcal{S}^{(1)}[x] = &-\frac{1}{2}\int_0^t dt_1 \int_{t_1}^t dt_2\, \frac{\partial_{t_1} x(t_1) \partial_{t_2} x(t_2)}{|t_1 - t_2|} \\ &- 2\mathcal{S}^{(0)}[x](1+\ln\tau).\end{aligned} \tag{30}$$

Note that we have introduced a regularization for coinciding times $t_1 = t_2 \to \ln|t_1 - t_2| = \ln\tau$ where $\tau > 0$ is the UV cutoff. A first-principles definition would necessitate a discretization in time. It is, however, sufficient to check that the law (6) is correctly reproduced and that the final result is cutoff independent.

B. Step 2: The propagator with an absorbing boundary

For a generic value of H, the propagator $Z_+(x_0,x,t)$, denoting the probability that the particle reaches x at time t, starting from x_0 at time 0, and staying positive over the interval $[0,t]$, can be written using standard path integral notation as

$$Z_+(x_0,x,t) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]\, e^{-\mathcal{S}[x]}\, \Theta[x]. \tag{31}$$

Here $\Theta[x]$ is an indicator function that is 1 if the path $x(t')$ stays positive over the interval $[0,t]$ and 0 otherwise. The action $\mathcal{S}[x]$ is given in (25). In the limit $x_0 \to 0$, we expect, as in the Brownian case ($H = 1/2$), the propagator to vanish as $x_0^{\phi_0}$, where the yet unknown exponent ϕ_0 depends on H. Note that for $H = 1/2$, $\phi_0 = 1$ [see Eq. (23)]. For $H = 1/2 + \epsilon$, we expect that $\phi_0 = 1 + a_1\epsilon + O(\epsilon^2)$, where a_1 is yet unknown. Analogous to Eq. (23) for $H = 1/2$, we define $Z_+(x,t)$ as

$$Z_+(x,t) = \lim_{x_0\to 0} \frac{1}{x_0^{\phi_0}} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]\, e^{-\mathcal{S}[x]}\, \Theta[x]. \qquad (32)$$

Using the expansion of the action given in Eq. (29) and $\phi_0 = 1 + a_1\epsilon$, we write to leading order in ϵ

$$\begin{aligned} Z_+(x,t) &= \lim_{x_0\to 0} \frac{1}{x_0^{1+a_1\epsilon}} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \left(1 - \epsilon \mathcal{S}^{(1)}[x]\right) e^{-\mathcal{S}^{(0)}[x]}\, \Theta[x] \\ &= \lim_{x_0\to 0} \Big\{ Z_+^{(0)}(x,t)\,[1 - a_1\epsilon \ln(x_0)] \\ &\quad - \frac{\epsilon}{x_0} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]\, \mathcal{S}^{(1)}[x]\, e^{-\mathcal{S}^{(0)}[x]}\, \Theta[x] \Big\} \\ &= Z_+^{(0)}(x,t) + \epsilon Z_+^{(1)}(x,t), \end{aligned} \qquad (33)$$

where $Z_+^{(0)}(x,t)$ is defined in Eq. (23) and $Z_+^{(1)}(x,t)$ is

$$\begin{aligned} Z_+^{(1)}(x,t) = \lim_{x_0\to 0} \Big\{ & \frac{-1}{x_0} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]\, \mathcal{S}^{(1)}[x]\, e^{-\mathcal{S}^{(0)}[x]}\, \Theta[x] \\ & - a_1 \ln(x_0) Z_+^{(0)}(x,t) \Big\}. \end{aligned} \qquad (34)$$

We see that for $\phi_0 = 1 - 4\epsilon$, that is, $a_1 = -4$, $Z_+^{(1)}(x,t)$ is independent of x_0.

C. Step 3: Calculation of $Z_+^{(1)}(x,t)$

The main achievement of this paper is the calculation of $Z_+^{(1)}(x,t)$ defined in Eq. (34). This calculation is rather involved, both conceptually and technically. Therefore, we relegate several technical calculations to Appendix B. Equation (34) can be divided into three pieces:

$$\begin{aligned} Z_+^{(1)}(x,t) &= Z_+^{A}(x,t) \\ &\quad + \lim_{x_0\to 0} [Z_+^{B}(x_0,x,t) - a_1 \ln(x_0) Z_+^{(0)}(x,t)], \end{aligned} \qquad (35)$$

$$\begin{aligned} Z_+^{A}(x,t) &= 2(1 + \ln\tau) \\ &\quad \times \lim_{x_0\to 0} \frac{1}{x_0} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \mathcal{S}^{(0)}[x]\, e^{-\mathcal{S}^{(0)}[x]}\, \Theta[x], \end{aligned} \qquad (36)$$

$$\begin{aligned} Z_+^{B}(x_0,x,t) &= \frac{1}{4} \int_0^t \int_0^t dt_1 dt_2 \\ &\quad \times \frac{1}{x_0} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \frac{\dot{x(t_1)}\dot{x(t_2)}}{|t_1 - t_2|}\, e^{-\mathcal{S}^{(0)}[x]}\, \Theta[x]. \end{aligned} \qquad (37)$$

The first term, $Z_+^{A}(x,t)$, is simple and is evaluated in Appendix B1. We now come to the evaluation of the contribution $Z_+^{B}(x_0,x,t)$, defined in Eq. (37). In Fig. 4 we show a path which contributes to $Z_+^{B}(x_0,x,t)$. The sum of all these paths is a product of transition probabilities. Explicitly, it reads, ordering $t_1 < t_2$, which gives an extra factor of 2 compared to (37)

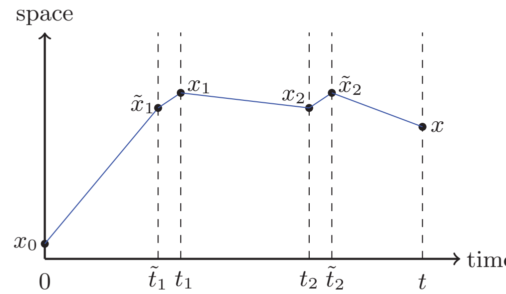

FIG. 4. Graphical representation of the path integral for $Z_+^{B}(x_0,x,t)$ given in Eq. (34).

$$\begin{aligned} Z_+^{B}&(x_0,x,t) \\ &= \frac{1}{2x_0} \int_0^t dt_2 \int_0^{t_2} dt_1 \int_{x_1>0} \int_{\tilde{x}_1>0} \int_{x_2>0} \int_{\tilde{x}_2>0} \\ &\quad \Big[Z_+^{(0)}(x_0,\tilde{x}_1,t_1) D(\tilde{x}_1,x_1) \frac{Z_+^{(0)}(x_1,x_2,t_2-t_1)}{|t_2 - t_1|} \\ &\quad \times D(x_2,\tilde{x}_2) Z_+^{(0)}(\tilde{x}_2,x,t-t_2) \Big]. \end{aligned} \qquad (38)$$

$Z_+^{(0)}(\tilde{x}_1,x_0,t_1)$, $Z_+^{(0)}(x_1,x_2,t_2-t_1)$, and $Z_+^{(0)}(\tilde{x}_2,x,t-t_2)$ are defined in (20). The factors $D(x_1,\tilde{x}_1)$ and $D(x_2,\tilde{x}_2)$ take into account the terms $\partial_{t_1} x(t_1)$ and $\partial_{t_2} x(t_2)$ in the action $S_1[x]$.

$$\begin{aligned} D(\tilde{x}_1,x_1) &= \lim_{dt\to 0} \frac{(x_1 - \tilde{x}_1)}{dt} Z_+^{(0)}(x_1,\tilde{x}_1,dt) \\ &= \lim_{dt\to 0} \frac{(x_1 - \tilde{x}_1)}{dt} \frac{e^{-\frac{(\tilde{x}_1 - x_1)^2}{4dt}}}{\sqrt{2\pi\, dt}} \\ &= -2\delta'(x_1 - \tilde{x}_1). \end{aligned} \qquad (39)$$

Finally, we have set $\tilde{t}_1 = t_1$ and $\tilde{t}_2 = t_2$, since we have taken the limit of their differences to 0. In order to perform the six integrations in Eq. (38) it turns out to be convenient to evaluate its Laplace transform, $\tilde{Z}_+^{B}(x_0,x,s)$. From now on, we always denote with $\tilde{f}(s)$ the Laplace transform of a function $f(t)$, defined as

$$\tilde{f}(s) := \int_0^\infty dt\, e^{-st}\, f(t). \qquad (40)$$

This Laplace transform leads to two important simplifications. The first simplification is that now the nested time integrals over t_1 and t_2 become a product. To see this, we remember that if f_1 and f_2 are two functions which depend on t, then the Laplace transform of their convolution is simply the product of their Laplace transforms,

$$\begin{aligned} &\int_0^\infty dt\, e^{-ts} \left[\int_0^t dt_1 f_1(t_1) f_2(t - t_1) \right] \\ &= \int_0^\infty dt \int_0^\infty dt_1 \int_0^\infty dt_2\, \delta(t - t_1 - t_2) \\ &\quad \times f_1(t_1) e^{-t_1 s} f_2(t_2) e^{-t_2 s} \\ &= \left[\int_0^\infty dt_1 f_1(t_1) e^{-t_1 s} \right] \left[\int_0^\infty dt_2 f_2(t_2) e^{-t_2 s} \right] \\ &= \tilde{f}_1(s) \tilde{f}_2(s). \end{aligned} \qquad (41)$$

This consideration generalizes to three and more times.

We obtain for the Laplace transform of (38)

$$\tilde{Z}_+^B(x_0,x,s) = -\frac{2}{x_0}\int_{x_1>0}\int_{x_2>0} \tilde{Z}_+^{(0)}(x_0,x_1,s)\tilde{Z}_+^{(0)}(x_2,x,s) \times \partial_{x_1}\partial_{x_2}\left[\int_0^\infty \mathrm{d}t\, e^{-st}\frac{Z_+^{(0)}(x_1,x_2,t)}{t}\right]. \tag{42}$$

The second simplification is even more important and is most easily understood on the example of the bulk propagator:

$$Z^{(0)}(x,y,t) := \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}}. \tag{43}$$

Its Laplace-transform is

$$\tilde{Z}^{(0)}(x,y,s) = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x-y|}. \tag{44}$$

While integrals over $x > 0$ involving (B19) give error functions, which are hard to integrate further, the same integrals over (44) remain similar exponential functions; the only complication is that one has to distinguish between x smaller or larger than y.

To evaluate (42), we now have to calculate the Laplace-transforms of its factors:

$$\tilde{Z}_+^{(0)}(x,y,s) = \frac{e^{-\sqrt{s}|x-y|} - e^{-\sqrt{s}(x+y)}}{2\sqrt{s}}. \tag{45}$$

Finally, the term in brackets in Eq. (42) can be rewritten, using a Fourier decomposition for $Z_+^{(0)}(x_2,x_1,t)$, as

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, e^{-st}\frac{Z_+^{(0)}(x_1,x_2,t)}{t} &= \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi}\int_0^\infty \mathrm{d}t\,\frac{e^{-(s+k^2)t}}{t}[e^{ik(x_1-x_2)} - e^{ik(x_1+x_2)}] \\ &= -\int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi}[e^{ik(x_1-x_2)} - e^{ik(x_1+x_2)}][\ln([s+k^2]\tau) + \gamma_{\mathrm{E}}].\end{aligned} \tag{46}$$

Note that the time integral in the second line of Eq. (46) is diverging at small times. Since the path integral is defined as discretized in time, a natural approach consists of discretizing this integral, with a step size τ. This would indeed be the only possible approach for stronger divergences, such as $1/t^2$. However, since our integral is only logarithmically diverging, we can take an easier path by using a small-time cutoff τ:

$$\int_0^\infty \frac{e^{-(s+k^2)t}}{t}\mathrm{d}t \to \int_\tau^\infty \frac{e^{-(s+k^2)t}}{t}\mathrm{d}t = -\ln([s+k^2]\tau) - \gamma_{\mathrm{E}} + O(\tau). \tag{47}$$

We note that the regularization by discretization gives the same result apart from the term $-\gamma_{\mathrm{E}}$. We check later that it only contributes to the normalization, which will drop from the final result.

Collecting the results of Eqs. (45) and (46) in Eq. (42), and doing the remaining space-derivatives, we find

$$\begin{aligned}\tilde{Z}_+^B(x_0,x,s) = {}& \frac{2}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}k^2[\ln(\tau(s+k^2)) + \gamma_{\mathrm{E}}] \\ &\times \int_{x_1>0}\int_{x_2>0}[e^{ik(x_1-x_2)} + e^{ik(x_1+x_2)}] \\ &\times \frac{e^{-\sqrt{s}|x-x_2|} - e^{-\sqrt{s}(x_2+x)}}{2\sqrt{s}} \\ &\times \frac{e^{-\sqrt{s}|x_0-x_1|} - e^{-\sqrt{s}(x_1+x_0)}}{2\sqrt{s}}.\end{aligned} \tag{48}$$

Performing the space integrations, we find

$$\begin{aligned}&\tilde{Z}_+^B(x_0,x,s) \\ &= \frac{4}{x_0}\sqrt{s}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx_0) - e^{-\sqrt{s}x_0}][\cos(kx) - e^{-\sqrt{s}x}] \\ &\quad\times \frac{k^2[\ln\left(\tau(s+k^2)\right) + \gamma_{\mathrm{E}}]}{(s+k^2)^2}.\end{aligned} \tag{49}$$

Note that this is (rescaling $k \to \sqrt{s}k$)

$$\begin{aligned}\tilde{Z}_+^B(x_0,x,s) = {}& \frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}] \\ &\times [\cos(kx_0\sqrt{s}) - e^{-x_0\sqrt{s}}] \\ &\times \frac{k^2[\ln(\tau s(1+k^2)) + \gamma_{\mathrm{E}}]}{\sqrt{s}(1+k^2)^2}.\end{aligned} \tag{50}$$

The next step is to invert this Laplace transform which is performed in Appendix Bc.

D. Step 4: The probability $P_+(x,t)$

The final result for $Z_+^{(1)}(x,t)$ is given in Eqs. (B60) and (B65) of Appendix Bd, expressed in terms of the scaling variable $z = x/\sqrt{2t}$. Note that setting $\phi_0 = 1 - 4\epsilon$, that is, $a_1 = -4$, the term $Z_+^{(1)}(x,t)$ does not depend on x_0:

$$\begin{aligned}\frac{Z_+^{(1)}(z,t)}{Z_+^{(0)}(z,t)} &= (z^2-2)[\ln(2z^2t) + \gamma_{\mathrm{E}}] - 2 + \mathcal{I}(z) + c(t), \\ c(t) &= \ln(t) - 2\gamma_{\mathrm{E}} + 2,\end{aligned} \tag{51}$$

where $\mathcal{I}(z)$ is defined in Eq. (B53). The first line is arranged as to not contribute to the normalization, whereas $c(t)$ is independent of z and will not appear in the final conditional probability. γ_{E} is Euler's constant. The probability distribution, $P_+(x,t)$, to find a non-yet-absorbed particle in the interval $(x,x+\mathrm{d}x)$ can be computed following the lines of Eq. (24) to order ϵ as

$$\begin{aligned}P_+(x,t) &= \frac{Z_+^{(0)}(x,t) + \epsilon Z_+^{(1)}(x,t)}{\int_0^\infty \mathrm{d}x\,(Z_+^{(0)}(x,t) + \epsilon Z_+^{(1)}(x,t))} = \frac{Z_+^{(0)}(x,t)}{\int_0^\infty \mathrm{d}x\, Z_+^{(0)}(x,t)} \\ &\quad\times\left[1 + \epsilon\left(\frac{Z_+^{(1)}(x,t)}{Z_+^{(0)}(x,t)} - \frac{\int_0^\infty \mathrm{d}x\, Z_+^{(1)}(x,t)}{\int_0^\infty \mathrm{d}x\, Z_+^{(0)}(x,t)}\right)\right].\end{aligned} \tag{52}$$

Note that the term proportional to $c(t)$ cancels in normalized objects such as $P_+(x,t)$. Therefore, we obtain

$$P_+(x,t)\,\mathrm{d}x = R_+^{(0)}(z)\mathrm{d}z\{1+\epsilon[(z^2-2) \times (\gamma_{\mathrm{E}}+\ln(2z^2t)) - 2 + \mathcal{I}(z)]\}, \quad (53)$$

where $R_+^{(0)}(z) = z\exp(-z^2/2)$, and $\mathcal{I}(z)$ is given in Eq. (B53). The result in Eq. (53) still involves both z and t. The reason is that for $H \neq 1/2$ the natural scaling variable is $y = x/(\sqrt{2}t^{1/2+\epsilon})$ instead of $z = x/\sqrt{2t}$, as can be seen from Eq. (2). To rewrite Eq. (53) in terms of $y = zt^{\epsilon}$, we note that

$$\begin{aligned} R_+^{(0)}(z)\mathrm{d}z &= R_+^{(0)}(yt^{\epsilon})t^{\epsilon}\mathrm{d}y \\ &= R_+^{(0)}(y)\left\{1+\epsilon\left[\frac{y\partial_y R_+^{(0)}(y)}{R_+^{(0)}(y)}+1\right]\ln t\right\} \\ &= R_+^{(0)}(y)\{1-\epsilon[y^2-2]\ln t\}. \end{aligned} \quad (54)$$

This gives for Eq. (53) up to terms of order ϵ^2

$$P_+(x,t)\,\mathrm{d}x = R_+^{(0)}(y)\mathrm{d}y \times \{1+\epsilon[(y^2-2)(\gamma_{\mathrm{E}}+\ln(2y^2)) - 2 + \mathcal{I}(y)]\}. \quad (55)$$

This is the final result announced in Eq. (9), with $\mathcal{I}(y)$ calculated in (B53) and below.

V. COMPARISON TO NUMERICS

In this section, we compare our analytical results with numerical simulations. More specifically, we consider the superdiffusive process with $H = \frac{2}{3}$.

A. Methodology of simulations

We aim to sample a fBm processes $x(t)$ at discrete times $t_1 = 1, t_2 = 2, \ldots, t_L = L$. The covariance matrix of $\{x_1,\ldots,x_i,\ldots,x_L\}$ coincides with the autocorrelation function of the original fBm process in Eq. (6), setting $D = 1$,

$$C_{i,j} = \langle x_i x_j\rangle = i^{2H} + j^{2H} - |i-j|^{2H}. \quad (56)$$

The $L \times L$ covariance matrix C is symmetric and has positive eigenvalues; it is thus possible to find a matrix A, positive and symmetric, such that $C = A^2$. Matrix A is called the square root of C.

One can simulate paths of a fBm using the standard procedure for Gaussian correlated processes: (i) Determine A, the square root of C. (ii) Each path $\vec{x} = \{x_1,\ldots,x_i,\ldots,x_L\}$ is given by the matrix multiplication $\vec{x} = A\vec{\eta}$. The vector $\vec{\eta} = \{\eta_1,\eta_2,\ldots,\eta_L\}$ is a set of L independent Gaussian numbers with unitary variance and zero mean. It is easy to check that these paths are characterized by the correct covariance matrix (56).

Unfortunately, this procedure is time consuming, as for step (i) it requires the full diagonalization of C. Better results are obtained by making use of the stationarity of the increments $\xi_i = x_i - x_{i-1}$ (we set $x_1 = \xi_1$). Using Eq. (56) we can compute $\tilde{C}$, the covariance matrix of the increments,

$$\tilde{C}_{i,i+k} := \langle \xi_i \xi_{i+k}\rangle = |k-1|^{2H} + (k+1)^{2H} - 2k^{2H}, \quad (57)$$

where $k = 0,\ldots,L-i$ and $\tilde{C}_{i+k,i} = \tilde{C}_{i,i+k}$. The matrix $\tilde{C}$ is symmetric and positive definite like the matrix C, but it also is a Toeplitz matrix. For Toeplitz matrices efficient numerical methods make it possible to avoid the full diagonalization of $\tilde{C}$. In particular, the Levinson algorithm (for a practical implementation of Levinson's algorithm, see [39] and [40]) is suitable for first passage problems, as it recursively generates the increment ξ_{i+1} given $\xi_1,\ldots,\xi_i$. The points of the fBm path are given by $x_i = \sum_{j=1}^{i}\xi_j$. In our simulation we are interested only in positive paths ($x_i > 0$ for all i). The Levinson method makes it possible to discard negative paths whenever a $x_i < 0$ is generated, without building the full path.

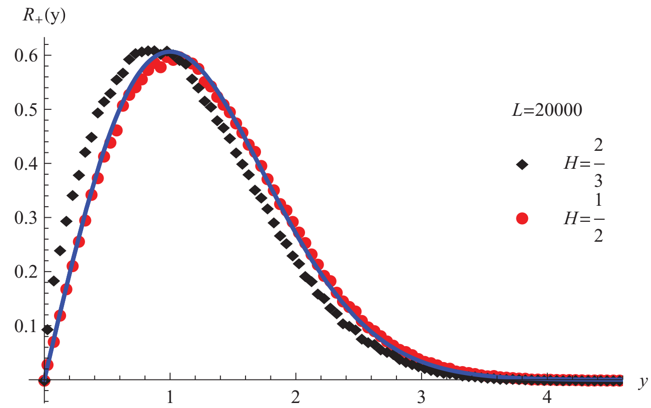

FIG. 5. (Color online) $R_+(y)$. Analytical result for Brownian motion $R_+^{(0)}(y) = ye^{-y^2/2}$ (solid blue line) and simulation data for $L = 20\,000$ and $H = \frac{1}{2}$ (red dots), as well as for the fBm with $H = \frac{2}{3}$ (black diamonds). Histograms are performed over 4×10^5 paths.

B. Simulation results

For each positive path we record the final position x_L. The histogram of the rescaled variable $y := x_L/(2L^H)$ is the scaling function $R_+(y)$. The results for $H = 2/3$ and the Markovian case $H = 1/2$ are presented in Figs. 5 and 6. For small y the scaling function, $R_+(y)$ behaves as a power law, with an exponent ϕ. For $H = 1/2$ we expect $\phi = 1$, for $H = 2/3$ we expect $\phi = 1/2$. Inspired by our perturbative calculation we predict that for $y \to \infty$, $R_+(y)$ behaves like $\sim y^{\gamma}e^{-y^2/2}$. In order to facilitate the comparison, we define the scaling function

$$r_+(y) := e^{\frac{y^2}{2}}R_+(y). \quad (58)$$

The numerical data for the scaling function $r_+(y)$ defined in Eq. (58) are shown in Fig. 7 for $H = 2/3$. They clearly show two distinctive power-law behaviors: For small y this power law is $\sim y^{\phi}$ with $\phi = \frac{1}{2}$, predicted by the scaling relation $\phi = \frac{1-H}{H}$. For large y a larger exponent $\gamma = 0.7 \pm 0.03$ is measured. This is consistent with the perturbative calculation, which suggests $\gamma > \phi$ for $H > 1/2$ and $\gamma < \phi$ for $H < 1/2$.

A more accurate comparison between the numerical data and the perturbation theory is possible. Our perturbative result given in Eq. (9) is equivalent to $r_+(y) = y[1+\epsilon W(y) + O(\epsilon^2)]$. In order to compare to numerics, we use

$$r_+^{\epsilon}(y) = y\,e^{\epsilon W(y)} + O(\epsilon^2). \quad (59)$$

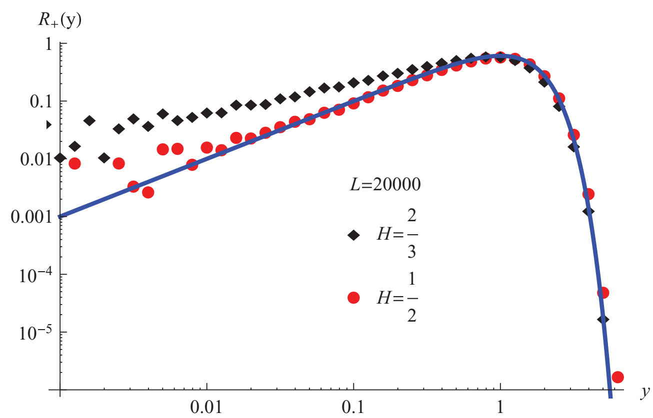

FIG. 6. (Color online) $R_+(y)$. Analytical result for Brownian motion $R_+^{(0)}(y) = ye^{-y^2/2}$ (solid blue line) and simulation data for $L = 20\,000$ and $H = \frac{1}{2}$ (red dots), as well as for the fBm with $H = \frac{2}{3}$ (black diamonds). Histograms are performed over 4×10^5 paths.

While the two expressions are equivalent to order ϵ, the latter (59) has the merit to resume the logarithms for small and large y into the power-law behavior

$$r_+^\epsilon(y) \sim \begin{cases} y^{\phi_\epsilon} & \text{for } y \to 0, \\ y^{\gamma_\epsilon} & \text{for } y \to \infty, \end{cases} \tag{60}$$

where the exponents are the order-ϵ results

$$\phi_\epsilon = 1 - 4\epsilon, \quad \gamma_\epsilon = 1 - 2\epsilon. \tag{61}$$

For $H = 2/3$, that is, $\epsilon = 1/6$, we predict a scaling $\sim y^{\gamma_\epsilon}$, $\gamma_\epsilon = \frac{2}{3}$, using (61). Note that the curve drawn is exactly the asymptotic behavior of our analytical result (59), using (10), thus also the *amplitude* and not only the exponent are estimated. This can more clearly be seen on Fig. 8, where the solid (blue) line represents the theoretical order-ϵ prediction, and the dashed line the asymptotic behaviors given in Eq. (60).

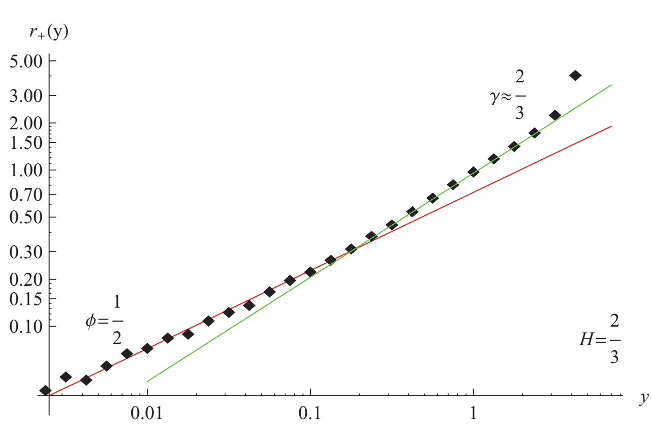

FIG. 7. (Color online) The numerically determined function $r_+(y)$, defined in Eq. (58) for the fBm with $H = \frac{2}{3}$ (black diamonds), using $L = 20\,000$ and 4×10^5 paths. The asymptotic small-y behavior is consistent with $\phi = \frac{1-H}{H} = \frac{1}{2}$. The large-$y$ asymptotics (including the amplitude) was taken from (59), with slope $\gamma \approx 1 - 2\epsilon = \frac{2}{3}$.

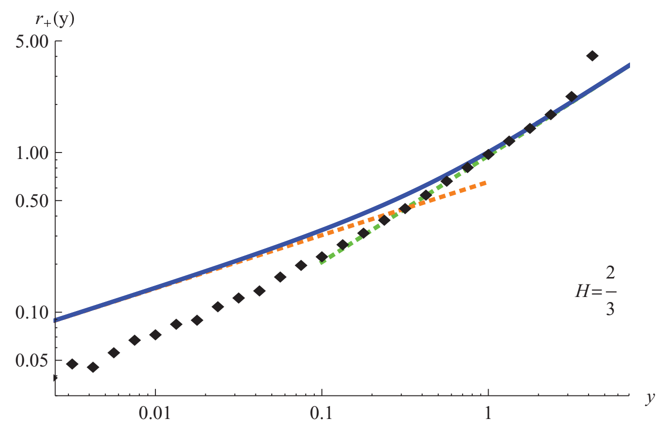

FIG. 8. (Color online) Blue solid line, the function $r_+^\epsilon(y)$, defined in Eq. (59) for $H = \frac{2}{3}$, that is, $\epsilon = \frac{1}{6}$. The dashed lines are the predicted asymptotic behaviors, $\sim y^\phi$ (for small y) and $\sim y^\gamma$ (for large y). Superimposed are the simulation data shown in Fig. 7. Note that there is no fitting parameter. The deviation at small y is due to the fact that ϵ is rather large, so the order-ϵ slope $\phi \approx 1 - 4\epsilon = \frac{1}{3}$ is smaller than the exact result $\phi = \frac{1-H}{H} \equiv \frac{1/2-\epsilon}{1/2+\epsilon}$, which evaluates to $\frac{1}{2}$. For large y, but smaller than 3, the effective cutoff of the simulation, both amplitude and slope are correctly predicted.

Conversely, relation (59) can be used to extract $W(y)$ from $r_+(y)$,

$$W(y) \approx \frac{1}{\epsilon} \ln\left(\frac{r_+(y)}{y}\right). \tag{62}$$

This relation should work the better, the smaller is ϵ. Using our numerical results for $H = \frac{2}{3}$, we obtain the curve presented in Fig. 9. The agreement is quite good for $1 \leqslant y \leqslant 2.5$. It breaks down for larger y due to numerical problems. For $y < 1$, the deviations can be attributed to the large value of ϵ.

VI. CONCLUSIONS

In this article, we develop a systematic scheme to calculate the corrections to the universal scaling function $R_+(y)$ for fBm in an $\epsilon = H - \frac{1}{2}$ expansion. We compute the full scaling function $R_+(y)$ to first order in ϵ. In particular we find that $R_+(y)$ behaves as $R_+(y) \sim y^\phi$ as $y \to 0$ (near the absorbing boundary), while $R_+(y) \sim y^\gamma \exp(-y^2/2)$ as $y \to \infty$ (far from the boundary), with, at the first order in ϵ, $\phi = 1 - 4\epsilon + O(\epsilon^2)$ and $\gamma = 1 - 2\epsilon + O(\epsilon^2)$. For small ϵ our results confirm the scaling relation found in Ref. [29]: $R_+(y) \sim y^\phi$ with $\phi = \theta/H$. For fBm it is known that $\theta = 1 - H$, so that $\phi = (1 - H)/H \approx 1 - 4\epsilon + \cdots$. Far from the boundary, that is, for large y, the leading behavior $R_+(y) \sim \exp(-y^2/2)$ recovers the Gaussian propagator (4) in absence of boundaries; our approach shows that $R_+(y)$ has a subleading power law prefactor y^γ, where γ is a new (independent) exponent.

Our numerical simulations show that the predictions of the asymptotic behavior of $R_+(y)$ hold at $H = 2/3$. In particular, the two exponents γ and ϕ have been measured and shown in Fig. 7.

Let us stress that few results are known about non-Markovian processes in presence of boundaries. Perturbation theory developed in this paper can provide substantial insight here. The method is versatile and can, in principle, be extended

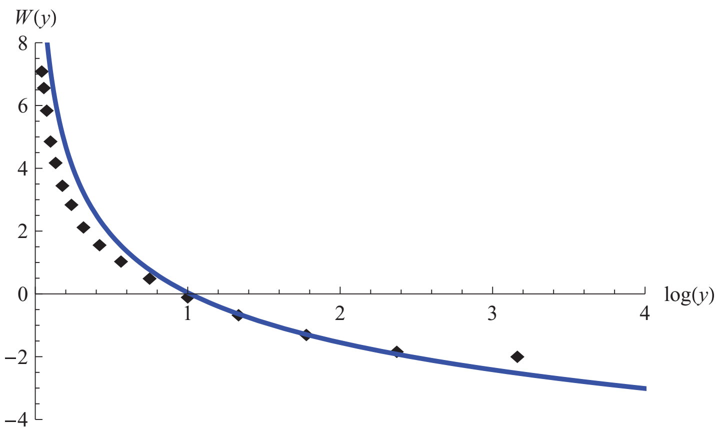

FIG. 9. (Color online) Blue solid line, the function $W(y)$, defined in Eq. (10). Black diamonds, estimation of $W(y)$ from the numerical data for $H = \frac{2}{3}$, using relation (62). The agreement is quite good for $1 \leqslant y \leqslant 3$. It breaks down for $y > 3$ due to numerical problems. For $y < 1$, the deviations can be attributed to the large value of ϵ.

to the calculation of other quantities such as the propagator for a process confined to a finite interval with absorbing boundaries, or alternatively with other, for example, reflecting boundary conditions. Particularly interesting for applications would be the hitting probability $Q(x,L)$, the probability that a generic stochastic process starting at x and evolving in a box $[0,L]$ hits the upper boundary at L before hitting the lower boundary at 0 [41]. In the context of polymer translocation, the hitting probability is the probability that a finite polymer chain will ultimately succeed in translocating through a pore.

In the more general framework of anomalous diffusion, presence of boundaries has been especially studied for non-Gaussian processes. For instance, Lévy flights are Markovian superdiffusive processes whose increments obey a Lévy stable (symmetric) law of index $0 \leqslant \mu \leqslant 2$. The Hurst exponent is $H = 1/\mu$ [42]. By virtue of the Sparre Andersen theorem [43], the persistence exponent is $\theta = 1/2$, independent of μ. The Laplace transform of the scaling function $R_+(y)$ has been computed in [44] for a generic value of μ. A scaling analysis of this Laplace transform shows that $R_+(y)$ behaves as $R_+(y) \sim y^{1/(2\mu)}$ as $y \to 0$ (this in in agreement with the scaling relation $\phi = \theta/H$), while far from the boundary the Lévy-stable behavior is recovered.

Increasing interest is devoted to Gaussian processes with self-affine anomalous displacements $\langle x^2(t)\rangle \sim t^{2H}$ with $0 < H < 1$ [16,45–49]. Our current results apply only to fBm, that is, self-affine Gaussian processes defined by the autocorrelation function (6). In particular, for fBm it is known that (i) the process has stationary increments, (ii) $\theta = 1 - H$, and (iii) $\phi = \theta/H$. For all other Gaussian processes with Hurst exponent H, (i) the increments are nonstationary, (ii) $\theta \neq 1 - H$, and we particularly emphasize that (iii) no scaling relation is known between ϕ and θ (unlike in fBm where $\phi = \theta/H$). Among such processes it is possible to show that the one, defined by the autocorrelation function

$$\langle x(t_1)x(t_2)\rangle \sim (t_1 + t_2)^{2H} - |t_1 - t_2|^{2H}, \tag{63}$$

describes the subdiffusive behavoir of a tagged monomer in an elastic interface which initially was flat [16]. For this process the persistence exponent is known only to first order in ϵ [16], whereas neither the exponents ϕ nor γ are known analytically. It would be interesting to determine the full scaling function $R_+(y)$ for this process within our perturbative framework.

ACKNOWLEDGMENTS

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APPENDIX A: THE ACTION

The aim of this appendix is to determine the action $\mathcal{S}^{(1)}[x]$, the first correction to the Brownian action, $\mathcal{S}^{(0)}[x]$, in the expansion of $\mathcal{S}[x]$ in Eq. (29). As a first step we expand the autocorrelation function (6) around $H = 1/2$, setting $D = 1$,

$$\begin{aligned}\langle x(t_1)x(t_2)\rangle &= G^{-1}(t_1,t_2)\\ &= [G^{(0)}]^{-1}(t_1,t_2) + \epsilon\, K(t_1,t_2) + O(\epsilon^2).\end{aligned} \tag{A1}$$

The first term is the autocorrelation function for $H = 1/2$,

$$[G^{(0)}]^{-1}(t_1,t_2) = 2\min(t_1,t_2), \tag{A2}$$

the second term gives the correction at first order in ϵ,

$$K(t_1,t_2) = 2\big[t_1 \ln(t_1) + t_2 \ln(t_2) - |t_1 - t_2| \ln|t_1 - t_2|\big]. \tag{A3}$$

Inverting Eq. (A1) and expanding up to order ϵ one gets

$$G = G^{(0)} + \epsilon G^{(1)} + O(\epsilon^2) \tag{A4}$$

$$G^{(1)} = -G^{(0)} K G^{(0)}, \tag{A5}$$

where $G^{(0)}(t_1,t_2)$ is defined as

$$\int_0^\infty dt' G^{(0)}(t_1,t')[G^{(0)}]^{-1}(t',t_2) = \delta(t_1 - t_2). \tag{A6}$$

One can check that the kernel of the Brownian action, $\mathcal{S}^{(0)}[x]$, that is,

$$G^{(0)}(t_1,t_2) = -\frac{1}{2}\,\delta''(t_1 - t_2), \tag{A7}$$

satisfies Eq. (A6), namely,

$$\begin{aligned}-\frac{1}{2}\int_0^\infty dt'\delta''(t_1-t')[G^{(0)}]^{-1}(t',t_2) &= -\int_0^\infty dt'\delta''(t_1-t')\min(t',t_2)\\ &= -\partial_{t_1}^2 \min(t_1,t_2) = \delta(t_1 - t_2).\end{aligned} \tag{A8}$$

It remains to compute the term $G^{(1)}$. Integrating by parts one has

$$G^{(1)}(t_1,t_2) = -\frac{1}{4}\int_0^t \int_0^t \mathrm{d}t'\,\mathrm{d}t''\,\delta'(t_1-t') \times \delta'(t_2-t'')\partial_{t'}\partial_{t''}K(t',t'') = \frac{1}{2}\int_0^t \int_0^t \mathrm{d}t'\,\mathrm{d}t''\,\delta'(t_1-t')\delta'(t_2-t'') \times \partial_{t'}\partial_{t''}(|t'-t''|\ln|t'-t''|), \tag{A9}$$

using that the first two terms in (A3) do not contribute since they only depend on one of the times. The derivative is

$$\partial_{t'}\partial_{t''}(|t'-t''|\ln|t'-t''|) = -\frac{1}{|t'-t''|} - 2\delta(t'-t'') \times(1+\ln|t'-t''|). \tag{A10}$$

The second term is not well defined. We decide to introduce a regularization for coinciding times $t = t' \to \ln|t-t'| = \ln\tau$ where $\tau > 0$ should be thought of as the time-discretization of the path integral. Let us first give the final result before commenting on this approximation:

$$G^{(1)}(t_1,t_2) = -\frac{1}{2}\int_0^t \int_0^t \mathrm{d}t'\,\mathrm{d}t''\,\delta'(t_1-t')\frac{1}{|t'-t''|}\delta'(t_2-t'') - 2(1+\ln\tau)[G^{(0)}]. \tag{A11}$$

This yields for the action

$$\mathcal{S}^{(1)}[x] = \int_0^t dt_1 \int_0^t dt_2\, \frac{1}{2}x(t_1)G^{(1)}(t_1,t_2)x(t_2) = -\frac{1}{4}\int_0^t dt_1 \int_0^t dt_2\, \frac{\partial_{t_1}x(t_1)\partial_{t_2}x(t_2)}{|t_1-t_2|} - 2\,\mathcal{S}^{(0)}[x](1+\ln\tau). \tag{A12}$$

We see that the only possibly ambiguous term, the term of order $\ln\tau$, is proportional to the zeroth-order action $\mathcal{S}^{(0)}[x]$, thus equivalent to a change in the diffusion constant D. Thus, its effect is easy to check in the final result, when looking at observables in a domain unaffected by the boundary.

APPENDIX B: EVALUATION OF $Z_+^{(1)}(x,t)$

A. Evaluation of $Z_+^A(x,t)$

This term is easily evaluated. Indeed, Eq. (36) can be recast in the following form:

$$\begin{aligned} Z_+^A(x,t) &= -2(1+\ln\tau) \times \lim_{x_0\to 0}\frac{1}{x_0}\frac{\partial}{\partial a}\bigg|_{a=1}\int_{x(0)=x_0}^{x(t)=x}\mathcal{D}[x]e^{-a\mathcal{S}^{(0)}[x]}\,\Theta[x] \\ &= -2(1+\ln\tau) \times \lim_{x_0\to 0}\frac{1}{x_0}\frac{\partial}{\partial a}\bigg|_{a=1}\sqrt{\frac{a}{4\pi t}}\left[e^{-\frac{a}{4t}(x-x_0)^2} - e^{-\frac{a}{4t}(x+x_0)^2}\right] \\ &= (1+\ln\tau)\frac{x}{\sqrt{4\pi t^3}}e^{-\frac{x^2}{4t}}\left(\frac{x^2}{2t}-3\right). \end{aligned} \tag{B1}$$

In going from the first to the second line we have used the expression of the propagator in the Brownian case in Eq. (20), introducing the factor of a from the observation that the latter appears together with x^2 and readjusting the normalization.

In terms of the variable $z = x/\sqrt{2t}$, this gives

$$Z_+^A(z,t) = Z_+^{(0)}(z,t)A(z), \tag{B2}$$

where $Z_+^{(0)}(z,t) = ze^{-z^2/2}/(\sqrt{2\pi}t)$ is defined in (23) and

$$A(z) = (1+\ln\tau)(z^2-3). \tag{B3}$$

B. $\tilde{Z}_+^B(x_0,x,s)$: The integration over k

We split $\tilde{Z}_+^B(x_0,x,s)$ into two parts:

$$\tilde{Z}_+^B(x_0,x,s) = \tilde{I}_1(x_0,x,s) + \tilde{I}_2(x_0,x,s), \tag{B4}$$

$$\tilde{I}_1(x_0,x,s) = \frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}] \times[\cos(kx_0\sqrt{s}) - e^{-x_0\sqrt{s}}] \times\frac{k^2\ln(1+k^2)}{\sqrt{s}(1+k^2)^2}, \tag{B5}$$

$$\tilde{I}_2(x_0,x,s) = \frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}] \times[\cos(kx_0\sqrt{s}) - e^{-x_0\sqrt{s}}] \times\frac{k^2[\ln(\tau s)+\gamma_{\mathrm{E}}]}{\sqrt{s}(1+k^2)^2}. \tag{B6}$$

1. $\tilde{I}_1(x_0,x,s)$

The expansion of this term for small x_0 must be done with care; when x_0 acts as a regulator, one cannot simply expand in it. We claim and show below that

$$\tilde{I}_1(x_0,x,s) = \tilde{I}_1^A(x_0,x,s) + \tilde{I}_1^B(x,s) + O(x_0), \tag{B7}$$

with

$$\tilde{I}_1^A(x_0,x,s) = -\frac{4e^{-x\sqrt{s}}}{x_0\sqrt{s}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx_0\sqrt{s}) - 1] \times\frac{k^2\ln(1+k^2)}{(1+k^2)^2}, \tag{B8}$$

$$\tilde{I}_1^B(x,s) = 4\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}] \times\frac{k^2\ln(1+k^2)}{(1+k^2)^2}. \tag{B9}$$

In order to prove this, we group the four terms in (B5) into two-times-two terms; the first combination is

$$\begin{aligned} &-\frac{4e^{-x_0\sqrt{s}}}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}]\frac{k^2\ln(1+k^2)}{(1+k^2)^2} \\ &= \left[-\frac{4}{x_0} + 4\sqrt{s} + O(x_0)\right] \times\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}]\frac{k^2\ln(1+k^2)}{(1+k^2)^2} \\ &= \tilde{I}_1^{\mathrm{div}}(x_0,x,s) + \tilde{I}_1^B(x,s) + O(x_0), \end{aligned} \tag{B10}$$

where the divergent contribution is

$$\tilde{I}_1^{\text{div}}(x_0,x,s) = -\frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}] \times \frac{k^2\ln(1+k^2)}{(1+k^2)^2}. \tag{B11}$$

This expansion in x_0 is justified since $\frac{e^{-x_0\sqrt{s}}}{x_0}$ stands outside the integrand, thus does not act as a regulator.

The second contribution to (B5) is

$$\begin{aligned}\frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}&[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}]\cos(kx_0\sqrt{s})\frac{k^2\ln(1+k^2)}{(1+k^2)^2}\\ = \frac{2}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}&[\cos(k(x+x_0)\sqrt{s}) + \cos(k(x-x_0)\sqrt{s})\\ -2e^{-x\sqrt{s}}&\cos(kx_0\sqrt{s})]\frac{k^2\ln(1+k^2)}{(1+k^2)^2}.\end{aligned} \tag{B12}$$

Since $x \gg x_0$, we can Taylor-expand $\cos\big(k(x+x_0)\sqrt{s}\big)$ and $\cos\big(k(x-x_0)\sqrt{s}\big)$, leading to

$$\begin{aligned}\frac{4}{x_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}&[\cos(kx\sqrt{s}) - e^{-x\sqrt{s}}\cos(kx_0\sqrt{s})]\\ &\times\frac{k^2\ln(1+k^2)}{(1+k^2)^2} + O(x_0)\\ &= -\tilde{I}_1^{\text{div}}(x_0,x,s) + \tilde{I}_1^{A}(x_0,x,s) + O(x_0).\end{aligned} \tag{B13}$$

The contributions proportional to $\tilde{I}_1^{\text{div}}$ cancel between (B10) and (B13), and we arrive at the decomposition (B7).

We now treat the two contributions to (B7). The first contribution $\tilde{I}_1^A(x_0,x,s)$ can be evaluated analytically. After integration over k we find a Bessel function, which can be expanded in x_0 as

$$\tilde{I}_1^A(x_0,x,s) = -4e^{-\sqrt{s}x}[\ln(x_0) + \tfrac{1}{2}\ln(s) + \gamma_{\mathrm{E}} - 1] + O(x_0). \tag{B14}$$

The second contribution $\tilde{I}_1^B(x,s)$ can be evaluated using the relation

$$\frac{k^2}{(1+k^2)^2}\ln(1+k^2) = \left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\frac{1}{(1+k^2)^{u+1}}. \tag{B15}$$

We rewrite $\tilde{I}_1^A(x,s)$ as

$$\tilde{I}_1^B(x,s) = 4\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{e^{ikx\sqrt{s}} - e^{-x\sqrt{s}}}{(1+k^2)^{u+1}}. \tag{B16}$$

It can be split into two parts,

$$\tilde{I}_1^B(x,s) = \tilde{I}_{1a}(x,s) + \tilde{I}_{1b}(x,s), \tag{B17}$$

$$\begin{aligned}\tilde{I}_{1a}(x,s) &= -4\,e^{-x\sqrt{s}}\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{(1+k^2)^{u+1}}\\ &= -[1+\ln(4)]\,e^{-x\sqrt{s}},\end{aligned} \tag{B18}$$

$$\begin{aligned}&\tilde{I}_{1b}(x,s)\\ &= \left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]4\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}e^{ikx\sqrt{s}}(1+k^2)^{-(u+1)}.\end{aligned} \tag{B19}$$

To do the k integral in $\tilde{I}_{1b}(x,s)$, it is useful to introduce the integral representation

$$(1+k^2)^{-(u+1)} = \frac{1}{\Gamma(1+u)}\int_0^{\infty}\mathrm{d}z\, z^u e^{-(1+k^2)z}. \tag{B20}$$

This gives

$$\begin{aligned}\tilde{I}_{1b}(x,s) = 4&\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\left[\frac{1}{\Gamma(1+u)}\right.\\ &\left.\times\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}e^{ikx\sqrt{s}}\int_0^{\infty}\mathrm{d}z\, z^u e^{-(1+k^2)z}\right],\end{aligned} \tag{B21}$$

and performing the Gaussian integral over k yields

$$\begin{aligned}\tilde{I}_{1b}(x,s) = 4&\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\left[\frac{1}{\Gamma(1+u)}\right.\\ &\left.\times\int_0^{\infty}\frac{\mathrm{d}z}{2\sqrt{\pi}}z^{u-1/2}e^{-\frac{sx^2}{4z}-z}\right].\end{aligned} \tag{B22}$$

2. $\tilde{I}_2(x_0,x,s)$

$\tilde{I}_2(x_0,x,s)$ can be calculated using residue calculus. We use $x_0 < x$ to expand the expression, choosing every pole in the half plane in which the corresponding exponential factor converges. The result is

$$\begin{aligned}\tilde{I}_2(x_0,x,s) = \frac{\gamma_{\mathrm{E}} + \ln(\tau s)}{2\sqrt{s}x_0}&\left[\frac{\sqrt{s}(x_0+x) - 1}{e^{\sqrt{s}(x+x_0)}}\right.\\ &\left.-\frac{\sqrt{s}(x-x_0) - 1}{e^{\sqrt{s}(x-x_0)}}\right].\end{aligned} \tag{B23}$$

Expanding for small x_0 yields

$$\tilde{I}_2(x_0,x,s) = e^{-\sqrt{s}x}(2 - \sqrt{s}x)[\ln(\tau s) + \gamma_{\mathrm{E}}] + O(x_0). \tag{B24}$$

3. Summary of all terms contributing to $\tilde{Z}_+^B(x_0,x,s)$

It is useful to reorganize

$$\begin{aligned}\tilde{Z}_+^B(x_0,x,s) = \tilde{I}_1^A(x_0,x,s) &+ \tilde{I}_{1a}(x,s) + \tilde{I}_{1b}(x,s)\\ &+ \tilde{I}_2(x,s) + O(x_0)\end{aligned} \tag{B25}$$

as the sum of three contributions:

$$\tilde{Z}_+^B(x_0,x,s) = \tilde{J}_0(x_0,x,s) + \tilde{J}_1(x,s) + \tilde{J}_2(x,s) + O(x_0). \tag{B26}$$

The first term depends on x_0,

$$\tilde{J}_0(x_0,x,s) = e^{-x\sqrt{s}}\,[3 - 2\gamma_{\mathrm{E}} + 2\ln(\tau/2) - 4\ln(x_0)]\,, \tag{B27}$$

while the other two terms are

$$\tilde{J}_1(x,s) = 4\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\left[\frac{1}{\Gamma(1+u)}\right.$$

$$\times \int_0^\infty \frac{\mathrm{d}z}{2\sqrt{\pi}} z^{u-1/2} e^{-\frac{sx^2}{4z}-z}\bigg], \tag{B28}$$

$$\tilde{J}_2(x,s) = -x\sqrt{s}e^{-x\sqrt{s}}\,[\gamma_{\mathrm{E}} + \ln(\tau s)]\,. \tag{B29}$$

C. $Z_+^B(x,t)$: The inverse Laplace transform of $\tilde{Z}_+^B(x,s)$

The inversion of $\tilde{J}_0(x_0,x,s)$ is done by observing that

$$\tilde{Z}_+^{(0)}(x,s) = \lim_{x_0\to 0}\frac{1}{x_0}Z_+(x_0,x,s) = e^{-\sqrt{s}x}. \tag{B30}$$

This yields

$$J_0(x_0,x,t) = Z_+^{(0)}(x,t)B_0(x_0), \tag{B31}$$

where $Z_+^{(0)}(x,t) = \frac{x}{2\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4t}}$, and

$$B_0(x_0) = 3 - 2\gamma_{\mathrm{E}} + 2\ln(\tau/2) - 4\ln x_0. \tag{B32}$$

The inverse Laplace transform of the second term can be done directly,

$$\begin{aligned}J_1(x,t) &:= \mathcal{L}_s^{-1}[\tilde{J}_1(x,s)]\\ &= 2\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\bigg[\frac{1}{\Gamma(1+u)}\\ &\quad\times\int_0^\infty \frac{\mathrm{d}z}{\sqrt{\pi}}z^{u-1/2}e^{-z}\delta\left(\frac{x^2}{4z}-t\right)\bigg].\end{aligned} \tag{B33}$$

We observe that $\delta(\frac{x^2}{4z}-t) = \delta(\frac{x^2}{4t}-z)z/t$ and obtain

$$J_1(x,t) = \frac{2}{\sqrt{\pi}t}e^{-\frac{x^2}{4t}}\left[\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=1} - \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0}\right]\frac{\left(\frac{x^2}{4t}\right)^{u+1/2}}{\Gamma(1+u)}. \tag{B34}$$

Finally,

$$\begin{aligned}J_1(x,t) &= \frac{1}{\sqrt{2\pi}t}\frac{x}{\sqrt{2t}}e^{-\frac{x^2}{4t}}\\ &\quad\times\left\{\frac{x^2}{2t}\left[\gamma_{\mathrm{E}} - 1 - \ln 2 + \ln\left(\frac{x^2}{2t}\right)\right]\right.\\ &\quad\left. - 2\left[\gamma_{\mathrm{E}} + \ln\left(\frac{x^2}{2t}\right) - \ln 2\right]\right\}.\end{aligned} \tag{B35}$$

Introducing the variable $z = x/\sqrt{2t}$ we have

$$J_1(z,t) = \frac{z}{\sqrt{2\pi}t}e^{-z^2/2}B_1(z) = Z_+^{(0)}(z,t)B_1(z), \tag{B36}$$

where $Z_+^{(0)}(z,t) = z\exp(-z^2/2)/\sqrt{2\pi}t$ [see Eq. (23)] and

$$B_1(z) = (z^2-2)(\gamma_{\mathrm{E}} - 1 + 2\ln z - \ln 2) - 2. \tag{B37}$$

This completes the Laplace inversion of $\tilde{J}_1(x,s)$.

1. $\tilde{J}_2(x,s)$

The Laplace inversion of the second term $\tilde{J}_2(x,s)$ is more complicated, and we split it as

$$\begin{aligned}\tilde{J}_2(x,s) &= -[\ln(\tau s) + \gamma_{\mathrm{E}}]x\sqrt{s}e^{-x\sqrt{s}}\\ &= x(\ln\tau + \gamma_{\mathrm{E}})\frac{\mathrm{d}}{\mathrm{d}x}e^{-x\sqrt{s}} + x\frac{\mathrm{d}}{\mathrm{d}x}(\ln s\, e^{-x\sqrt{s}})\\ &= \tilde{J}_{2a}(x,s) + \tilde{J}_{2b}(x,s).\end{aligned} \tag{B38}$$

It is easy to perform the Laplace inversion of the first term:

$$\begin{aligned}J_{2a}(x,t) &= (\ln\tau + \gamma_{\mathrm{E}})\frac{x}{\sqrt{4\pi t^3}}\frac{\mathrm{d}}{\mathrm{d}x}xe^{-\frac{x^2}{4t}}\\ &= (\ln\tau + \gamma_{\mathrm{E}})\frac{x}{\sqrt{4\pi t^3}}e^{-\frac{x^2}{4t}}\left(1 - \frac{x^2}{2t}\right).\end{aligned} \tag{B39}$$

The inverse Laplace transform of the second term can be written as

$$J_{2b}(x,t) = x\frac{\mathrm{d}}{\mathrm{d}x}f(x,t), \tag{B40}$$

where

$$\int_0^\infty e^{-st}f(x,t)\mathrm{d}t = e^{-x\sqrt{s}}\ln s = \tilde{g}_1(s)\tilde{g}_2(s), \tag{B41}$$

$$\tilde{g}_1(s) = \sqrt{s}e^{-x\sqrt{s}}, \tag{B42}$$

$$\tilde{g}_2(s) = \frac{\ln s}{\sqrt{s}}. \tag{B43}$$

The idea is to inverse-Laplace transform $\tilde{g}_1(s)$ and $\tilde{g}_2(s)$, and then to calculate $f(x,t)$ as convolution of $g_1(t)$ and $g_2(t)$, using (41). These inverses are

$$g_1(t) = \frac{1}{2\sqrt{\pi t^3}}\left(\frac{x^2}{2t} - 1\right)e^{-\frac{x^2}{4t}}, \tag{B44}$$

$$g_2(x,t) = -\frac{\ln(4t) + \gamma_{\mathrm{E}}}{\sqrt{\pi t}}. \tag{B45}$$

The convolution is

$$\begin{aligned}f(x,t) &= \int_0^t g_1(t')g_2(t-t')\mathrm{d}t' \qquad \text{(B46)}\\ &= -\int_0^t \frac{\mathrm{d}t'}{2\pi t'^{3/2}}\left(\frac{x^2}{2t'} - 1\right)e^{-\frac{x^2}{4t'}}\frac{\ln(4[t-t']) + \gamma_{\mathrm{E}}}{\sqrt{t-t'}}.\end{aligned}$$

Using (B40) we have

$$J_{2b}(x,t) = \frac{x^2}{4\pi}\int_0^t \frac{\mathrm{d}t'}{t'^{5/2}}\left[\frac{x^2}{2t'} - 3\right]e^{-\frac{x^2}{4t'}}\frac{\ln(4[t-t']) + \gamma_{\mathrm{E}}}{\sqrt{t-t'}}. \tag{B47}$$

Making a change of variables $t' = ut$, and using $z = x/\sqrt{2t}$, this gives

$$\begin{aligned}J_{2b}(z,t) &= \frac{z^2}{2\pi t}\int_0^1 \frac{\mathrm{d}u}{u^{5/2}\sqrt{1-u}}\left[\frac{z^2}{u} - 3\right]\\ &\quad\times e^{-\frac{z^2}{2u}}\,[\ln(4t) + \gamma_{\mathrm{E}} + \ln(1-u)]\,.\end{aligned} \tag{B48}$$

The integral contains two pieces, which we note

$$J_{2b}(z,t) = \frac{[\ln(4t) + \gamma_{\mathrm{E}}]F_2(z) + F_3(z)}{t}. \tag{B49}$$

The first piece is

$$\begin{aligned}F_2(z) &:= \frac{z^2}{2\pi}\int_0^1 \frac{\mathrm{d}u}{u^{5/2}\sqrt{1-u}}\left(\frac{z^2}{u} - 3\right)e^{-\frac{z^2}{2u}}\\ &= e^{-\frac{z^2}{2}}\frac{z}{\sqrt{2\pi}}(z^2 - 1).\end{aligned} \tag{B50}$$

The second integral,

$$F_3(z) := \frac{z^2}{2\pi}\int_0^1 \frac{\mathrm{d}u}{u^{5/2}\sqrt{1-u}}\ln(1-u)\left(\frac{z^2}{u}-3\right)e^{-\frac{z^2}{2u}}, \tag{B51}$$

is more difficult, but can be performed using MATHEMATICA. A convenient substitution $\alpha = z^2(1/u-1)$ makes it possible to write

$$F_3(z) = e^{-\frac{z^2}{2}}\frac{z}{\sqrt{2\pi}}\mathcal{I}(z), \tag{B52}$$

where

$$\begin{aligned}\mathcal{I}(z) &= \frac{1}{\sqrt{2\pi}z^2}\int_0^\infty \frac{\mathrm{d}\alpha}{\sqrt{\alpha}}\ln\Big(\frac{\alpha}{z^2+\alpha}\Big)(z^2+\alpha)(z^2+\alpha-3)e^{-\frac{\alpha}{2}}\\ &= \frac{z^4}{6}\,{}_2F_2\Big(1,1;\frac{5}{2},3;\frac{z^2}{2}\Big)+\pi(1-z^2)\mathrm{erfi}(z/\sqrt{2})\\ &\quad -3z^2+\sqrt{2\pi}e^{\frac{z^2}{2}}z+2.\end{aligned} \tag{B53}$$

erfi is the imaginary error function,

$$\mathrm{erfi}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}z\, e^{z^2}. \tag{B54}$$

The hypergeometric function ${}_2F_2(1,1;\frac{5}{2},3;z^2/2)$ can be defined by its series expansion

$${}_2F_2\left(1,1;\frac{5}{2},3;\frac{z^2}{2}\right) = 24\sum_{n=0}^{\infty}\frac{n!(2z^2)^n}{(2n+4)!}. \tag{B55}$$

The error function and the exponential function can be combined in another converging series,

$$\begin{aligned}&e^{\frac{z^2}{2}}z-\sqrt{\frac{\pi}{2}}\left(z^2-1\right)\mathrm{erfi}\Big(\frac{z}{\sqrt{2}}\Big)\\ &\quad = -\sum_{n=0}^{\infty}\frac{2^{1-n}z^{2n+1}}{(2n-1)(2n+1)n!}.\end{aligned} \tag{B56}$$

While problems of numerical precision appear for $y > 7$, we can use the asymptotic expansion

$$\mathcal{I}(z) = 1-\gamma_{\mathrm{E}}-\ln(2z^2)+\frac{1}{2z^2}-\frac{1}{2z^4}+\frac{5}{4z^6}+O(z^{-8}). \tag{B57}$$

At $z = 7$, the relative numerical agreement of (B57) and (B53) is about 10^{-6}.

Note that $\int_0^\infty \mathrm{d}z\, ze^{-z^2/2}\mathcal{I}(z) = 0$; thus, $\mathcal{I}(z)$ does not contribute to the normalization.

2. $J_2(x,t) = J_{2a}(x,t) + J_{2b}(x,t)$

The sum $J_2(x,t) = J_{2a}(x,t) + J_{2b}(x,t)$ can be expressed using the variable $z = x/\sqrt{2t}$ as

$$J_2(z,t) = Z_+^{(0)}(z,t)B_2(z,t), \tag{B58}$$

where $Z_+^{(0)}(z,t) = ze^{-z^2/2}/(\sqrt{2\pi}\,t)$ and

$$B_2(z) = (z^2-1)\ln(4t/\tau)+\mathcal{I}(z). \tag{B59}$$

D. Summary of all terms

In summary,

$$\begin{aligned}Z_+^{(1)}(z,t) &= Z_+^{(0)}(z,t)[A(z)+B_0(x_0)+B_1(z)\\ &\quad +B_2(z)-a_1\ln x_0],\end{aligned} \tag{B60}$$

where $Z_+^{(0)}(z,t) = ze^{-z^2/2}/(\sqrt{2\pi t})$ is defined in Eq. (23). The terms in question are given in Eqs. (B3), (B32), (B37), and (B59), and repeated here:

$$A(z) = (1+\ln\tau)\left(z^2-3\right), \tag{B61}$$

$$B_0(x_0) = 3-2\gamma_{\mathrm{E}}+2\ln(\tau/2)-4\ln x_0, \tag{B62}$$

$$B_1(z) = (z^2-2)(\gamma_{\mathrm{E}}-1+2\ln z-\ln 2)-2, \tag{B63}$$

$$B_2(z) = \left(z^2-1\right)\ln(4t/\tau)+\mathcal{I}(z). \tag{B64}$$

Their sum is

$$\begin{aligned}&A(z)+B_0+B_1(z)+B_2(z)-a_1\ln x_0\\ &\quad = \{(z^2-2)[\ln(2z^2t)+\gamma_{\mathrm{E}}]-2\}+\mathcal{I}(z)\\ &\qquad -(4+a_1)\ln x_0+c(t),\\ &c(t) = \ln(t)+2-2\gamma_{\mathrm{E}}.\end{aligned} \tag{B65}$$

The result is arranged such that the term in the curly brackets, when multiplied by $Z_+^{(0)}(z,t)$, integrates to zero, as does $Z_+^{(0)}(z,t)\mathcal{I}(z)$. The propagator $Z_+^{(1)}(z,t)$ becomes independent of x_0 if $a_1 = -4$, equivalent to $\phi_0 = 1-4\epsilon+O(\epsilon^2)$. As expected, $\phi_0 = \phi$ [see Eq. (15)].

Since $c(t)$ only contributes to the (time-dependent) normalization, it does not enter the scaling function $R_+(y)$.

On the other hand, the only contribution to the normalization of the propagator $Z_+(x,t)$ comes from $c(t)$. Since $Z_+^{(0)}(x,t)$ integrated over x equals 1, we conclude that the survival-probability is

$$\begin{aligned}S(x_0,t) &= t^{-\frac{1}{2}}[1+\epsilon(2-2\gamma_{\mathrm{E}}+\ln t)] \sim t^{-\theta},\\ \theta &= \frac{1}{2}-\epsilon+O(\epsilon^2),\end{aligned} \tag{B66}$$

in agreement with $\theta = 1-H$. This is a nontrivial check of our calculations.

APPENDIX C: SCALING ARGUMENTS

Consider a process $x(t')$, starting at $x(0) = x_0$, and arriving at x at time t, without having crossed zero, that is, $x(t') > 0$ for all $t' < t$. Denote $Z_+(x_0,x,t)$ its arrival probability density at x. Further denote

$$S(x_0,t) := \int_0^\infty \mathrm{d}x\, Z_+(x_0,x,t) \tag{C1}$$

the survival probability or the persistence up to time t. At late times and fixed x_0, for many processes, this survival probability decays algebraically,

$$S(x_0,t) \sim t^{-\theta}, \tag{C2}$$

where θ is the persistence exponent [3]. Let us now assume that the process $x(t)$ is self-affine. This simply means that the process is characterized by a single growing length scale $x \sim t^H$, where H is the Hurst exponent of the process. For example, ordinary Brownian motion is a self-affine process with $H = 1/2$. Since the only length scale is $x \sim t^H$, the survival probability $S(x_0,t)$ is a function of only the scaled variable $y_0 = x_0/t^H$, that is, $S(x_0,t) = G\left(\frac{x_0}{t^H}\right)$. In order that $S(x_0,t) \sim t^{-\theta}$ for large t and fixed x_0, the scaling function $G(y)$, for small y, must behave as

$$G(y_0) \sim y_0^{\phi}, \qquad \text{where } \phi = \frac{\theta}{H}. \tag{C3}$$

We next define $p_{x_0}(x,t)$ as the conditional probability density of finding the walker, given that it has not been absorbed at any previous time:

$$p_{x_0}(x,t) = \frac{Z_+(x_0,x,t)}{\int_0^\infty \mathrm{d}x\, Z_+(x_0,x,t)} = \frac{Z_+(x_0,x,t)}{S(x_0,t)}. \tag{C4}$$

Note that following Eq. (22), the probability distribution of a nonadsorbed particle is for $x_0 \to 0$:

$$P_+(x,t) = p_0(x,t) = \lim_{x_0\to 0} \frac{Z_+(x_0,x,t)}{\int_0^\infty \mathrm{d}x\, Z_+(x_0,x,t)}. \tag{C5}$$

We anticipate the following scaling form for $Z_+(x_0,x,t)$:

$$Z_+(x_0,x,t) = \frac{1}{t^H} F\left(\frac{x_0}{t^H},\frac{x}{t^H}\right). \tag{C6}$$

In terms of the scaling variables $y = x/t^H$ and $y_0 = x_0/t^H$ we get from (C4) and (C6)

$$F(y,y_0) = G(y_0) p_{y_0}(y), \tag{C7}$$

where $p_{y_0}(y)$ is the conditional probability density (C4) expressed in terms of the rescaled variables. In the long-time limit, $y_0 \to 0$ and $F(y,y_0)$ can be factorized as

$$F(y,y_0) \sim y_0^{\theta/H} p_0(y) = y_0^{\theta/H} R_+(y). \tag{C8}$$

Let us now consider the limit $y \to 0$ and suppose that $p_0(y) = R_+(y) \sim y^{\phi}$. The process is time-reversible invariant, since its increments are stationary; that is, a path from x_0 to x forward in time plays the same role as a path from x to x_0 backward in time. As a consequence, $F(y,y_0)$ is a symmetric function, $F(y,y_0) = F(y_0,y)$. Factorization of probabilities for x and x_0 to zero and symmetry thus implies $F(y,y_0) \sim (y_0 y)^{\theta/H}$ and it follows the proposed scaling relation $\phi = \theta/H$.

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