Generalized Langevin equation with multiplicative noise: Temporal behavior of the autocorrelation functions

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The temporal behavior of the mean-square displacement and the velocity autocorrelation function of a particle subjected to a periodic force in a harmonic potential well is investigated for viscoelastic media using the generalized Langevin equation. The interaction with fluctuations of environmental parameters is modeled by a multiplicative white noise, by an internal Mittag-Leffler noise with a finite memory time, and by an additive external noise. It is shown that the presence of a multiplicative noise has a profound effect on the behavior of the autocorrelation functions. Particularly, for correlation functions the model predicts a crossover between two different asymptotic power-law regimes. Moreover, a dependence of the correlation function on the frequency of the external periodic forcing occurs that gives a simple criterion to discern the multiplicative noise in future experiments. It is established that additive external and internal noises cause qualitatively different dependences of the autocorrelation functions on the external forcing and also on the time lag. The influence of the memory time of the internal noise on the dynamics of the system is also discussed.

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I. INTRODUCTION

In complex systems an ensemble of conditions far from thermal equilibrium and the influence of environmental fluctuations may give rise to unexpected phenomena, which are ruled out by the second law of thermodynamics under equilibrium conditions [1–3]. Among them we can mention the ratchet effect [1,4], stochastic resonance [3,5], noise-induced phase transitions in spatially extended systems [2,6], noise-enhanced stability [7], and hypersensitive response [8], to name a few. Particularly, the study of anomalous diffusion in complex or disordered media has achieved substantial progress during recent years [9–16]. It is well known that the conventional Brownian motion theory cannot account for anomalous diffusion processes in which the mean-square displacement is not proportional to time. Examples of such systems are supercooled liquids, glasses, colloidal suspensions, polymer solutions [12,13], viscoelastic media [14], and amorphous semiconductors [15]. Even anomalous diffusive dynamics of atoms in biological macromolecules and intrinsic conformational dynamics of proteins can be subdiffusive [9,16].

There are several approaches to describe anomalous diffusion processes, where the dynamical origin of the phenomenon is considered as a nonlocality, either in space or in time [10]. One of the possibilities for modeling such processes can be formulated in the framework of the generalized Langevin equation (GLE) [9–11,17]. The dynamical equation for such systems is in most cases obtained by replacing the usual friction term by a generalized friction term with a power-law-type memory [9–11,17]. Physically such a friction term has, due to the fluctuation-dissipation theorem, its origin in a non-ohmic thermal bath whose influence on the dynamical system is described with a power-law correlated additive noise in the GLE [9,10]. In spite of the fact that a GLE with a power-law-type friction kernel is very useful for modeling anomalous diffusion processes, the corresponding power-law correlated noise has some nonphysical properties, e.g., absence of a characteristic memory time and infinite variance. Thus, recently Viñales and Despósito have introduced a more general noise with a Mittag-Leffler correlation function (called Mittag-Leffler noise) in the GLE [18]. Notably, for certain values of the parameters that characterize this noise one can produce a power-law correlation function, a standard Ornstein-Uhlenbeck noise with an exponential correlation function, and a white noise. Although the behavior of the GLE with an additive noise for a particle in viscoelastic media under the influence of a trapping harmonic potential has been investigated in some detail [17–19], it seems that analysis of the potential consequences of interplay between a multiplicative noise and memory effects is still rather rare in literature [20,21]. This is quite unjustified in view of the fact that the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g., living cells, has been well recognized [14,22].

Thus motivated, the authors of [20] have recently considered a fractional oscillator with a power-law memory kernel subjected to a harmonic potential as well as to an external periodic force. The influence of the fluctuating environment was modeled by a multiplicative white noise and by an additive noise. This model demonstrates that an interplay of multiplicative noise and memory can generate a variety of cooperative effects, such as memory-enhanced energetic stability and a resonance-like behavior of the variance and signal-to-noise ratio (SNR) as functions of the memory exponent. However, in this work the authors have confined themselves to investigating the dependence of the variance and SNR on the memory exponent, and thus the possible influence of a multiplicative noise on the temporal behavior of the autocorrelation functions of the output signal is not considered.

It is of interest, both from theoretical and experimental viewpoints, e.g., in active and passive microrheology

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experiments to study the mechanical properties of the intracellular environment [23–25], to know the behavior of the autocorrelation functions of the output in the case of similar model systems. We emphasize that in experiments information about the observed dynamical behavior of particles is usually extracted from the mean-square displacement and/or from the velocity autocorrelation function [19,23–25].

Motivated by the results of [20] and [19,23], the present paper considers a model similar to the one presented in [20], except that the power-law-type friction kernel in the GLE is replaced with a Mittag-Leffler memory kernel. So in the present paper we considerably generalize the model used in [20], increasing the dimension of the system parameter space by one. Namely, in our calculations we allow the characteristic memory time of the memory kernel to take any values. This extra degree of freedom can prove useful in modeling the viscoelastic properties of real media.

The main purpose of this paper is to provide exact formulas for the analytic treatment of the dependence of the mean-square displacement and the one-time velocity autocorrelation function on the lag-time. We show that the dependence of the normalized autocorrelation functions on the lag-time depends crucially on the physical nature of the additive driving noise, i.e., the results are qualitatively different for internal and external noises. We also demonstrate that the presence of a multiplicative noise in the GLE has a profound effect on the behavior of autocorrelation functions: first, their dependence on the frequency of the external periodic forcing occurs that enables easy verification of the presence of a multiplicative noise in experiments; second, at sufficiently large values of the lag-time the model predicts, for correlation functions, a crossover between two different asymptotic power-law regimes, which may be important for correct interpretation of experimental data in microrheology.

The structure of the paper is as follows. In Sec. II we present the model investigated. Exact formulas are found for the analysis of the behavior of autocorrelation functions. In Sec. III we analyze the dependence of the output characteristics on the system parameters and on the time lag. Section IV contains some brief concluding remarks. Some formulas are delegated to the Appendix.

II. MODEL AND THE EXACT MOMENTS

A. Model

We start from the traditional GLE model for a particle of the unit mass ($m = 1$) in the fluctuating harmonic potential subjected to a linear friction with a memory kernel $\eta(t)$, a multiplicative white noise $Z(t)$, an external periodic force, and an additive random force $\xi(t)$ of zero mean:

$$\ddot{X} + \int_0^t \eta(t-t')\dot{X}(t')dt' + [\omega^2 + Z(t)]X = A_0 \sin(\Omega t) + \xi(t), \quad (1)$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the particle displacement, while A_0 and Ω are the amplitude and the frequency of the harmonic driving force, respectively. The fluctuations of the eigenfrequency ω of the binding harmonic field are expressed as a Gaussian white noise $Z(t)$ with a zero mean and a δ-correlated correlation function:

$$\langle Z(t)\rangle = 0, \quad \langle Z(t)Z(t')\rangle = 2D\delta(t-t'), \quad (2)$$

where D is the noise intensity. The zero-centered random force $\xi(t)$ with a stationary correlation function

$$C(|t-t'|) := \langle \xi(t)\xi(t')\rangle, \quad \langle \xi(t)\rangle = 0 \quad (3)$$

is assumed as statistically independent from the noise $Z(t)$. Depending on the physical situation, the driving noise $\xi(t)$ can be regarded either as an internal noise, in which case its stationary correlation function satisfies Kubo's second fluctuation-dissipation theorem [26] expressed as

$$C(|t|) = k_B T \eta(|t|) \quad (4)$$

(here T is the absolute temperature of the heat bath, and k_B is the Boltzmann constant), or as an external noise, in which case the driving noise $\xi(t)$ and the dissipation may have different origins and no fluctuation-dissipation relation holds, i.e., $\xi(t)$ is not related to the memory kernel $\eta(t)$. Henceforth, in this work the random force $\xi(t)$ is assumed to be the sum of two uncorrelated contributions

$$\xi(t) = \xi_1(t) + \xi_2(t), \quad (5)$$

where $\xi_1(t)$ is the internal noise due to thermal activity, and $\xi_2(t)$ is an external white noise with an intensity D_1 and the correlation function

$$\langle \xi_2(t)\xi_2(t')\rangle = 2D_1\delta(t-t'), \quad \langle \xi_2(t)\rangle = 0. \quad (6)$$

In models for oscillatory systems with memory, strongly coupled with a noisy viscoelastic environment, usually a power-law correlation function for the internal noise is employed to model the memory effects [9,10,17]. As in Ref. [18], in this paper we assume a more general correlation function modeled as

$$\langle \xi_1(t)\xi_1(t')\rangle = k_B T \eta(|t-t'|) = \frac{\gamma k_B T}{\tau_c^\alpha} E_\alpha\left[-\left(\frac{|t-t'|}{\tau_c}\right)^\alpha\right], \quad (7)$$

where τ_c acts as the characteristic memory time, γ is a constant (called a friction constant), and the memory exponent α can be taken as $0 < \alpha < 1$, which is determined by the dynamical mechanism of the physical processes considered. The $E_\alpha(y)$ function denotes the Mittag-Leffler function [27], which behaves as a stretched exponential for short times and as an inverse power law in the long time regime. Note that if $\alpha = 1$, the correlation function (7) reduces to an exponential form which describes a standard Ornstein-Ulenbeck process [28]. In the limit $\tau_c \to 0$ the proposed correlation function reproduces a power-law correlation function

$$\langle \xi_1(t)\xi_1(t')\rangle = \frac{\gamma k_B T}{\Gamma(1-\alpha)|t-t'|^\alpha}, \quad (8)$$

where $\Gamma(y)$ is the gamma function, which has been previously used to model the viscoelastic properties of a medium [9,10,17]. Moreover, by taking the limit $\alpha \to 1$ in Eq. (8) we see that the noise $\xi_1(t)$ corresponds to white noise and consequently, to nonretarded friction.

It is worth emphasizing that counterparts of the model (1) without multiplicative noise are widely used in fitting experimental data from intracellular microrheology and from single-molecule experiments probing conformational fluctuations in proteins (see, e.g., [9,23]). For example, in Ref. [9] Xie and co-workers succeeded in modeling the motion of the donor-acceptor distance within a protein as the coordinate of a fictitious particle diffusing in a harmonic potential according to a GLE (i.e., with the help of a model similar to Eq. (1) without multiplicative noise), while the memory exponent $\alpha \approx 0.51$ was deduced from experimental observations.

As most of the results of the present paper reflect the influence of a multiplicative noise, included in a GLE, on the dynamics of a particle, let us now briefly discuss the possible physical origin of such a noise in the cytoplasm of cells. Recent studies using microrheology techniques have shown that adenosin-triphosphate-consuming molecular motors can drive the system out of thermal equilibrium. The corresponding violations of the fluctuation-dissipation theorem have been observed inside cells [23,25]. As collective action of molecular motors is a stochastic process, its influence on the dynamics of a particle embedded in the cytoplasm can be represented in a GLE as external noise [23]. In Ref. [29] it is pointed out that a GLE cannot always be expressed by means of a deterministic drift term, supplemented by internal and external additive noises. It is argued that in complex systems (e.g., the cytoplasm of cells) the external noise emerges in a phenomenological GLE rather as a multiplicative noise or as a combination of multiplicative and additive noises due to the elimination of fast degrees of freedom [29]. Multiplicative noise also arises in case one of the externally controlled system parameters in a phenomenological GLE is subjected to fluctuations [e.g., trap stiffness in experiments with optical tweezers [24,30], i.e., the quantity ω^2 in Eq. (1)].

B. First moments

By means of the calculation scheme represented in [20] one can easily obtain, from Eq. (1), a formal expression for the particle displacement $X(t)$ in the following form:

$$X(t) = \langle X(t)\rangle + \int_0^t H(t-t')[\xi(t') - X(t')Z(t')]dt', \quad (9)$$

where the average $\langle X(t)\rangle$ is given by

$$\langle X(t)\rangle = y_0 H(t) + x_0\left[1-\omega^2\int_0^t H(t')dt'\right] + A_0\int_0^t H(t-t')\sin(\Omega t')dt', \quad (10)$$

with the deterministic initial conditions $X(0)=x_0$ and $\dot{X}(0)=y_0$. The kernel $H(t)$ with the initial condition $H(0)=0$ is the Laplace inversion of

$$\hat{H}(s) = \frac{1}{s^2 + s\hat{\eta}(s) + \omega^2}, \quad (11)$$

where

$$\hat{H}(s) = \int_0^\infty e^{-st}H(t)dt, \quad (12)$$

and

$$\hat{\eta}(s) = \frac{\gamma s^{\alpha-1}}{1+(\tau_c s)^\alpha}. \quad (13)$$

An integral representation of the relaxation function $H(t)$ is given by Eqs. (A1)–(A5) in the Appendix. In the long-time limit, $t\to\infty$, the memory about the initial conditions will vanish and the asymptotic formula for the average particle displacement, $\langle X(t)\rangle_{as} := \langle X(t)\rangle_{|t\to\infty}$, reads as

$$\langle X(t)\rangle_{as} = A\sin(\Omega t + \varphi), \quad (14)$$

where the amplitude A and the phase shift φ can be represented as

$$A = |\hat{H}(-i\Omega)|, \varphi = \arctan\left\{-\frac{\mathrm{Im}\left[\hat{H}(-i\Omega)\right]}{\mathrm{Re}\left[\hat{H}(-i\Omega)\right]}\right\}. \quad (15)$$

The explicit dependence of A and φ on the system parameters is given by Eqs. (A8)–(A10) (see Appendix).

C. Second moments

From an experimental point of view, the information about the observed diffusive behavior of particles is extracted from the mean-square displacement $\rho(t,\tau)$, which in long-time measurements is determined by [19,23]

$$\rho_{as}(t,\tau) = \lim_{t\to\infty}\langle [X(t+\tau)-X(t)]^2\rangle, \quad (16)$$

where τ is the so-called time lag. Alternative information about the dynamics can be extracted from the asymptotic velocity autocorrelation function [19]

$$K_{as}(t,\tau) = \lim_{t\to\infty}\langle[\dot{X}(t+\tau) - \langle\dot{X}(t+\tau)\rangle][\dot{X}(t)-\langle\dot{X}(t)\rangle]\rangle. \quad (17)$$

In the limit $t\to\infty$ the mean-square displacement $\rho_{as}(t,\tau)$ and the autocorrelation function $K_{as}(t,\tau)$ depend on both of the times t and τ and become periodic functions of t with the period of the external driving, $\mathcal{T} = 2\pi/\Omega$. Thus as in [20,31], we define the one-time counterparts of $\rho_{as}(t,\tau)$ and $K_{as}(t,\tau)$ as an average of the corresponding two-time quantity over a period of the external driving, i.e.,

$$\rho(\tau) = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}\rho_{as}(t,\tau)dt, \qquad K(\tau) = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}K_{as}(t,\tau)dt. \quad (18)$$

Starting from Eq. (9) we obtain

$$\rho(\tau) = 2\sigma^2\left[1-\left(1-\frac{k_BT}{\omega^2\sigma^2}\right)\frac{\psi(\tau)}{\psi(0)} - \frac{k_BT}{\omega^2\sigma^2}F(\tau)\right], \quad (19)$$

$$K(\tau) = \sigma_v^2\left[\left(1-\frac{k_BT}{\sigma_v^2}\right)\frac{\ddot{\psi}(\tau)}{\ddot{\psi}(0)} + \frac{k_BT}{\sigma_v^2}\dot{H}(\tau)\right], \quad (20)$$

where σ^2 and σ_v^2 are the time-homogenous parts of the variance of the particle displacement X and the velocity $\dot{X}$, respectively, i.e.,

$$\sigma^2 := \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}\langle[X(t)-\langle X(t)\rangle]^2\rangle_{as}dt, \quad (21)$$

$$\sigma_v^2 := \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \langle [\dot{X}(t) - \langle \dot{X}(t)\rangle]^2\rangle_{as} dt. \tag{22}$$

It should be emphasized that the functions $\psi(\tau)$ and $F(\tau)$, defined as

$$\psi(\tau) := \int_0^\infty H(t+\tau)H(t)dt, \tag{23}$$

and

$$F(\tau) := 1 - \omega^2 \int_0^\tau H(t)dt, \tag{24}$$

respectively, are independent of the driving force parameters A_0 and Ω as well as of the noise intensities D, D_1, and T. The exact formulas useful for a numerical treatment of the functions $\psi(\tau)$ and $F(\tau)$ are given by Eqs. (A11) and (A12) in the Appendix. Using the formula (9) and the results of [20], one gets

$$\sigma^2 = \frac{1}{D_{cr} - D}\left[\frac{A^2 D}{2} + D_1 + \frac{k_B T D_{cr}}{\omega^2}\right], \tag{25}$$

$$\sigma_v^2 = k_B T - 2D_{cr}\ddot{\psi}(0)\left[\sigma^2 - \frac{k_B T}{\omega^2}\right], \tag{26}$$

where the critical noise intensity D_{cr} reads

$$D_{cr} = \frac{1}{2\psi(0)}. \tag{27}$$

From Eq. (25) we can see that the stationary regime is possible only if

$$D < D_{cr}. \tag{28}$$

As the intensity of the multiplicative noise D tends to the critical value D_{cr}, the variance σ^2 increases to infinity. This is an indication that for $D > D_{cr}$ energetic instability appears, which manifests itself as an unlimited increase of second-order moments of the output of the system with time, while the mean value of the particle displacement remains finite [32].

The analytical expressions (19), (20), (25), and (26) with Eqs. (A11) and (A12) belong to the main results of this work. Finally, we emphasize that these results are physically meaningful only if the inequality Eq. (28) holds, i.e., if the system is energetically stable.

III. RESULTS

A. Variance of the output signal

In Figs. 1(a) and 1(b) we depict the behavior of the critical noise intensity D_{cr} and the variance σ^2 by variations of the memory exponent α. Figure 1(a) shows a typical phenomenon of the memory-enhanced energetic stability considered previously in [20]. As a rule the maximal value of $D_{cr}(\alpha)/\gamma$ increases as the value of the characteristic memory time τ_c decreases, while the positions of the maxima are monotonically shifted to a greater α as τ_c rises. In the case considered in Fig. 1(b) the intensity of the multiplicative noise is in the interval $\omega^2\gamma < D < D_{cr_{\max}}$, where $D_{cr_{\max}}$ is the maximal value of $D_{cr}(\alpha)$ by variations of α. In this case the variance σ^2 decreases rapidly from infinity at α_1, $D_{cr}(\alpha_1) = D$, to a minimum and next increases to infinity at α_2, $D_{cr}(\alpha_2) = D$. Thus the system is energetically stable only in the interval $\alpha_1 < \alpha < \alpha_2$. From Fig. 1(b) one can see that the values of the variance σ^2 depend on the frequency Ω of the harmonic driving force. Note that in the case without multiplicative noise, such a dependence is absent [see Eqs. (25) and (A9)]. Finally, it should be mentioned that the behavior of the variance of particle velocities σ_v^2 is similar to the behavior of σ^2 described above [cf. also Eq. (26)].

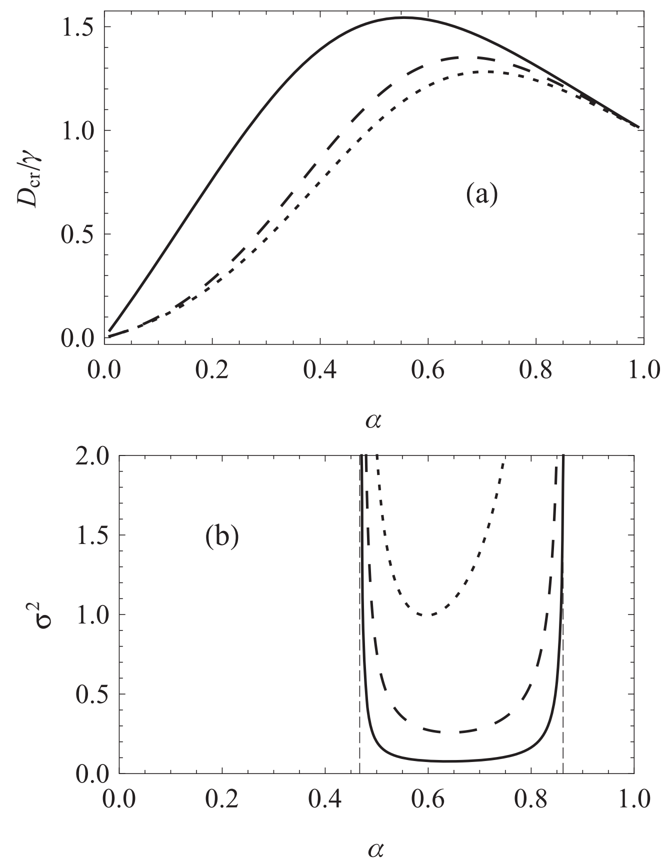

FIG. 1. Dependence of the critical noise intensity D_{cr} and the variance σ^2 computed from Eqs. (27), (A11), (25), and (A9) on the memory exponent α. Parameter values: $A_0 = \omega = 1$, $\gamma = 4$. Panel (a): D_{cr}/γ vs α at different values of the memory time τ_c. Solid line: $\tau_c = 0$; dashed line: $\tau_c = 0.025$; dotted line: $\tau_c = 0.05$. Panel (b): σ^2 vs α at different values of the driving frequency Ω. System parameter values: $\tau_c = 0.01$, $D = 4.8$, $D_1 = 0.01$, and $k_B T = 0.01$. Solid line: $\Omega = 10$; dashed line: $\Omega = 1$; dotted line: $\Omega = 0.1$. The thin dashed lines depict the positions of the critical memory exponents $\alpha_1 \approx 0.467$ and $\alpha_2 \approx 0.862$ between which the system is energetically stable.

B. Temporal behavior of the autocorrelation function

Now we consider the behavior of the normalized autocorrelation functions $K_{xn}(\tau)$ and $K_{vn}(\tau)$, where

$$K_{xn}(\tau) = 1 - \frac{\rho(\tau)}{2\sigma^2}, \qquad K_{vn}(\tau) = \frac{K(\tau)}{\sigma_v^2}. \tag{29}$$

In contrast to the results for the variances σ^2 and σ_v^2, here the role of the additive driving noise $\xi(t)$ is crucial. If the driving

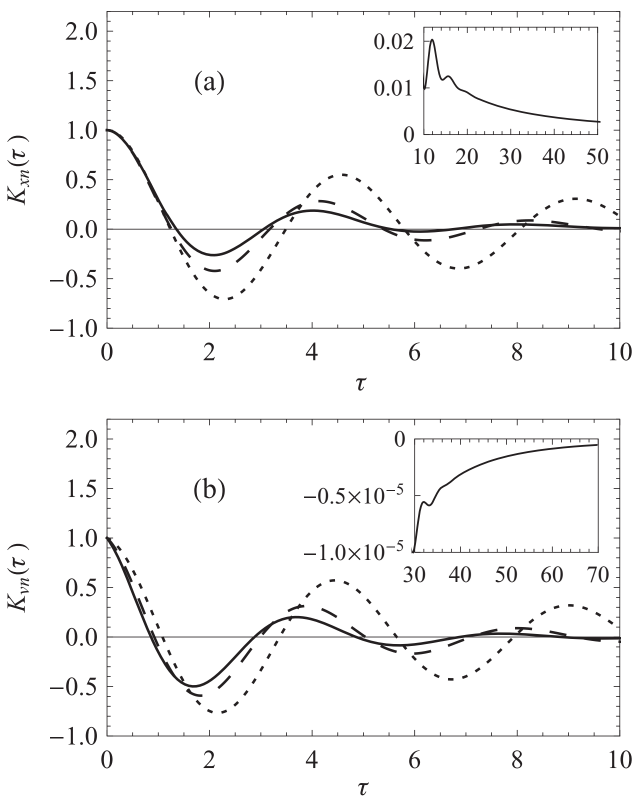

FIG. 2. Normalized autocorrelation functions $K_{xn}(\tau)$ and $K_{vn}(\tau)$ vs the time lag τ in the case of external noise (i.e., $T = 0$). System parameter values: $\omega = 1$, $\gamma = 1.6$, $\alpha = 0.5$, $D = 0.3$, and $D_1 = 0.1$. Solid line, $\tau_c = 0.01$; dashed line, $\tau_c = 0.1$; dotted line, $\tau_c = 1$. The insets depict the behavior of K_{xn} and K_{vn} at large values of τ; $\tau_c = 0.01$.

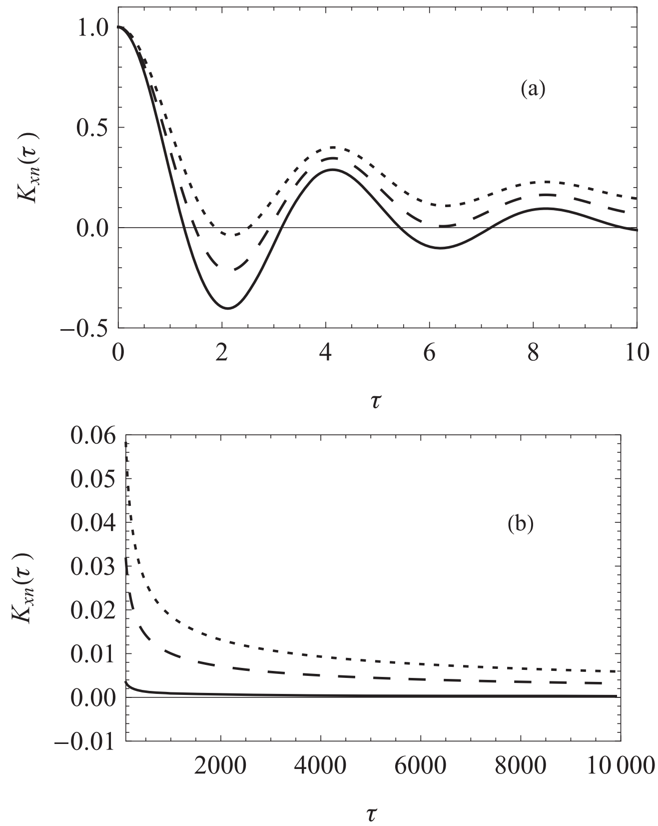

FIG. 3. Dependence of the normalized autocorrelation function $K_{xn}(\tau)$ on the time lag τ at several values of the driving frequency Ω in the case of internal noise (i.e., $D_1 = 0$). Parameter values: $A_0 = \omega = 1$, $\gamma = 1.6$, $\alpha = 0.5$, $\tau_c = 0.1$, $k_B T = 0.01$, and $D = 0.3$. Solid line, $\Omega = 1.5$; dashed line, $\Omega = 2.5$; dotted line $\Omega = 4$. Panel (b) the behavior of K_{xn} at large values of τ.

noise is external (i.e., $T = 0$), the typical forms of the graphs $K_{xn}(\tau)$ and $K_{vn}(\tau)$ are represented in Fig. 2. Note that the exact solution exhibits exponentially damped oscillations around a curve which for large τ decays absolutely monotonically like a power law. Consequently, the normalized autocorrelation functions K_{xn} and K_{vn} have only a finite number of zeros and they decay, in the long-time-lag regime, as $\tau^{-(1+\alpha)}$ and as $\tau^{-(3+\alpha)}$, respectively. Note that in this case both of the normalized autocorrelation functions are independent of the driving force parameters A_0 and Ω.

In the case of an internal noise $\xi(t)$ (i.e., $T \neq 0$), the picture of the dependence of K_{xn} and K_{vn} on τ is different (see Fig. 3). First, the autocorrelation functions $K_{xn}(\tau)$ and $K_{vn}(\tau)$ relax asymptotically like $\tau^{-\alpha}$ and $\tau^{-(2+\alpha)}$, respectively. This is in sharp contrast with the results for the external noise that exhibits a much faster decay. Second, the most important difference is the dependence of K_{xn} and K_{vn} on the amplitude A of the output signal [cf. Eqs. (19), (20), (25), (26), and (29)]. So, in the case of an internal noise the exact form of the function $K_{xn}(\tau)$ is sensitive to the values of the frequency Ω of the external harmonic driving force (see also Fig. 3). Thus the formulas (19), (20), (25), and (26) provide some simple criteria that enable us to verify the presence of multiplicative and internal noises by manipulating an external field in active microrheology experiments. First, if the experimental data show at moderate lag-times a dependence of the mean-square displacement $\rho(\tau)$ or the velocity autocorrelation function $K(\tau)$ on the frequency Ω (or amplitude A_0) of the external force, then an influence of a multiplicative noise in the system dynamics can be assumed. Second, if additionally the normalized autocorrelation functions show a dependence on the parameters of the external field, then the contribution of an internal noise is significant.

In Fig. 2 we have plotted the normalized autocorrelation functions at several values of the characteristic memory time τ_c. It can be seen that in the displayed range of the time lag τ the function $K_{vn}(\tau)$ shows larger oscillations if the memory time τ_c increases. Moreover, it exhibits more zero crossings, which represent transitions between a positive velocity correlation and velocity anticorrelations. This behavior is related to the so-called whip-back effect [19,33].

C. Characteristic time scales

In experimental realizations, the time lag is $\tau_{\min} \leqslant \tau \leqslant \tau_{\max}$, where $\tau_{\min}$ is the acquisition time interval and $\tau_{\max}$ is the measurement time. Therefore it is important to analyze the behavior of the normalized correlation functions for several time scales involved in the model (1).

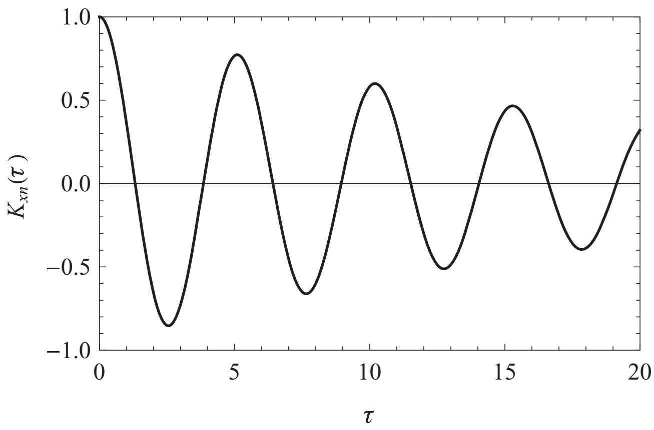

FIG. 4. Normalized autocorrelation function $K_{xn}(\tau)$ vs the time lag τ in the time regime $\tau < \tau_1$ [cf. Eqs. (19), (20), and (A11)]. Parameter values: $A_0 = \omega = \gamma = \tau_c = 1$, $\alpha = 0.3$, $\Omega = 2$, $D_1 = k_B T = 0.1$, and $D = 0.1$. The characteristic time $\tau_1 = 1/\beta \approx 20.02$.

There are two important characteristic times for $K_{xn}(\tau)$:

$$\tau_1 = \frac{1}{\beta}, \tag{30}$$

$$\tau_2 = \frac{2\alpha D D_{cr}}{\omega^4 (D_{cr} - D)} \left[1 + \frac{\omega^2}{2k_B T} \left(A^2 + \frac{2D_1}{D} \right) \right]. \tag{31}$$

Below we consider mainly the case

$$\tau_{\min} < \tau_1 \ll \tau_2 \ll \tau_{\max} \tag{32}$$

to allow the following separation of time scales: (i) $\tau \lesssim \tau_1$; (ii) $\tau_1 \ll \tau \ll \tau_2$; (iii) $\tau \gg \tau_2$.

From Eq. (A2) it follows that τ_1 depends only on the parameters of the memory kernel η, and ω. Particularly, as a rule τ_1 increases as τ_c increases or as γ and α decrease. Note that in the time region $\tau \lesssim \tau_1$ the oscillatory behavior of the correlation functions is significant [see Fig. 4, cf. also Eqs. (A11) and (A12)] and thus it should be used by interpretation of experimental results. Notably, the usually employed overdamped approximation is not applicable in this situation.

The other characteristic time τ_2 is related to the asymptotic regimes of the normalized autocorrelation function $K_{xn}(\tau)$ and is defined as the time at which a crossover from a power-law regime with the exponent $-(1+\alpha)$, to a power-law regime with the exponent $-\alpha$ occurs (see Fig. 5).

The time τ_2 can fulfill the inequalities (32) only in the presence of a multiplicative noise or, if the multiplicative noise is absent, in the presence of an additive external noise with an intensity $D_1 \neq 0$. In that case

$$\tau_2 = \frac{2D_1\alpha}{\omega^2 k_B T}, \tag{33}$$

and consequently it characterizes the relative contributions of the external and internal noises to the system dynamics. It is obvious that τ_2 tends to a very large value if internal noise is negligible, i.e.,

$$\frac{2k_B T}{\omega^2} \ll DA^2 + 2D_1, \tag{34}$$

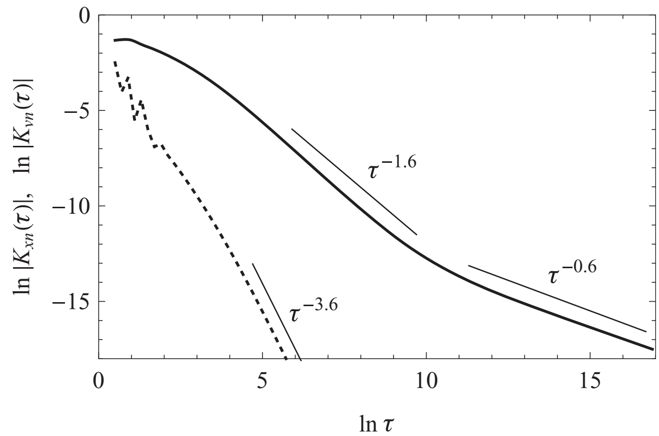

FIG. 5. A logarithmic plot of the asymptotic dependence of the normalized autocorrelation functions $K_{xn}(\tau)$ and $K_{vn}(\tau)$ on the time lag τ. System parameter values: $A_0 = \omega = 1$, $\gamma = 4$, $\alpha = 0.6$, $D_1 = 0$, $k_B T = 0.01$, $\Omega = 0.1$, $\tau_c = 0.01$, and $D = 5.6$. Solid line, $\ln|K_{xn}(\tau)|$ vs $\ln\tau$; dotted line $\ln|K_{vn}(\tau)|$ vs $\ln\tau$. Note a crossover between two asymptotic power-law regimes, $\tau^{-1.6}$ and $\tau^{-0.6}$, at the characteristic lag-time value $\ln\tau_2 = 9.86$.

or in the vicinity of energetic instability, $D \to D_{cr}$. If $\tau_2 > \tau_{\max}$, then the asymptotic monotonic decay of the correlation functions should be used with care in the interpretation of experimental data, even if both the external additive noise and external driving force are absent.

For example, although in the presence of an internal noise with a memory exponent α $K_{xn}(\tau)$ asymptotically decays as $\tau^{-\alpha}$, in the case of $\tau_2 > \tau_{\max}$ a naive interpretation of experimental data shows a power-law decay like $\tau^{-(1+\alpha)}$ which corresponds to the memory exponent $1+\alpha$ for an internal noise. Thus, the genuine memory exponent α for the internal noise appears only in the time scale $\tau \gg \tau_2$.

Finally, we emphasize that at $D \approx D_{cr}$, i.e., in a vicinity of energetic instability, the characteristic time τ_2 is very large and consequently, the long-time behavior of the correlation functions is similar to the case of an external additive noise, even if both the external additive noise and the external periodic driving force are absent.

D. Memory-induced trapping

Next we consider the behavior of output characteristics [$K_{xn}(\tau)$, $K_{vn}(\tau)$, and σ^2] without the harmonic trapping field, i.e., that of Eq. (1) with a zero eigenfrequency, $\omega = 0$. It can be shown that if the additive driving noise includes an internal component, $T \neq 0$, the asymptotic behavior of the model (1) with $\omega = 0$ is subdiffusive, $\sigma^2 \sim t^\alpha$, and a stationary regime is impossible (see also [20,34]), which renders the formulas (19), (20), (25), and (26) physically meaningless. If the driving noise $\xi(t)$ is external, ($T = 0$, $D_1 \neq 0$), and if the memory exponent is sufficiently small, $\alpha < 1/2$, a stationary regime is possible and the formulas (19), (20), (25), and (26) are applicable (see Fig. 6, cf. also [20]). This behavior of the model (1) is in agreement with the previous results given in Ref. [34], where the long-time behavior of a particle governed by the GLE (1) with $A_0 = \omega = Z(t) = 0$ has been considered. Particularly, in [34] it was shown that if the intensity of an external noise $\xi(t)$ defined by

$$D_1 := \int_0^\infty C(t)dt$$

is finite and nonvanishing, and if the friction kernel $\eta(t)$ decays as

$$\eta(t) \sim \frac{1}{t^\alpha}, \quad t \to \infty,$$

where $0 < \alpha < \frac{1}{2}$, then the process $X(t)$ becomes asymptotically stationary with a finite variance. In the case of $\frac{1}{2} < \alpha < 1$ the long-time behavior of the particle is subdiffusive.

Note that in the case of a power-law memory kernel (i.e., in the limit $\tau_c \to 0$) the phenomenon of energetic stability for an unbounded system [Eq. (1) with $\omega = 0$] has been previously considered in [20], where it was shown that this phenomenon agrees well with the description of the friction force for small α as an elastic force due to the cage effect. For small α the friction force induced by the medium is not just slowing down the particle but also causing the particle to undergo a rattling motion, which can be explained by the harmonic motion of the particle in a cage formed by the surrounding particles [17]. In this sense, at small α the medium is binding the particle preventing diffusion but forcing oscillations. The resonantlike behavior of the critical intensity of the multiplicative noise $D_{cr}(\alpha)$ versus α, which is seen in Fig. 6, is the manifestation of the cage effect, which is contained in Eq. (1) due to the friction memory kernel.

In Fig. 7 three graphs depict, in the case of memory-induced trapping ($\omega = 0$, $T = 0$, i.e., internal noise is absent), the behavior of the mean-square displacement $\rho(\tau)$ for different values of the driving frequency Ω [see Eq. (19)]. As can be expected from the discussion in subsection B, these graphs show a dependence of the mean-square displacement on the frequency Ω of the external force. Here we remind to reader that as the internal noise is absent, such a multiplicative-noise-induced dependence on Ω is excluded for the normalized autocorrelation functions $K_{xn}(\tau)$ and $K_{vn}(\tau)$ [see Eqs. (19), (20), and (29)].

It is remarkable that in the case of $\omega = 0$ the characteristic time $\tau_1 = 1/\beta$ can, at low and large values of the memory time τ_c, be represented with the following simple formulas:

$$\tau_1 \approx \frac{1}{\gamma^{\frac{1}{2-\alpha}} \cos\left(\frac{\pi(1-\alpha)}{2-\alpha}\right)}, \quad \tau_c \to 0, \tag{35}$$

$$\tau_1 \approx \frac{2\tau_c^\alpha}{\sin\left(\frac{\alpha\pi}{2}\right)} \left(\frac{\tau_c^\alpha}{\gamma}\right)^{\frac{1-\alpha}{2}}, \quad \tau_c \to \infty. \tag{36}$$

Thus in the cases of a low memory exponent α or large values of the memory time τ_c the characteristic time τ_1 is very large and the oscillatory parts of $K_{xn}(\tau)$ and $K_{vn}(\tau)$ decay very slowly.

Finally, due to the cage effect the dependence of the spectral amplification A^2/A_0^2 on the frequency Ω exposes a bona fide resonance even when the binding harmonic field is absent, $\omega = 0$ (see Fig. 8). It is important that for any $\alpha < 0.441$ and for any values of γ and τ_c the dependence of A^2 on Ω is always nonmonotonic with a resonance peak, which apparently gets

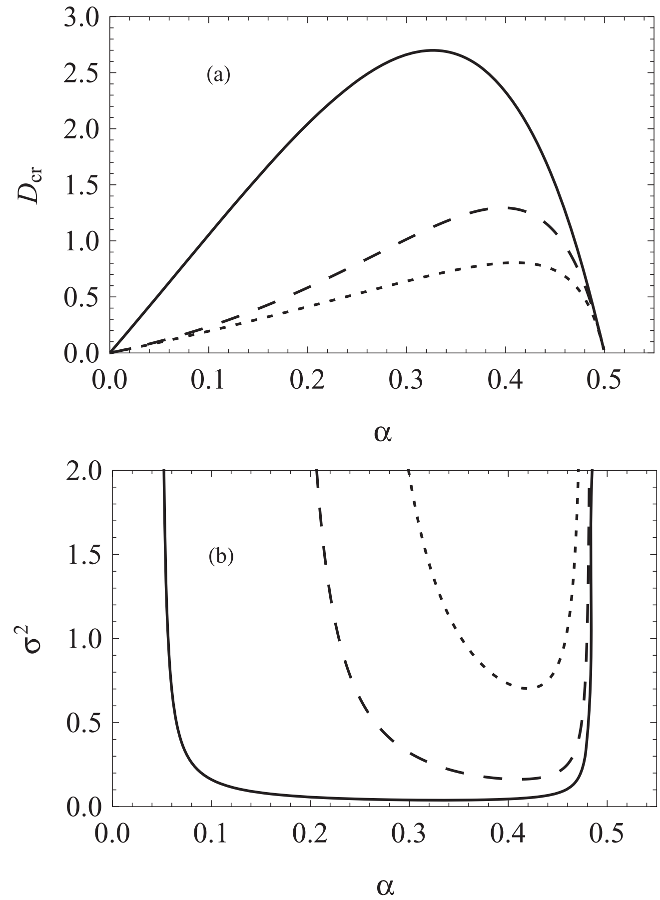

FIG. 6. The critical noise intensity D_{cr} and the variance σ^2 as functions of the memory exponent α by absence of the harmonic potential, $\omega = 0$ [Eqs. (27), (A11), (25), and (A9)]. Parameter values: $A_0 = \Omega = 1$, $T = 0$, $D_1 = 0.05$, $\gamma = 3.5$, and $D = 0.5$. Solid line, $\tau_c = 0$; dashed line, $\tau_c = 0.1$; dotted line, $\tau_c = 0.5$. Note that the critical memory exponent $\alpha_{cr} = 1/2$, at which the critical noise intensity D_{cr} tends to zero.

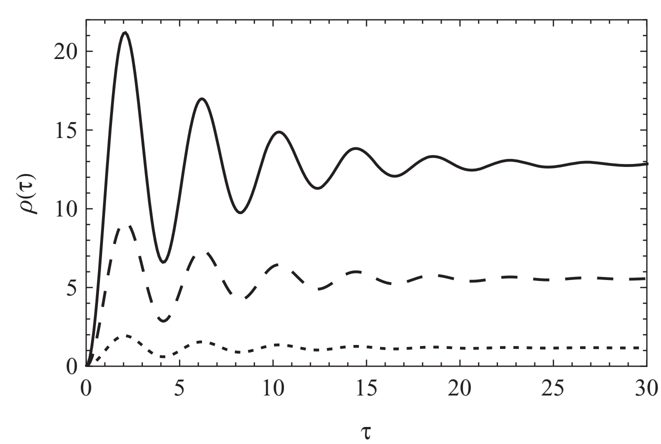

FIG. 7. Mean-square displacement ρ vs time lag τ in the case of external noise (i.e., $k_B T/\omega^2 = 0$). System parameter values: $\omega = 0$, $A_0 = 1$, $D_1 = 0.01$, $D = 0.5$, $\gamma = 3.5$, $\alpha = 0.22$, and $\tau_c = 0.1$. Solid line, $\Omega = 1.5$; dashed line, $\Omega = 1.3$; dotted line, $\Omega = 2.0$.

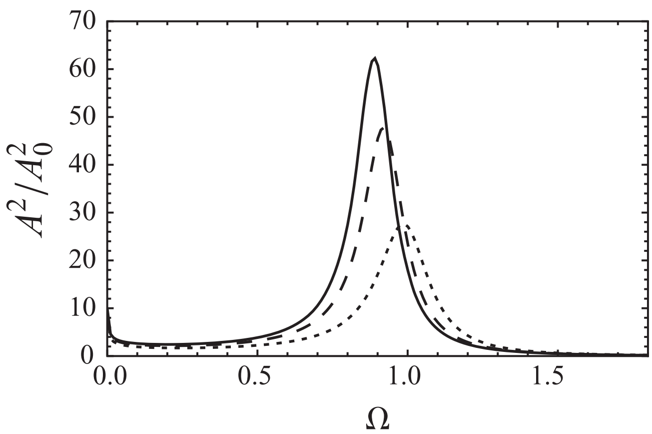

FIG. 8. The spectral amplification A^2/A_0^2 vs the driving frequency Ω computed from Eqs. (A8) and (A9) for $\omega = 0$, $\gamma = 1.6$, and $\alpha = 0.2$. Solid line, $\tau_c = 1$; dashed line, $\tau_c = 0.5$; dotted line, $\tau_c = 0.1$.

more and more pronounced as the memory time τ_c increases (for $\tau_c = 0$, cf. [17,21]).

IV. CONCLUSIONS

In the present work we have analyzed, in the long-time regime $t \to \infty$, the dependence of the mean-square displacement and the velocity autocorrelation function for a harmonically trapped particle in a viscoelastic medium on the time lag τ. Starting from a generalized Langevin equation with memory driven by an external sinusoidal forcing and by additive and multiplicative noises [Eq. (1)], we have been able to derive exact analytic expressions for the one-time autocorrelation functions in the case of a Mittag-Leffler type memory kernel.

As one of our main results we have established that in the presence of a multiplicative noise the autocorrelation functions depend on the parameters of external sinusoidal forcing. Since without a multiplicative noise such a dependence is absent, this effect gives, in active microrheology experiments, a simple criterion to determine whether there is a multiplicative noise influencing the dynamics of the system or not. Moreover, it is remarkable that in the case of an additive external noise and a sufficiently strong memory, a related phenomenon involving memory-induced trapping occurs for an unbound system [i.e., in Eq. (1), the harmonic binding potential is absent]. Note that for an internal noise the behavior of such an unbound system is always subdiffusive [20,34], i.e., the trapping is absent.

As another main result we have shown that in the case of an additive external noise the dependence of the normalized autocorrelation functions on the time lag is independent of external periodic forcing. This contrasts with the case of internal noise, where the dependence of the normalized autocorrelation functions on a periodic forcing is significant.

Thus we have found two experimentally convenient criteria that enable us to verify the presence of a multiplicative noise and make a clear distinction between contributions of an external noise and an internal noise on the dynamics of the systems described by Eq. (1). The advantage of these criteria is that the control parameter is the frequency of the external periodic force, which can be easily varied in possible experiments.

Our main generic result is that the dynamics with a multiplicative noise and without multiplicative noise are profoundly different, in spite of some superficial similarities. Particularly, perhaps the most important result in view of experiments is that the model with multiplicative and internal noises predicts a crossover between two different asymptotic power-law regimes for the correlation function: $\tau^{-(\alpha+1)}$ and $\tau^{-\alpha}$ (between $\tau^{-(3+\alpha)}$ and $\tau^{-(2+\alpha)}$ for the velocity correlation function). In some cases, e.g., if the measurement time $\tau_{\max}$ is comparable to or smaller than the crossover time τ_2 [see Eqs. (31) and (33)], this circumstance might be of importance for correct interpretation of experimental data.

Finally, we have presented a theoretical analysis to explain the effects of multiplicative noise and the trapping potential in the anomalous behavior of the mean-square displacement and the normalized velocity autocorrelation function of a particle embedded in a viscoelastic environment. How such effects might have affected the previously reported measurements of the viscoelastic properties of cytoplasm (or conformational fluctuations in proteins) remains to be specified. Undoubtedly, ultimate verification of the importance of multiplicative noise in microrheology experiments lies with experimentalists. Active microrheology techniques using optical or magnetic tweezers [24] will be valuable tools for exploring multiplicative-noise-induced effects.

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APPENDIX: FORMULAS FOR THE RELAXATION FUNCTION

1. Time dependence of the relaxation function $H(t)$

The relaxation function $H(t)$ in Eq. (9) can be obtained by means of the Laplace transformation technique. To evaluate the inverse Laplace transform of $\hat{H}(s)$ [see Eq. (11)], we use the residue theorem method described in [35]. The inverse Laplace transform gives

$$H(t) = \frac{2}{\sqrt{u^2+v^2}} e^{-\beta t} \sin(\omega^* t + \Theta) + \frac{\gamma \sin(\alpha\pi)}{\pi} \int_0^\infty \frac{r^\alpha e^{-rt} dr}{B(r)}, \tag{A1}$$

where $s_{1,2} = -\beta \pm i\omega^*$, $(\beta > 0,\ \omega^* > 0)$, are the pair of conjugate complex zeros of the equation

$$G(s) \equiv s^2 + \frac{\gamma s^\alpha}{1 + (\tau_c s)^\alpha} + \omega^2 = 0; \tag{A2}$$

here, $G(s)$ is defined by the principal branch of s^α. The quantities u, v, Θ, and $B(r)$ are determined by

$$
\begin{aligned}
u &= -2\beta + \frac{\gamma\alpha\{\cos[(1-\alpha)\varphi^*] + \tau_c^{2\alpha}(\beta^2+\omega^{*2})^\alpha \cos[(1+\alpha)\varphi^*] + 2\tau_c^\alpha(\beta^2+\omega^{*2})^{\frac{\alpha}{2}}\cos\varphi^*\}}{(\beta^2+\omega^{*2})^{\frac{1-\alpha}{2}}[1+\tau_c^{2\alpha}(\beta^2+\omega^{*2})^\alpha + 2\tau_c^\alpha(\beta^2+\omega^{*2})^{\frac{\alpha}{2}}\cos(\alpha\varphi^*)]^2},\\
v &= 2\omega^* - \frac{\gamma\alpha\{\sin[(1-\alpha)\varphi^*] + \tau_c^{2\alpha}(\beta^2+\omega^{*2})^\alpha \sin[(1+\alpha)\varphi^*] + 2\tau_c^\alpha(\beta^2+\omega^{*2})^{\frac{\alpha}{2}}\sin\varphi^*\}}{(\beta^2+\omega^{*2})^{\frac{1-\alpha}{2}}[1+\tau_c^{2\alpha}(\beta^2+\omega^{*2})^\alpha + 2\tau_c^\alpha(\beta^2+\omega^{*2})^{\frac{\alpha}{2}}\cos(\alpha\varphi^*)]^2}
\end{aligned}
\tag{A3}
$$

with

$$
\varphi^* = \pi + \arctan\left(-\frac{\omega^*}{\beta}\right),\quad \Theta = \arctan\left(\frac{u}{v}\right),
\tag{A4}
$$

and

$$
B(r) = \{(r^2+\omega^2)[\cos(\alpha\pi) + (\tau_c r)^\alpha] + \gamma r^\alpha\}^2 + (r^2+\omega^2)^2\sin^2(\alpha\pi).
\tag{A5}
$$

The relaxation function $H(t)$ can be represented via Mittag-Leffler-type special functions [27]. But as in the last case the numerical calculations are very complicated, so we suggest, apart from possible representations via Mittag-Leffler functions, a numerical treatment of Eq. (A1).

2. Complex susceptibility

Here the exact formulas for the imaginary part χ'' and for the real part χ' of the complex susceptibility $\chi(\Omega) = \hat{H}(-i\Omega)$ are presented. From Eqs. (11) and (13) one can conclude that the quantities χ'' and χ' are given by

$$
\chi' = \frac{\omega^2 - \Omega^2 + f_1}{(\omega^2-\Omega^2+f_1)^2 + f_2^2},
\tag{A6}
$$

$$
\chi'' = \frac{f_2}{(\omega^2-\Omega^2+f_1)^2 + f_2^2},
\tag{A7}
$$

where

$$
f_1 = \frac{\gamma\Omega^\alpha\left[\cos\left(\frac{\alpha\pi}{2}\right) + (\tau_c\Omega)^\alpha\right]}{1+(\tau_c\Omega)^{2\alpha} + 2(\tau_c\Omega)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)},\quad
f_2 = \frac{\gamma\Omega^\alpha\sin\left(\frac{\alpha\pi}{2}\right)}{1+(\tau_c\Omega)^{2\alpha} + 2(\tau_c\Omega)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}.
\tag{A8}
$$

Thus, for the amplitude A and the phase shift φ in Eq. (14) [see also Eq. (15)] we obtain that

$$
A^2 = \frac{A_0^2}{(\omega^2-\Omega^2+f_1)^2+f_2^2}
\tag{A9}
$$

and

$$
\varphi = \arctan\left(\frac{f_2}{\Omega^2-\omega^2-f_1}\right).
\tag{A10}
$$

3. The relaxation functions for second moments

Now we present the time dependence of the relaxation functions $\psi(\tau)$ and $F(\tau)$ [see Eqs. (19) and (20)]. From Eqs. (23), (24), and (A1) we obtain

$$
\begin{aligned}
\psi(\tau) = &\frac{e^{-\beta\tau}}{u^2+v^2}\left\{\frac{1}{\beta}\cos(\omega^*\tau) - \frac{1}{\beta^2+\omega^{*2}}[\beta\cos(\omega^*\tau+2\Theta) - \omega^*\sin(\omega^*\tau+2\Theta)]\right\}\\
&+\frac{\gamma\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{r^\alpha dr}{B(r)}\left\{\frac{e^{-r\tau}[1+(\tau_c r)^\alpha]}{(r^2+\omega^2)[1+(\tau_c r)^\alpha]+\gamma r^\alpha}\right.\\
&\left.+\frac{2e^{-\beta\tau}[\omega^*\cos(\omega^*\tau+\Theta) + (r+\beta)\sin(\omega^*\tau+\Theta)]}{\sqrt{u^2+v^2}[(r+\beta)^2+\omega^{*2}]}\right\}
\end{aligned}
\tag{A11}
$$

and

$$
\begin{aligned}
F(\tau) = &\frac{2\omega^2 e^{-\beta\tau}}{\sqrt{u^2+v^2}(\beta^2+\omega^{*2})}[\omega^*\cos(\omega^*\tau+\Theta)\\
&+\beta\sin(\omega^*\tau+\Theta)] + \frac{\omega^2\gamma\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{e^{-r\tau}dr}{r^{1-\alpha}B(r)},
\end{aligned}
\tag{A12}
$$

where the quantities u, v, Θ, and $B(r)$ are determined by Eqs. (A3)–(A5).

From Eqs. (A1), (A11), and (A12) it follows that for large τ the relaxation functions $\psi(\tau)$, $F(\tau)$, and $H(\tau)$ decay as a power law. Namely, at a long-time limit ($\tau\to\infty$), the asymptotic behavior of $\psi(\tau)$, $F(\tau)$, and $H(\tau)$ read as

$$
\psi(\tau) \sim \frac{\gamma\alpha}{\omega^6\Gamma(1-\alpha)}\tau^{-(1+\alpha)},
\tag{A13}
$$

$$
F(\tau) \sim \frac{\gamma}{\omega^2\Gamma(1-\alpha)}\tau^{-\alpha},
\tag{A14}
$$

$$
H(\tau) \sim \frac{\gamma\alpha}{\omega^4\Gamma(1-\alpha)}\tau^{-(1+\alpha)}.
\tag{A15}
$$

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