Prescription-induced jump distributions in multiplicative Poisson processes

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Generalized Langevin equations (GLE) with multiplicative white Poisson noise pose the usual prescription dilemma leading to different evolution equations (master equations) for the probability distribution. Contrary to the case of multiplicative Gaussian white noise, the Stratonovich prescription does not correspond to the well-known midpoint (or any other intermediate) prescription. By introducing an inertial term in the GLE, we show that the Itô and Stratonovich prescriptions naturally arise depending on two time scales, one induced by the inertial term and the other determined by the jump event. We also show that, when the multiplicative noise is linear in the random variable, one prescription can be made equivalent to the other by a suitable transformation in the jump probability distribution. We apply these results to a recently proposed stochastic model describing the dynamics of primary soil salinization, in which the salt mass balance within the soil root zone requires the analysis of different prescriptions arising from the resulting stochastic differential equation forced by multiplicative white Poisson noise, the features of which are tailored to the characters of the daily precipitation. A method is finally suggested to infer the most appropriate prescription from the data.

I. INTRODUCTION

Intense and concentrated state-dependent forcing events may often be modeled as multiplicative random jumps, taking place according to an underlying point process. Unlike the additive case, which counts a relatively vast literature [1–6], state-dependent jumps have been less investigated [7–11] and, usually, the state dependency is assumed to be in the frequency of the jump occurrence rather than in its amplitude. The generalized Langevin equation (GLE) for white multiplicative noise $\zeta(t)$, which can be either Gaussian or non- Gaussian,

$$\dot{x}(t) = a(x,t) + b(x)\zeta(t) \qquad (1)$$

is ill defined unless a prescription for the evaluation of the stochastic term $b(x)\zeta(t)$ is specified [12]. While this issue is well understood for Gaussian white noise (GWN) [13], a precise characterization of the noise prescriptions and a clear connection between the different interpretations are still missing for other kinds of noises.

The last term in Eq. (1) for the white Poisson (WP) process can be written as $\zeta(t) = \xi_\rho(\nu,t) = \sum_i w_i \delta(t - t_i)$, where the t_i are the times at which jumps occur, δ is the Dirac delta function, and the probability that n jumps occur during a time interval Δt is given by the Poisson distribution $P_n(t) = \exp(-\nu\Delta t)(\nu\Delta t)^n/n!$. The jump heights w are independent and identically distributed random variables with a probability distribution function (PDF) $\rho(w)$. We note that the multiplicative case $b(x)\zeta(t)$ of Eq. (1) is a special case in which the x dependence of a more general state-dependent white noise $\zeta(x,t)$ can be factorized out. Note that, while it is always possible to reduce the state-dependent noise as in (1) for GWN, because GWN is fully characterized by its mean and variance, this is not the case for the WP process.

This paper is organized as follows. First, we show how different prescriptions corresponding to the Itô (I) and Stratonovich (S) interpretation of a stochastic differential equation (SDE) arise naturally for multiplicative jumps, depending on the relevant time scales of the process. In Sec. III, we present the master equation (ME) for a GLE with multiplicative compound Poisson process in both the I and S prescriptions. The core of this work is presented in Sec. IV where we show how, in the linear case $b(x) \propto x$, the difference between prescriptions is properly interpreted as a transformation of the jump size PDFs. We demonstrate the relevance of these effects on a minimalist model of soil salinization, describing possible long-term accumulation of salt in soils in arid and semi-arid regions. In this problem, the salt mass balance equation is characterized by state-dependent losses concentrated in negative jumps due to the leaching of salt produced by intense rainfall events. The stochastic equation is solved analytically obtaining explicitly the jump distributions that arise in connection to the different noise interpretations.

II. CONNECTION BETWEEN DIFFERENT PRESCRIPTIONS OF A GLE AND TIME SCALES OF THE PROCESS

We begin with a pedagogical example of a particle that experiences multiplicative impulsive forcing events, proportional to $\dot{\Theta}_\tau(t)$, of duration τ, in a field characterized by a friction coefficient ψ. Our analysis is inspired by the work in Refs. [14,15]. We choose $\Theta_\tau(t) = \vartheta(t/\tau)$ with $\vartheta(z) \to 1\,(0)$ in the limit $z \to \infty\,(-\infty)$ so that $\dot{\Theta}_\tau(t) \to \delta(t)$ in the $\tau \to 0$ limit (in the distribution sense). We first consider the case

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of a single jump event at $t = t_0 > 0$, where the dynamics is described by the Newton equation

$$m\,\ddot{x}(t) \;=\; -\psi\dot{x} \,+\, \psi b(x)w\dot{\Theta}_\tau(t-t_0), \qquad (2)$$

where the random jump w is drawn from the jump size PDF $\rho(w)$. Thus, in Eq. (2), we have two time scales $\sigma = m/\psi$ and τ. The former is associated with the relaxation time toward stationarity, while the latter is related to the characteristic duration of the impulsive forcing. Different prescriptions of Eq. (1) arise depending on how the two emerging time scales σ and τ in Eq. (2) go to zero, i.e., $\sigma \to 0$ followed by $\tau \to 0$ or vice versa (see Fig. 1). For this reason, writing $\dot{x}(t) = b(x)w\delta(t-t_0)$ is ambiguous, being the result of two different limit procedures with different physical and mathematical meaning.

When $\sigma \ll \tau$ and then the zero limit of τ is taken in Eq. (2), the S prescription of the SDE (1), which preserves the usual rules of calculus, is obtained [16,17]. For example, if $b(x) = x$, the resulting S equation $d\ln(x)/dt = w\dot{\Theta}_\tau(t-t_0)$, after performing the limit $\tau \to 0$, has the formal solution $x(t) = [1 + \Theta(t-t_0)(e^w - 1)]x_0$, where $x_0 = x(0)$ and Θ is the Heaviside function. The corresponding PDF is

$$p^S(x,t) = \delta(x - x_0)[1 - \Theta(t - t_0)]$$
$$+ \Theta(t-t_0)\frac{\rho\!\left[\ln\!\left(\frac{x}{x_0}\right)\right]}{x}\Theta\!\left(\frac{x}{x_0}\right), \qquad (3)$$

with initial condition $\rho^S(x,0) = \delta(x - x_0)$. If otherwise $\tau \ll \sigma$, then Eq. (2) becomes $\sigma\ddot{x} + \dot{x} = b(x)w\delta(t-t_0)$. Imposing the conditions of continuity and right and left differentiability in t_0, the initial conditions $x(t_0^-)$ and $\dot{x}(t_0^-)$, and taking the limit $\sigma \to 0$, the solution is [again for the case $b(x) = x$] $x(t) = x_0 + x_0 w\Theta(t - t_0)$. Note that the latter corresponds to the solution in the Itô prescription of the SDE (2). From the
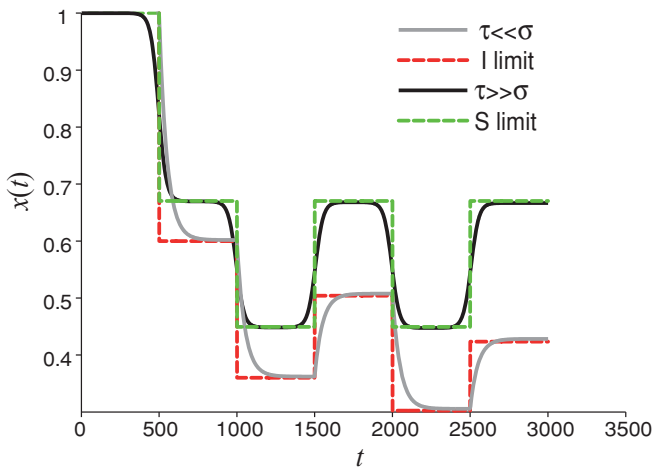

FIG. 1. (Color online) Comparison between trajectories of a particle that undergoes impulsive multiplicative forcing in a viscosity field for different time scales (τ and σ), and the trajectories that result from the SDE $\dot{x}(t) = -x(t)\sum_{i=1}^{5} w_i\delta(t - t_i)$ interpreted in the I and S prescriptions. The jumps in this case are given by $w_i = \pm 0.4$.

formal Itô solution of Eq. (2), we obtain the corresponding PDF in the I sense:

$$p^I(x,t) = \delta(x - x_0)[1 - \Theta(t - t_0)] + \Theta(t-t_0)\frac{\rho\!\left(\frac{x-x_0}{x_0}\right)}{x_0}. \qquad (4)$$

The latter equation can not be made to correspond to Eq. (3) for any choice of $\Theta(0)$. It is, in fact, interesting to observe that, if we set $\Theta(0) = \alpha$, then the parameter α defines where the $b(x)$ that multiplies the jump is evaluated: When $\alpha = 0$, $b(x)$ is evaluated before the jump, while $\alpha = 1/2$ corresponds to calculating $b(x)$ in the middle of the jump. In the literature on GWN, these choices are associated to the I and S prescriptions, respectively [16,18]. Conversely, as just seen for a discrete jump process, the S interpretation of the SDE (2) does not correspond to any of the α prescriptions. In other words, there is not an immediate intuitive interpretation of the S prescription.

III. MULTIPLICATIVE COMPOUND POISSON NOISE

We generalize now our analysis to a process described by the following SDE:

$$\dot{x}(t) \;=\; a(x,t) \,+\, b(x)\xi_\rho^\tau(\nu,t), \qquad (5)$$

where $\xi_\rho^\tau(\nu,t) = \sum_{i=1}^{N(t)} w\dot{\Theta}_\tau(t - t_i)$ is a colored compound Poisson process (CP), with jump heights w, each time drawn from a generic PDF $\rho(w)$, and $\{t_i\}$ are random times, the sequence of which is drawn from a homogeneous Poisson counting process $\{N(t),t \geqslant 0\}$ of rate ν. The case in Sec. II corresponds to the special case of a finite deterministic number of jumps. As before, the I interpretation consists of taking $\tau = 0$ and, should a jump occur at time t, evaluating $b(x)$ at the right-hand side of Eq. (5) before the jump occurrence, i.e., $x = x(t^-)$, while the S interpretation of Eq. (5) corresponds to performing the zero limit of the correlation time τ of the colored Poisson noise.

The S ME associated with the GLE (5) can be derived through the generating function of $\xi_\rho^\tau(\nu,t)$ (see Appendix A), or in a more formal way [8,19] as

$$\frac{\partial P^S(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}a(x,t) + \nu\big\langle e^{-w\frac{\partial}{\partial x}b(x)} - 1\big\rangle_{\rho(w)}\right] P^S(x,t), \qquad (6)$$

where $\langle\cdots\rangle$ denotes the ensemble average operator. A simpler alternative derivation of the ME (6) can be obtained using the fact that, in the S prescription, the rules of calculus are preserved. Defining the function $\eta(x) = \int^x \frac{dx'}{b(x')}$, the ME can be also written as (see Appendix B)

$$\frac{\partial P^S(x,t)}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)P^S(x,t)]$$
$$+ \nu\int_{-\infty}^{\infty}\frac{\rho[\eta(x) - \eta(x')]}{|b(x)|}P^S(x',t)dx' - \nu P^S(x,t). \qquad (7)$$

In the I prescription, $x(t)$ at time t does not depend on the noise $\xi_\rho^{\tau=0}(v,t) \equiv \xi_\rho(v,t)$ at the same time [20]. From this, it follows that

$$\langle b(x)\xi_\rho(v,t)\rangle = \langle b(x)\rangle\langle\xi_\rho(v,t)\rangle. \qquad (8)$$

Therefore, if (5) with $\tau = 0$ is interpreted in the I sense, we can change the size of the jumps from w to $b(x)w$, and the corresponding ME can be derived without ambiguity [11,19] as

$$\frac{\partial P^I(x,t)}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)P^I(x,t)] + v\int_{-\infty}^{\infty}\rho\left(\frac{x-x'}{b(x')}\right)$$
$$\times \frac{P^I(x',t)}{|b(x')|}dx' - vP^I(x,t). \qquad (9)$$

Alternatively, we achieved a different form of the I ME (9), which is the I analogous of the S ME (6):

$$\frac{\partial P^I(x,t)}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)P^I(x,t)]$$
$$+ v\langle : e^{-w\frac{\partial}{\partial x}b(x)} : -1\rangle_{\rho(w)}P^I(x,t), \qquad (10)$$

where $: \cdots :$ is an operator (analogous to the normal order operator in quantum field theory) that indicates that all the derivatives must be placed on the left of the expression, i.e., $: e^{-w\frac{\partial}{\partial x}b(x)}F(x) := \sum_{n=0}^{+\infty}\frac{(-w)^n}{n!}(\frac{\partial}{\partial x})^n[b(x)^nF(x)]$. For details, see Appendix C.

When $b(x) = b$ is constant, by using $e^{-bw\frac{\partial}{\partial x}}P^S(x,t) = P^S(x-bw,t)$, the I and S MEs become coincident, as expected. In Appendix D, we also show that, taking the limit $v \to \infty$, $\langle w\rangle \to 0$, i.e., infinite frequency and infinitesimally small jumps, such that $v\langle w\rangle^2 = D$ remains constant, Eqs. (6) and (10) reduce to the well-known I and S Fokker-Planck equation (FPE) for GWN [16,18], respectively.

IV. PRESCRIPTION-INDUCED JUMP DISTRIBUTIONS

It is clear from the previous MEs (7) and (9) that the I and S prescriptions of the GLE

$$\dot{x}(t) = a(x,t) + b(x)\xi_\rho(v,t) \qquad (11)$$

lead to different MEs. We now want to determine the connection between the two different interpretations. Specifically, we seek the two jump PDFs in the I and S interpretation, ρ_I and ρ_S, which give rise to the same process. We also seek how to obtain one form when the other is given. To this purpose, it is sufficient to equate the two MEs (7) and (9) for simplicity, from which

$$\frac{1}{|b(x')|}\rho_I\left(\frac{x-x'}{b(x')}\right) = \frac{1}{|b(x)|}\rho_S[\eta(x)-\eta(x')]. \qquad (12)$$

As a result, if Eq. (12) can be solved, given the jumps PDF and choosing the S (I) prescription for Eq. (11), the solutions ρ_I (ρ_S) of Eq. (12) give the equivalent corresponding I (S) GLE and ME. This is one of the main results of this paper and it provides the connection between the prescription-induced jump distributions ρ_I and ρ_S, allowing the link to the Itô ME and the Stratonovich ME corresponding to a GLE with multiplicative white Poisson noise.

The previous equation, however, has a solution only when $b(x)$ is a linear function of x. To show this, we rewrite Eq. (12) as

$$\rho_I(y) = \frac{|b(x')|}{|b(x)|}\rho_S[\eta(x)-\eta(x')] \equiv F(x',y), \qquad (13)$$

where $y = (x-x')/b(x')$. Because the left-hand side of Eq. (13) does not depend on x', we must have $\frac{\partial F}{\partial x'} = 0$. If the latter condition holds for all ρ_S, then we get $b''(x) = 0$, the solution of which is $b(x) = kx + c$ (see Appendix E for details). For other functional shapes of $b(x)$, the jumps PDF $\rho_I(w)$ depend also on the state of the system, i.e., the dependence on x of $\rho_I(w|x)$ can not be factored out. In this case, is not even clear what a Stratonovich prescription would correspond to.

Finally, we derive the distribution of the impulses that may be measured from the time series of the process (see inset in Fig. 2). In fact, if a random jump [drawn from $\rho(w)$] occurs at time t, then the size of the impulse that the whole process experiences is $y_t = x(t+dt) - x(t)$. From the GLE (5), we know that with probability $v\,dt$, $\dot{x} = b(x)\,w\,\Theta_\tau(t)$. Taking the limit $\tau \to 0$, and using the definition of $\eta(x)$, we have (see Appendix B)

$$y(t) = \begin{cases} w\,b(x), & (I) \\ \eta^{-1}[\eta(x)+w]-x, & (S) \end{cases} \qquad (14)$$

and thus we obtain

$$\hat{P}^I(y,t) = \langle\delta(y-y_t)\rangle = \int_{-\infty}^{+\infty}dx\,dw\,\frac{1}{|b(x)|}$$
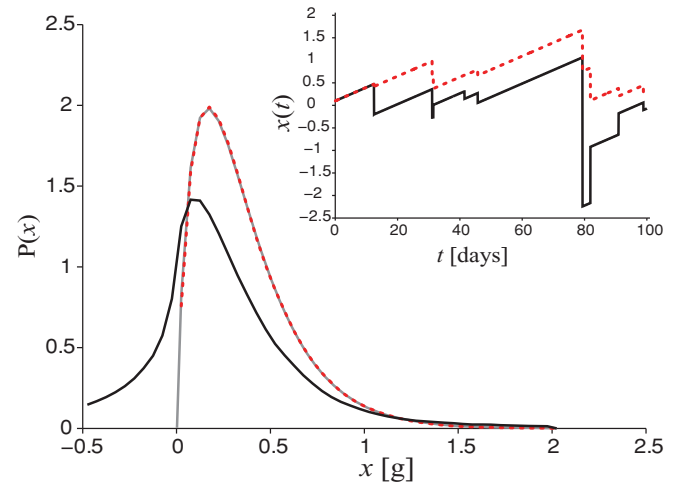$$\times P^I(x,t)\rho(w)\delta[w-y/b(x)], \qquad (15)$$

FIG. 2. (Color online) Comparison of the steady state PDF of Eq. (17): S solution [solid line, obtained analytically from Eq. (18)], I solution (dashed-dotted line, from numerical simulation), and I solution using the jump distribution given by Eq. (20) (dotted line, from numerical simulation). The numerical simulations confirm our analytical results. Inset: simulated trajectory of the salt mass under the two different prescriptions. Note that, if artificial reflecting barriers are not imposed, the salt mass given by the I prescription of Eq. (17) may assume unphysical negative values. The parameters used for the simulation are $\mu = 0.463$, $v = 0.15$ day^{-1}, and $\Upsilon = 30$ mg/day.

$$\hat{P}^S(y,t) = \int_{-\infty}^{+\infty} dw\, dx\, P^S(x,t)\rho(w)$$
$$\times \frac{\delta\{w - [\eta(x+y) - \eta(x)]\}}{|b(x+y)|}, \qquad (16)$$

i.e., the prescriptions characterize the PDF of the impulses of the whole process.

V. APPLICATION TO SOIL SALINIZATION

The above mathematical problems naturally arise in the context of the process of soil salinization. This is an extremely relevant environmental problem as 4 million km^2 in arid and semi-arid lands are affected by soil salinization, causing vegetation die-off and possible desertification [21,22]. In natural salinization (unlike the anthropogenic one due to irrigation), salt may accumulate in surface soils by dry and wet deposition due to wind and rain. In this problem, state-dependent Poisson jumps arise naturally when writing the salt mass balance equation at the daily-to-monthly time scale for soil root zone used as the control volume [22]. Salt inputs due to rainfall and wind act almost continuously in time, while the state-dependent losses of salt occur through negative jumps due to the leaching caused by intense rainfall events. Schematically, the salt mass at time $x(t)$ in the root zone is described by the GLE

$$\frac{dx}{dt} = \Upsilon - x\,\xi_\rho(\nu,t), \qquad (17)$$

where Υ is the time-averaged salt mass input flux and $\xi_\rho(\nu,t)$ is the leaching flux toward deeper layers, which can be approximated by a WP process with $\rho(w) = \mu \exp(-\mu w)\Theta(w)$. The leaching parameters ν (frequency of leaching events) and μ (mean jump) can be expressed in terms of the climatic, soil, and vegetation properties [22]. Because the typical duration of leaching events is on the order of a few hours, while the equilibration times of salt in the soil solution (proportional to the inverse of its dissolution rate) tend to be smaller (minutes to hours), this means that the inertia in the dynamics is small ($\sigma \ll \tau$) and the physically correct interpretation is likely to be the Stratonovich one.

The stationary solution of Eq. (6) in the S prescription is a gamma distribution (Fig. 2) [7,22]

$$P^S(x) = \mathcal{N}\, e^{-(x\,\nu/\Upsilon)}\, x^{1/\mu} \qquad (18)$$

for $x > 0$ and where $\mathcal{N} = (\frac{\nu}{\Upsilon})^{\frac{1+\mu}{\mu}} / \Gamma(\frac{1+\mu}{\mu})$ is the normalization constant and $\Gamma(z)$ is the complete gamma function of argument z. Equation (18) summarizes the soil salinity statistics as a function of climate, soil, and vegetation parameters, which may in turn be used in conjunction with the soil moisture statistic to obtain a full characterization of the salt concentration in the root zone and the ensuing risk of salinization [22].

From Eq. (16), it is possible to derive the PDF of the impulses of the process for the S interpretation as

$$\hat{P}^S(y) = \epsilon e^{\epsilon y}\Theta(-y), \qquad (19)$$

which is an exponential distribution controlled by the parameter $\epsilon = \nu/\Upsilon$, given by the ratio between the rate of leaching events and the average rate of salt input. Thus, if
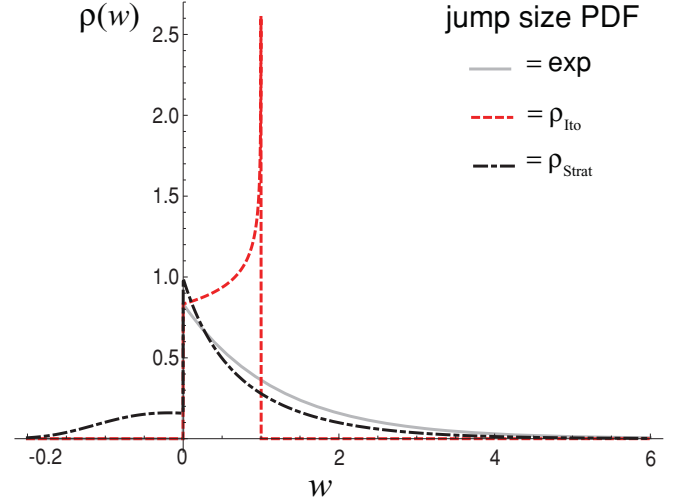

FIG. 3. (Color online) Comparison between a jump exponential distribution $\rho(w)$ with mean $1/\gamma = 0.8$, and the solutions $\rho_I(z)$, $\rho_S(z)$ of the prescription induced jumps corresponding to Eqs. (20) and (21) respectively, corresponding to the given $\rho(w)$.

time series of the process are available, the Stratonovich assumption can be checked by backtracking information on the physical time scales involved in the process via a comparison with experimental data. A further support for the S interpretation of Eq. (17) is given by the fact that x must remain positive after a jump, a fact that is not ensured by the I interpretation unless a reflecting boundary in $x = 0$ is imposed (see Fig. 2). We also computed the prescription-induced jump distribution correspondence for this case [$b(x) = -x$], which is $\rho_S(\ln|\frac{x'}{x}|) = |\frac{x}{x'}|\rho_I(1 - \frac{x}{x'})$, where x' and x are the variables before and after the jump, respectively. By taking into account that in the S prescription $x, x' > 0$, the I-jump PDF equivalent to $\rho_S(w) = \gamma e^{-\gamma w}\Theta(w)$ is

$$\rho_I(z) = \gamma(1-z)^{\gamma-1}, \qquad z \in [0,1]. \qquad (20)$$

This equivalence is indeed remarkable because it considerably facilitates the numerical simulation of the salinity equation in the S formulation (see Fig. 2). On the other hand, if the GLE (17) were interpreted in the I sense, the ratio x/x' could also be negative and the solution of Eq. (12), for $\rho_I = \gamma\Theta(w)e^{-\gamma w}$, would read as

$$\rho_S(w) = \gamma e^{-\gamma - w}\big[\Theta(w)e^{\gamma e^{-w}} + e^{-\gamma e^{-w}}\big], \quad w \in (-\infty, +\infty). \qquad (21)$$

This implies that possible negative jumps (that occur for $x < 0$) in the I prescription for the given $\rho_I(w)$ would be explicitly present in the corresponding equivalent S-jump PDF $\rho_S(w)$ (see Fig. 3).

VI. CONCLUSIONS

In this paper, we have proposed an alternative approach to solve the Itô-Stratonovich (I-S) dilemma for GLE with multiplicative WP noise. We have shown how different interpretations lead to different results and that choosing between the I and S prescriptions is crucial to describe correctly the dynamics of the model systems, and how this choice can be

determined by physical information about the time scales involved in the process. Moreover, we have addressed the related issue of finding a connection between the I and S interpretations in the case of linear WP noise. Differently from the introduction of a drift previously proposed [9,10], we have found such connection in a transformation of the jumps PDFs and tested our results numerically. Our results are also consistent with the physics of the random forcing, which takes place at specific points in time, whereas a continuously acting spurious drift would conceptually violate the causality of the process. In particular, once the GLE (11) is given, its I and S interpretations are shown to be equivalent if ρ_I and ρ_S satisfy the prescription-induced jumps PDF [Eq. (12)]. The case of nonlinear multiplicative WP noise will be studied elsewhere. We have applied our results to the geophysical problem of soil salinization by solving a minimalist model that describes the salt mass and concentration in a soil control volume as a function of climatic and ecohydrological parameters.

APPENDIX A

The stochastic process under study is described by the GLE (5) presented in the main text. For simplicity, in the following, we have set $\xi_\rho^\tau(t,\nu) = \xi(t)$. The CP is characterized by the correlation structure ($\langle \cdots \rangle$ denotes the ensemble average)

$$\langle \xi(t)\xi(t + \tau) \rangle \sim e^{-\frac{t}{\tau}}, \tag{A1}$$

where τ is the characteristic time of the process and we have omitted all the subscripts and superscripts to simplify the notation. If Φ_t is the generating function of CP at time t, then

$$\Phi_t[v] = \left\langle e^{i \int_0^t v(s)\xi(s)ds} \right\rangle = e^{\Psi_t[v]}$$

$$= \sum_{n=0}^{+\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \int dw\, \rho(w)$$

$$\times \int \prod_{j=1}^n \frac{dt_j}{t} e^{i \sum_{j=1}^n w_j \int_0^t v(s)\Theta_\tau(s-t_j)ds}$$

$$= \sum_{n=0}^{+\infty} e^{-\nu t} \frac{\nu^n}{n!} \left[\int_0^t dr \int dw\, \rho(w) \right.$$

$$\left. \times \exp\left(i \sum_{j=1}^n w_j \int_0^t v(s)\Theta_\tau(s-t_j)ds \right) \right]. \tag{A2}$$

Moreover, if we define $\hat{\rho} = \int_{-\infty}^{+\infty} e^{ivw}\rho(w)dw$ as the

characteristic function of $\rho(w)$, then we have

$$\Phi_t[v] = \exp\left[-\nu t + \nu \int_0^t dr\, \hat{\rho}\left(\int_0^t v(s)\Theta_\tau(s-\tau)d\tau \right) \right], \tag{A3}$$

and thus

$$\Psi_t[v] = \ln \Phi_t[v] = \nu \int_0^t dr \left[\hat{\rho}\left(\int_0^t v(s)\Theta_\tau(s-\tau)d\tau \right) - 1 \right]. \tag{A4}$$

The Stratonovich interpretation of Eq. (11) arises when the limit $\Theta_\tau(t - \tau) \to \delta(t - \tau)$ is taken [13,17], which is considering a white Poisson process (WP) as the zero limit of the correlation time of the corresponding CP. For a WP, the logarithm of the generating function thus reads as

$$\Psi_t[v] = \nu \int_0^t dr[\hat{\rho}(v(r)) - 1]. \tag{A5}$$

Finally, because of the Kubo theorem [23]

$$\Psi_t[v] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_0^t ds_1,\ldots,ds_n v(s_1),\ldots,v(s_n)$$

$$\times \langle\langle \xi(t_1),\ldots,\xi(t_n) \rangle\rangle_n, \tag{A6}$$

where $\langle\langle \cdots \rangle\rangle_j$ is the jth cumulant, i.e., $\langle\langle \cdot \rangle\rangle_1 = \langle \cdot \rangle$, $\langle\langle \cdots \rangle\rangle_2 = \langle \cdot \cdot \rangle - \langle \cdot \rangle\langle \cdot \rangle$, etc.

From Eqs. (A5) and (A6), we obtain the explicit formula to calculate the cumulants:

$$\langle\langle \xi(t_1) \rangle\rangle_1 = \frac{\delta\Psi_t}{i\delta v(t_1)} = \frac{\nu}{i}\hat{\rho}'(v)_{|v=0}, \tag{A7}$$

$$\langle\langle \xi(t_1)\xi(t_2) \rangle\rangle_2 = \frac{\delta^2\Psi_t}{i^2\delta v(t_1)\delta v(t_2)} = \frac{\nu}{i^2}\hat{\rho}''(v)_{|v=0}\delta(t_2 - t_1), \tag{A8}$$

$$\langle\langle \xi(t_1),\ldots,\xi(t_n) \rangle\rangle_n = \frac{\delta^n\Psi_t}{i^n\delta v(t_1),\ldots,\delta v(t_n)}$$

$$= \frac{\nu}{i^n}\hat{\rho}^{(n)}(v)_{|v=0}\delta(t_2 - t_1),\ldots,\delta(t_n - t_{n-1}). \tag{A9}$$

In this way, once $\rho(w)$ is given, we have a complete description of the WP. For example, in the case of exponential distributed jumps, i.e., $\rho(w) = \frac{1}{\langle w \rangle}e^{-\frac{w}{\langle w \rangle}}$, the WP is fully characterized by the moments

$$\langle\langle \xi(t) \rangle\rangle_1 = \nu\langle w \rangle, \tag{A10}$$

$$\langle\langle \xi(t_1)\xi(t_2) \rangle\rangle_2 = \nu\langle w^2 \rangle\delta(t_1 - t_2), \tag{A11}$$

$$\langle\langle \xi(t_1),\ldots,\xi(t_n) \rangle\rangle_n = \nu\langle w^n \rangle\delta(t_1 - t_2),\ldots,\delta(t_{n-1} - t_n). \tag{A12}$$

Once we have calculated all the moments of the WP process, we can easily achieve the ME corresponding to the GLE (11). For a given realization of ξ, the solution of Eq. (5) is

$$p^S(x,t|\xi) = \delta[x - x(t)]. \tag{A13}$$

To obtain the general solution of Eq. (11), we simply have to take the ensemble average of different trajectories

$$\langle p^S(x,t|\xi)\rangle = P^S(x,t). \quad (A14)$$

By differentiating both sides of Eq. (A13) and using Eq. (11), we have

$$\partial_t p^S(x,t|\xi) = \partial_x \delta[x - x(t)][-\dot{x}(t)] \quad (A15)$$
$$= -\partial_x \delta[x - x(t)]\{a[x,t] + b[x]\xi(t)\}, \quad (A16)$$

and thus we obtain a forward ME for the PDF conditioned by a given realization of the WP:

$$\frac{\partial}{\partial t} p^S(x,t|\xi) = -\mathcal{O}(x,\partial_x,t)\, p^S(x,t|\xi), \quad (A17)$$

where $\mathcal{O}(x,\partial_x,t) = \partial_x\{a[x,t] + b[x]\xi(t)\}$ is the forward time evolution operator. The solution of Eq. (A17) for the initial condition $p^S[x(0),0|\xi] = \delta[x - x(0)]$ is

$$p^S(x,t|\xi) = T\left[\exp\left(-\int_0^t \{\partial_x a(x,\tau) + \partial_x b[x(\tau)]\xi(\tau)\}d\tau\right)\right]$$
$$\times \delta[x - x(0)], \quad (A18)$$

where T is the T-product operator. By using Eq. (A14) and the Kubo relation (A6), an explicit formula for the general formal solution of the GLE (11) in the Stratonovich prescription is obtained as

$$P^S(x,t) = T\left[\exp\left(-\int_0^t \partial_x a(x,\tau)d\tau - \sum_{n=1}^{\infty} \int_0^t dt_1, \ldots,\right.\right.$$
$$\times \int_0^t dt_n \partial_x b[x(t_1)], \ldots, \partial_x b[x(t_n)] \quad (A19)$$
$$\left.\left.\times \langle\langle\xi(t_1), \ldots, \xi(t_n)\rangle\rangle\right)\right]\delta[x - x(0)]. \quad (A20)$$

Thanks to Eqs. (A7), (A8), and (A9), we have a complete characterization of the cumulants and, thus, substituting Eq. (A9) into Eq. (A19), we obtain

$$P^S(x,t) = T\left[\exp\left(-\int_0^t \partial_x a(x,\tau)d\tau\right.\right.$$
$$\left.+ \sum_{n=1}^{\infty} \nu \int_0^t \{-\partial_x b[x(\tau)]\}^n \rho(0)^{(n)} d\tau\right)\right]$$
$$= T\left[\exp\left(-\int_0^t \partial_x a(x,\tau)d\tau\right.\right.$$
$$\left.\left.- \nu \int_0^t d\tau \langle e^{-\partial_x b[x(\tau)]} - 1\rangle_{\rho(w)}\right)\right]. \quad (A21)$$

Eventually, differentiating Eq. (A21) with respect to t, we obtain the ME corresponding to the GLE (11) in the Stratonovich interpretation

$$\frac{\partial P^S(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} a(x,t) + \nu\langle e^{-w\frac{\partial}{\partial x} b(x)} - 1\rangle_{\rho(w)}\right] P^S(x,t), \quad (A22)$$

which is the ME (6) reported in the main text.

APPENDIX B

We now show the derivation of the S ME (7) in the main text and its equivalence with Eq. (6). We can write the GLE (5) as

$$\dot{x}(t) = \begin{cases} a(x,t), & \text{with probability } 1 - \nu\, dt \\ b(x)\, w\, h_\tau(t), & \text{with probability } \nu\, dt, \end{cases} \quad (B1)$$

where $h_\tau(t) = \dot{\Theta}_\tau(t)$. We now consider only the effect of the jumps on x. From Eq. (B1), we have that $dx/b[x(t)] = w\, h_\tau(t)dt$, and setting

$$\frac{d\eta(x)}{dx} = \frac{1}{b(x)} \quad \Rightarrow \quad \eta(x) = \int^x \frac{dx'}{b(x')}, \quad (B2)$$

Eq. (B1) becomes

$$d\eta[x(t)] = w\, h_\tau dt, \quad (B3)$$

which, integrated between t and $t + dt$, reads as

$$\eta[x(t + dt)] = \eta[x(t)] + w\,\Delta\Theta_\tau(t)$$
$$\Rightarrow x(t + dt) = \eta^{-1}\{\eta[x(t)] + w\,\Delta\Theta_\tau(t)\}, \quad (B4)$$

where $\Delta\Theta_\tau(t) = \Theta_\tau(t + dt) - \Theta_\tau(t)$.

Finally, we can write the discrete ME corresponding to the GLE (11) interpreted in the Stratonovich sense:

$$P^S(x,t + dt) = (1 - \nu)dt \int_0^\infty dx'\, P^S(x',t)\delta\{x$$
$$- [a(x')dt + x']\} + \nu dt \int_0^\infty \int_0^\infty \rho(w)$$
$$\times P^S(x',t)\delta(x - \{\eta^{-1}[\eta(x') + w]\})dw\, dx', \quad (B5)$$

where we have performed the limit $\tau \to 0$ of the GLE (5) and used the fact that $\lim_{\tau\to 0}\Delta\Theta_\tau(t) = 1$. The integral in the right-hand side of Eq. (B5) can be rewritten, inverting the Dirac delta with respect to w and using the rule of the inverse function, as $\int_0^\infty \int_0^x \rho(w)P^S(x')\frac{\delta\{w - [\eta(x) - \eta(x')]\}}{|1/\eta'(x)|}\, dw\, dx'$ and, thus, after taking the continuum time limit, the master equation (B5) becomes

$$\frac{\partial P^S(x,t)}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)P^S(x,t)] + \nu \int_0^\infty \frac{\rho[\eta(x) - \eta(x')]}{|b(x)|}$$
$$\times P^S(x',t)dx' - \nu\, P^S(x,t), \quad (B6)$$

which is Eq. (7) reported in the main text.

In order to show the equivalence between Eqs. (6) and (7), we define

$$Q(x,w) = \int_0^\infty P^S(x')\delta(x - \{\eta^{-1}[\eta(x') + w]\})\, dx', \quad (B7)$$

so we have that the integral in Eq. (B6) is simply $\int_0^\infty Q(x,w)\rho(w)dw$.

By differentiating Eq. (B7) with respect to w, we obtain the partial differential equation

$$\partial_w Q(x,w) = -\partial_x b(x) Q(x,w) = -\mathcal{H}Q, \qquad \text{(B8)}$$

where we used Eqs. (B2) and (B7) and the definition of the derivative of the inverse function. The solution of Eq. (B8) is

$$Q(x,w) = e^{-w\mathcal{H}} Q(x,0) = e^{-w\partial_x b(x)} P^S(x). \qquad \text{(B9)}$$

We thus have

$$\int_0^\infty \frac{\rho[\eta(x) - \eta(x')]}{|b(x)|} P^S(x',t) dx' = \langle e^{-w\partial_x b(x)} \rangle_\rho P^S(x,t), \qquad \text{(B10)}$$

which substituted in Eq. (B6) proves the equivalence between the MEs (6) and (7) in the main text.

APPENDIX C

We present in this appendix the derivation for the I ME (10) and its equivalence with Eq. (9). We first note that the integral in the ME (9) can be rewritten as

$$\int_{-\infty}^\infty \rho\left(\frac{x - x'}{b(x')}\right) \frac{P^I(x',t)}{|b(x')|} dx'$$
$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \rho(w)\delta[x - x' - w b(x')] P^I(x',t) dx' dw. \quad \text{(C1)}$$

Formally expanding the Dirac delta

$$\delta[x - x' - w b(x')] = \sum_{n=0}^{+\infty} \frac{(-w)^n}{n!}\left(\frac{\partial}{\partial x}\right)^n b(x')^n \delta(x - x')$$
$$\text{(C2)}$$

and substituting Eq. (C2) in (C1), we have

$$\int_{-\infty}^\infty \rho\left(\frac{x - x'}{b(x')}\right) \frac{P^I(x',t)}{|b(x')|} dx' = \langle (:e^{-w\frac{\partial}{\partial x} b(x)}:) P^I(x,t) \rangle_\rho, \qquad \text{(C3)}$$

where $(:e^{-w\frac{\partial}{\partial x} b(x)}:) P^I(x,t) \equiv \sum_{n=0}^{+\infty} \frac{(-w)^n}{n!}(\frac{\partial}{\partial x})^n b(x)^n P^I(x,t)$. Using the expression (C3) in Eq. (9), we obtain the ME (10).

APPENDIX D

We now derive the well-known FPE corresponding to the GLE (5) when $\xi(t)$ is a GWN with mean $\langle \xi(t) \rangle = 0$ and correlation $\langle \xi(t) \xi(s) \rangle = 2D\delta(t - s)$, from the MEs (6) and (10) presented in the main text. We generalize our results to any jump size PDF of the form

$$\rho(w) = \gamma f(\gamma w), \qquad \text{(D1)}$$

with $\gamma > 0$ and $\int w^n \rho(w) dw = \langle w^n \rangle_\rho < \infty \quad \forall n$. We note that the latter condition implies $\gamma \int dw\, w^n f(\gamma w) = \gamma^{-n} \int dz\, z^n f(z) = \gamma^{-n} \langle z^n \rangle_f < \infty \,\forall n$.

Stratonovich equation. The case for the Stratonovich prescription has been first presented in [7]. The FPE corresponding to multiplicative GWN process interpreted in the Stratonovich sense is

$$\frac{\partial}{\partial t} P^S(x,t) = -\frac{\partial}{\partial x}[a(x,t) P^S(x,t)]$$
$$+ D\frac{\partial}{\partial x} b(x) \frac{\partial}{\partial x} b(x) P^S(x,t). \qquad \text{(D2)}$$

Once we consider a zero mean WP process, the ME (6) reads as [7]

$$\frac{\partial P^S(x,t)}{\partial t} = -\frac{\partial}{\partial x}[[a(x,t) - \nu\langle w \rangle b(x)] P^S(x,t)]$$
$$+ \nu\langle e^{-w\frac{\partial}{\partial x} b(x)} - 1 \rangle_\rho P^S(x,t) \qquad \text{(D3)}$$

$$= -\frac{\partial}{\partial x}[a(x,t) P^S(x,t)] + \nu \sum_{n=1}^{+\infty}\left(-\frac{1}{\gamma}\right)^n \frac{\langle z^n \rangle_f}{n!}$$
$$\times \left(\frac{\partial}{\partial x} b(x)\right)^n P^S(x), \qquad \text{(D4)}$$

where the integral in the right-hand side of Eq. (D3) has been expanded as

$$\langle e^{-w\frac{\partial}{\partial x} b(x)} \rangle_\rho P^S(x) = \sum_{n=0}^{+\infty}(-1)^n \frac{\langle w^n \rangle}{n!}\left(\frac{\partial}{\partial x} b(x)\right)^n P^S(x)$$
$$= \sum_{n=0}^{+\infty}\left(-\frac{1}{\gamma}\right)^n \frac{\langle z^n \rangle_f}{n!}\left(\frac{\partial}{\partial x} b(x)\right)^n P^S(x). \qquad \text{(D5)}$$

Taking the limit $\nu, \gamma \to \infty$, such that $\frac{\nu}{\gamma^2} = D'$, then $\frac{\nu}{\gamma^n} \to 0$ for $n > 2$ and the latter ME (D4) corresponds exactly to the FPE (D2) with $D = D'\frac{\langle z \rangle_f}{2}$.

Itô equation. The FPE corresponding to multiplicative GWN process interpreted with the Itô prescription is

$$\frac{\partial}{\partial t} P^I(x,t) = -\frac{\partial}{\partial x}[a(x,t) P(x,t)], D\frac{\partial^2}{\partial x^2}[b(x)^2 P^I(x,t)]. \qquad \text{(D6)}$$

We now repeat the same procedure as before, starting from the zero mean I ME

$$\frac{\partial P^I(x,t)}{\partial t} = -\frac{\partial}{\partial x}\{[a(x,t) - \nu\langle w \rangle b(x)] P^I(x,t)\}$$
$$+ \nu\langle (:e^{-w\frac{\partial}{\partial x} b(x)}:) - 1 \rangle_\rho P^I(x,t). \qquad \text{(D7)}$$

We can expand the right-hand side remembering that the operator $: \cdots :$ means that all the derivatives must be placed on the left of the expression:

$$\nu\langle (:e^{-w\frac{\partial}{\partial x} b(x)}:) - 1 \rangle_\rho P^I(x,t)$$
$$= -\nu\langle w \rangle_\rho \frac{\partial}{\partial x}[b(x) P^I(x,t)] + \nu \sum_{n=2}^{+\infty}\left(-\frac{1}{\gamma}\right)^n \frac{\langle z^n \rangle_f}{n!}$$
$$\times \left(\frac{\partial}{\partial x}\right)^n [b(x)^n P^I(x,t)]. \qquad \text{(D8)}$$

Eventually, by inserting Eq. (D8) in the I ME (D7) and taking $\nu, \gamma \to \infty$ with $\frac{\nu}{\gamma^2} = D'$ and $D = D'\frac{\langle z \rangle_f}{2}$, we obtain the I FPE (D6).

APPENDIX E

In this appendix, we show how a solution of Eq. (12), rewritten as

$$\rho_I(y) = \frac{|b(x')|}{|b(x)|}\rho_S[\eta(x) - \eta(x')] \equiv F(x',y), \qquad (E1)$$

where $y = (x - x')/b(x')$, exists only if b is a linear function.

Because the left-hand side of Eq. (E1) does not depend on x', we must have $\frac{\partial F}{\partial x'} = 0$, which explicitly reads as

$$0 = \rho_S[\eta(x) - \eta(x')]\left(\text{sgn}[b(x')]\frac{b'(x')}{|b(x)|} - \text{sgn}[b(x)]\right.$$
$$\times \frac{b'(x)}{|b(x)|^2}|b(x')|[1 + yb'(x')]\Big) + \rho_S'[\eta(x) - \eta(x')]$$
$$\times \frac{|b(x')|}{|b(x)|}\{\eta'(x)[1 + yb'(x) - \eta'(x')]\}. \qquad (E2)$$

The latter, using Eq. (B2), can be expressed as

$$\frac{\rho_S'[\eta(x) - \eta(x')]}{\rho_S[\eta(x) - \eta(x')]}\left(\frac{1}{b(x)}[1 + yb'(x')] - \frac{1}{b(x')}\right)$$
$$+ \frac{b'(x')}{b(x')} - \frac{b'(x)}{b(x)}[1 + yb'(x')] = 0. \qquad (E3)$$

Equation (E3) must hold for all ρ_S, then the solution of Eq. (E3) is given by the function b, which satisfies the conditions

$$b(x')[1 + yb'(x')] = b(x), \qquad (E4)$$

$$b(x')b'(x)[1 + yb'(x')] = b(x)b'(x'). \qquad (E5)$$

By combining Eqs. (E4) and (E5) and using $x = b(x')y + x'$, we obtain the equation $b'[b(x')y + x'] = b'(x')$. If we take the derivative of both sides with respect to the independent variable y, then we have $b''[b(x')y + x']b(x') = 0$. This implies $b''(x) = 0 \,\forall x$, the solution of which is $b(x) = kx$ (with k any constant).

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