Product of Ginibre matrices: Fuss-Catalan and Raney distributions

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Squared singular values of a product of s square random Ginibre matrices are asymptotically characterized by probability distributions $P_s(x)$, such that their moments are equal to the Fuss–Catalan numbers of order s. We find a representation of the Fuss-Catalan distributions $P_s(x)$ in terms of a combination of s hypergeometric functions of the type ${}_sF_{s-1}$. The explicit formula derived here is exact for an arbitrary positive integer s, and for $s = 1$ it reduces to the Marchenko-Pastur distribution. Using similar techniques, involving the Mellin transform and the Meijer G function, we find exact expressions for the Raney probability distributions, the moments of which are given by a two-parameter generalization of the Fuss-Catalan numbers. These distributions can also be considered as a two-parameter generalization of the Wigner semicircle law.

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I. INTRODUCTION

Random matrices of various ensembles find numerous applications in several fields of statistical physics. In the general class of non-Hermitian random matrices an important role is played by the *Ginibre ensemble* [1]. A matrix G of size N of such an ensemble consists of N^2 independent random complex numbers, drawn according to the Gaussian distribution with zero mean and a fixed variance [2,3]. Such matrices are used to describe nonunitary dynamics of chaotic systems and open quantum systems [4]. This ensemble of random matrices can also be used to analyze the human electroencephalogram data [5], for telecommunication applications based on the scattering of electromagnetic waves on random obstacles [6], or in mathematical finance to describe correlation matrices of various stocks [7,8].

The spectrum of a non-Hermitian matrix G belongs to the complex plane. Spectral density of random matrices of the suitably normalized Ginibre ensemble is described by the Girko circular law [9], as in the limit $N \to \infty$ it covers uniformly the unit disk. The random matrix $W = GG^\dagger$, called a *Wishart matrix*, is positive. Hence its eigenvalues λ_i, $i = 1, \ldots, N$, are real and nonnegative. Introducing a rescaled eigenvalue, $x = N\lambda$, one can show that in the limit of the large matrix size the spectral density $P(x)$ converges to the *Marchenko-Pastur* (MP) distribution [10].

In general, products of random matrices have been a subject of an intensive research for many years [11]. Recent studies on products of Ginibre matrices concern multiplicative diffusion processes [12], correlation matrices used in macroeconomic time series [13], a random matrix approach to quantum chromodynamics [14], and lattice gauge field theories [15]. Properties of the complex spectra of products of random Ginibre matrices were recently analyzed in Ref. [16].

It is also interesting to study singular values of a product of s independent Ginibre matrices, $X = G_1 \cdots G_s$. Note that a squared singular value of the product X equals the corresponding eigenvalue of the Wishart-like matrix $W = XX^\dagger$. For $s = 2$, positive random matrices of the form $W_2 = G_1G_2(G_1G_2)^\dagger$ found their applications in finance [13]. Matrices of the form $W_s = G_1 \cdots G_s(G_1 \cdots G_s)^\dagger$ for an arbitrary s were used to describe random quantum states associated with certain graphs [17] and quantum states obtained by orthogonal measurements in a product of maximally entangled bases [18].

The corresponding asymptotic-level density distribution $P_s(x)$ is called a *Fuss-Catalan distribution* of order s, since its moments are given [19–21] by the Fuss-Catalan numbers [22,23] (also called Fuss[1] numbers [24]). Strangely enough, the Fuss-Catalan numbers generalize the Catalan numbers, although the work of Fuss [25] was done much earlier then the contribution of Catalan [26]. The Catalan number can be defined as a number of different bracketings of a product of $n + 1$ numbers, or the number of possible n foldings of a map that contains $n + 1$ pages in a row [27].

The Fuss-Catalan distribution describes asymptotically statistics of singular values of the sth power of random Ginibre matrices. This result obtained recently by Alexeev *et al.* [28] was derived by estimating the moments of the distribution of squared singular values of a power G^s of a random matrix and showing that these moments converge asymptotically to the Fuss-Catalan numbers. This is true under rather weak assumptions: All entries of the matrix G are independent random variables characterized by the zero mean, variance set to unity, and finite fourth moment.

The Fuss-Catalan distribution can be considered as a generalization of the MP distribution, which is obtained for $s = 1$. Moreover, the distribution $P_s(x)$ belongs to the class of free Meixner measures [29], and in terms of free probability theory it appears as the *free* multiplicative convolution product

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[1]When Leonard Euler, after an eye operation in 1772, became almost completely blind, he asked Daniel Bernoulli in Basel to send a young assistant, well trained in mathematics, to him in St. Petersburg. It was Nikolaus Fuss, who arrived in St. Petersburg in May 1773 [49].

of s copies of the MP distribution [21,30], which is written as $P_s(x) = [P_1(x)]^{\boxtimes s}$.

An explicit form in the case $s = 2$ was derived in Ref. [31] in context of construction of generalized coherent states from combinatorial sequences. The spectral distribution of $P_s(x)$ for a product of an arbitrary number of s random Ginibre matrices was recently analyzed by Burda *et al.* [32] also in the general case of rectangular matrices. The distribution was expressed as a solution of a polynomial equation, and it was conjectured that the finite size effects can be described by a simple multiplicative correction. Another recent work by Liu *et al.* [33] provides an integral representation of the distribution $P_s(x)$ derived in the case of s square matrices of size N, which is assumed to be large. However, these recent contributions do not provide an explicit form of the distribution $P_s(x)$.

The aim of this note is to derive exact and explicit formulas for the Fuss-Catalan distribution $P_s(x)$, which can be represented as a combination of s hypergeometric functions. The derivation is presented in Sec. II, and auxiliary information on special functions and the proof of positivity of $P_s(x)$ are provided in Appendices A and B, respectively. In Sec. III we discuss a certain two-parameters generalization of Fuss-Catalan numbers. As these numbers quantify generalized Raney sequences [22], the corresponding probability measures $W_{p,r}(x)$ will be called *Raney distributions*. As special cases they include the Marchenko-Pastur distribution, Fuss-Catalan distributions, and Wigner semicircle law. We find exact expressions for Raney distributions corresponding to integer parameter values in the general case and provide explicit formulas in the simplest cases of small values of the integer parameters $1 \leqslant r \leqslant p$.

II. FUSS-CATALAN DISTRIBUTIONS

For any integer number s one can use the binomial symbol to define a sequence of integers denoted by $C_s(n)$,

$$C_s(n) := \frac{1}{sn+1}\binom{sn+n}{n}. \tag{1}$$

Here $n = 0,1,\ldots$, while $s = 1,2,\ldots$, and these numbers are called the *Fuss-Catalan* numbers of order s [22].

We enumerate some of these sequences for $n = 0,\ldots,7$:

$$\begin{aligned} C_1(n) &= 1,1,2,5,14,42,132,427,\ldots,\\ C_2(n) &= 1,1,3,12,55,273,1428,7752,\ldots,\\ C_3(n) &= 1,1,4,22,140,969,7084,53\,820,\ldots,\\ C_4(n) &= 1,1,5,35,285,2530,23\,751,23\,1880,\ldots. \end{aligned}$$

The above sequences are contained in the Online Encyclopedia of Integer Sequences (OEIS) [34] under the labels (A000108), (A001724), (A002293), and (A002294), respectively. These sequences can be considered as a generalization of the sequence $C_1(n)$, which consists of *Catalan numbers*, $C_1(n) = \frac{1}{n+1}\binom{2n}{n}$.

We will show that for any given s there exists a density distribution $P_s(x)$, which satisfies

$$\int_0^{K_s} x^n P_s(x)\,dx = C_s(n), \quad n = 0,1,\ldots, \tag{2}$$

where

$$K_s := (s+1)^{s+1}/s^s. \tag{3}$$

In other words, we are looking for a positive density $P_s(x)$ that satisfies the above infinite system of equations. As the density turns out to be defined in a finite segment $[0, K_s]$, the solution of this *Hausdorff moment problem* [35] associated with the Fuss-Catalan numbers is unique. An explicit proof of positivity of $P_s(x)$ is provided in Appendix B.

We employ the method of the inverse Mellin transform, which was previously used to construct explicit solutions of the Hausdorff moment problem [36,37] and to derive explicit form of the Lévy-stable distributions [38]. The *Mellin transform* $\mathcal{M}$ of a function $f(x)$ and its inverse $\mathcal{M}^{-1}$ are defined by a pair of equations:

$$f^*(\sigma) := \mathcal{M}[f(x);\sigma] = \int_0^\infty x^{\sigma-1} f(x)\,dx \tag{4}$$

and

$$f(x) := \mathcal{M}^{-1}[f^*(\sigma);x] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-\sigma} f^*(\sigma)\,d\sigma, \tag{5}$$

with complex σ. In (5) this variable is integrated over a vertical line in the complex plane [39]. For discussion concerning the role of c, see Refs. [39–41]. Therefore the solution of Eq. (2) can be obtained by extending integer variable n to complex σ by a substitution $n \to \sigma - 1$. The desired form of the distribution $P_s(x)$ can be formally written as an inverse Mellin transform:

$$P_s(x) \;=\; \mathcal{M}^{-1}[C_s(\sigma);x]\,. \tag{6}$$

To find such a transform we will bring the Fuss-Catalan numbers into a more suitable form. Representing the binomial symbol in (1) by the ratio of Euler's gamma function, one obtains

$$C_s(\sigma) = \frac{\Gamma\left[(s+1)\left(\sigma - \frac{s}{s+1}\right)\right]}{\Gamma\left[s\left(\sigma - \frac{s-2}{s}\right)\right]\Gamma(\sigma)}\,. \tag{7}$$

After applying twice the Gauss-Legendre formula (A1) for multiplication of the argument of the gamma function one arrives at

$$\begin{aligned} C_s(\sigma) = {} & \frac{1}{\sqrt{2\pi}}\left[\frac{(s+1)^{s+1}}{s^s}\right]^\sigma \\ & \times \frac{s^{s-3/2}}{(s+1)^{s+1/2}}\left[\prod_{j=0}^{s-1}\frac{\Gamma\left(\sigma + \frac{j-s}{s+1}\right)}{\Gamma\left(\sigma + \frac{2+j-s}{s}\right)}\right]. \end{aligned} \tag{8}$$

Obtaining the above form of the C_s numbers, in which a ratio of products of the gamma functions of a shifted argument appears, is a key step of our reasoning. It allows us to represent the inverse Mellin transform of Eq. (8) as a certain special function. To see this, recall that the *Meijer G function* of the argument z can be defined by the inverse Mellin transform [40]:

$$\begin{aligned} & G^{m,n}_{p,q}\left(z\left|{\alpha_1\cdots\alpha_p \atop \beta_1\cdots\beta_q}\right.\right) \\ & = \mathcal{M}^{-1}\left[\frac{\prod_{j=1}^m \Gamma(\beta_j+\sigma)\prod_{j=1}^n \Gamma(1-\alpha_j-\sigma)}{\prod_{j=m+1}^q \Gamma(1-\beta_j-\sigma)\prod_{j=n+1}^p \Gamma(\alpha_j+\sigma)};z\right]. \end{aligned} \tag{9}$$

This definition involves four lists of parameters, which can be represented by p complex numbers α_j and other q complex numbers β_j. Integers numbers p and q can be equal to zero, and it is assumed that $0 \leqslant m \leqslant q$ and $0 \leqslant n \leqslant p$, so that possibly empty products in this form are taken to be equal to unity. A detailed description of the integration contours of the Mellin transform (9), general properties of the Meijer functions, and their special cases can be found in Ref. [40].

Direct comparison of expression (8) for the Fuss Catalan numbers and the Mellin transform as the Meijer G function (9) allows us to represent the Fuss-Catalan distribution $P_s(x)$ by a Meijer G function,

$$P_s(x) = \frac{1}{\sqrt{2\pi}} \frac{s^{s-3/2}}{(s+1)^{s+1/2}} G^{s,0}_{s,s}\left(z \Big| \begin{matrix} \alpha_1 \cdots \alpha_p \\ \beta_1 \cdots \beta_q \end{matrix}\right) \tag{10}$$

of the argument $z = x s^s (s+1)^{-(s+1)}$. Looking at the range of the parameter j in (8) we see that the numbers of parameters of the Meijer G function have to be set to $n = 0$, $p = s$, and $m = q = s$. Hence this function involves $2s$ parameters, which read $\alpha_j = (1 + j - s)/s$ and $\beta_j = (j - 1 - s)/(s + 1)$ for $j = 1, \ldots, s$.

As there are only two products of gamma functions in (8), in contrast to four in (9), which is equivalent to setting $n = 0$ and $m = q$, a further simplification of the above formula is possible. In this case the Meijer G function can be written as a combination of hypergeometric functions of the same argument z [see Eq. (5.2.11), p. 146 of Ref. [41]].

Let ${}_pF_q([\{a_j\}_{j=1}^p],[\{b_j\}_{j=1}^q]; x)$ denote the hypergeometric function [42] of the type ${}_pF_q$ with p "upper" parameters a_j and q "lower" parameters b_j of the argument x. The symbol $\{a_i\}_{i=1}^r$ represents the list of r elements, $a_1, \ldots, a_r$. Then formula (10) for the Fuss-Catalan distribution can be rewritten as

$$P_s(x) = \sum_{k=1}^{s} \Lambda_{k,s} x^{\frac{k}{s+1}-1} {}_sF_{s-1}\left\{\left[\left(1 - \frac{1+j}{s} + \frac{k}{s+1}\right)_{j=1}^{s}\right], \left[\left(1 + \frac{k-j}{s+1}\right)_{j=1}^{k-1}, \left(1 + \frac{k-j}{s+1}\right)_{j=k+1}^{s}\right]; \frac{s^s}{(s+1)^{s+1}} x\right\}. \tag{11}$$

where the coefficients $\Lambda_{k,s}$ read for $k = 1, 2, \ldots, s$,

$$\Lambda_{k,s} := s^{-3/2} \sqrt{\frac{s+1}{2\pi}} \left(\frac{s^{s/(s+1)}}{s+1}\right)^k \times \frac{\left[\prod_{j=1}^{k-1} \Gamma\left(\frac{j-k}{s+1}\right)\right]\left[\prod_{j=k+1}^{s} \Gamma\left(\frac{j-k}{s+1}\right)\right]}{\prod_{j=1}^{s} \Gamma\left(\frac{j+1}{s} - \frac{k}{s+1}\right)}. \tag{12}$$

This formula, along with Eq. (22), constitutes the key result of the present note. It gives an exact result for the Fuss-Catalan (FC) distribution $P_s(x)$ for an arbitrary natural s. The FC distribution describes the density of squared singular values of a product of s independent square Ginibre matrices in the limit of a large matrix size.

The convergence conditions of the hypergeometric series ${}_sF_{s-1}$ immediately yield the support of $P_s(x)$, which is equal to $[0, (s+1)^{s+1}/s^s]$. For small values of x the distribution behaves as $x^{-s/(s+1)}$. It is comforting to see that in the simplest case $s = 1$ the above complicated form reduces indeed to the Marchenko-Pastur distribution,

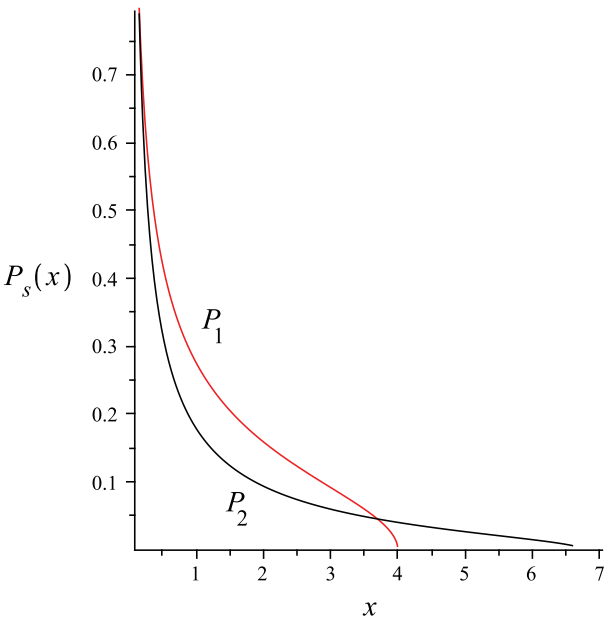

FIG. 1. (Color online) Marchenko-Pastur distribution $P_1(x)$ compared with the Fuss-Catalan distribution $P_2(x)$. The singularity at $x \to 0$ is of the type $P_s(x) \sim x^{-s/(s+1)}$.

$$P_1(x) = \frac{1}{\pi\sqrt{x}} {}_1F_0\left(\left[-\frac{1}{2}\right],[\,]; \frac{1}{4}x\right) = \frac{\sqrt{1 - x/4}}{\pi\sqrt{x}}. \tag{13}$$

Furthermore, the distribution $P_2(x)$, shown in Fig. 1,

$$P_2(x) = \frac{\sqrt{3}}{2\pi x^{2/3}} {}_2F_1\left(\left[-\frac{1}{6}, \frac{1}{3}\right], \left[\frac{2}{3}\right]; \frac{4}{27}x\right) - \frac{\sqrt{3}}{6\pi x^{1/3}} {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right], \left[\frac{4}{3}\right]; \frac{4}{27}x\right) \tag{14}$$

is equivalent to the form

$$P_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{[\sqrt[3]{2}(27 + 3\sqrt{81 - 12x})^{\frac{2}{3}} - 6\sqrt[3]{x}]}{x^{\frac{2}{3}}(27 + 3\sqrt{81 - 12x})^{\frac{1}{3}}}, \tag{15}$$

valid for $x \in [0, 27/4]$ and obtained first in Ref. [31] in the context of construction of generalized coherent states from combinatorial sequences. The distribution $P_3(x)$, plotted in Fig. 2, is given by a sum of three terms:

$$P_3(x) = \frac{1}{\sqrt{2}\pi x^{3/4}} {}_3F_2\left(\left[-\frac{1}{12}, \frac{1}{4}, \frac{7}{12}\right], \left[\frac{1}{2}, \frac{3}{4}\right]; \frac{27}{256}x\right) - \frac{1}{4\pi x^{1/2}} {}_3F_2\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right], \left[\frac{3}{4}, \frac{5}{4}\right]; \frac{27}{256}x\right) - \frac{\sqrt{2}}{64\pi x^{1/4}} {}_3F_2\left(\left[\frac{5}{12}, \frac{3}{4}, \frac{13}{12}\right], \left[\frac{5}{4}, \frac{3}{2}\right]; \frac{27}{256}x\right). \tag{16}$$

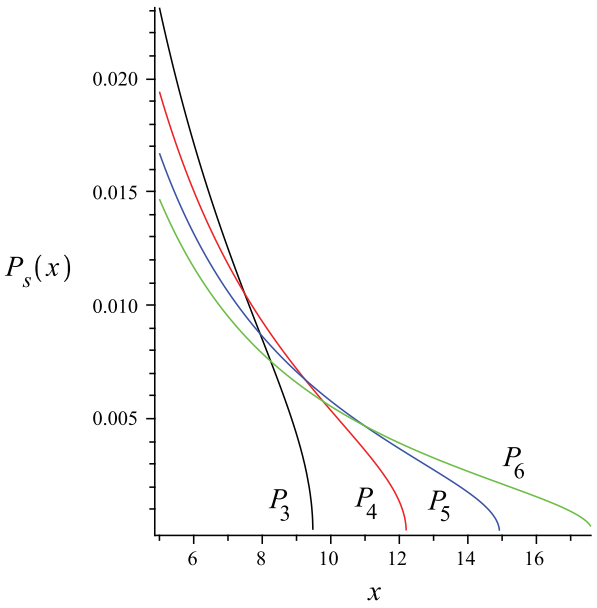

FIG. 2. (Color online) Fuss-Catalan distributions $P_s(x)$ plotted for parameter s equal to 3, 4, 5, and 6 are supported on the interval $[0,(s+1)^{s+1}/s^s]$. To show behavior at the right side of the support the figure is plotted for $x \geqslant 5$.

In Fig. 2 we present the Fuss-Catalan distributions $P_s(x)$ for $s = 3, 4, 5$, and 6.

III. RANEY DISTRIBUTIONS

The Fuss-Catalan numbers $C_s(n)$ defined in (1) can be considered as a special cases of a larger family of sequences,

$$R_{p,r}(n) := \frac{r}{pn+r}\binom{pn+r}{n}, \tag{17}$$

defined for $n = 0,1,\ldots$. Here p and r are treated as integer parameters, $p \geqslant 2$, $r = 1,2,\ldots$. Setting $r = 1$ and $p = s+1$ we have $\frac{1}{np+1}\binom{np+1}{n} = \frac{1}{n(p-1)+1}\binom{np}{n}$, so the numbers $R_{s+1,1}(n)$ are equal to $C_s(n)$. Further relations involving the sequences $R_{p,r}(n)$ are

$$R_{p+1,p+1}(n) = C_p(n+1) \tag{18}$$

and

$$R_{p,p}(n) = R_{p,1}(n+1), \tag{19}$$

which can be verified directly from their definitions (1) and (17).

The Raney lemma [43] implies that the number of the Raney sequences of order p and length $pn+1$, for which all partial sums are positive, is given by the Fuss-Catalan numbers $C_{p-1}(n) = R_{p,1}(n)$. Furthermore, as the number of positive generalized Raney sequences is equal to $R_{p,r}(n)$ [22] we will refer to $R_{p,r}(n)$ defined in (17) as *Raney numbers*. These numbers appear as coefficients in a generalized binomial series [22]. Some representative examples of sequences $R_{p,r}(n)$, for $n = 0,1,\ldots,7$ are given together with their OEIS labels [34]:

$$R_{4,2}(n) = 1,2,9,52,340,2394,17\,710,135\,720,\ldots\,(A069271),$$
$$R_{5,2}(n) = 1,2,11,80,665,5980,56\,637,556\,512,\ldots\,(A118969),$$

whereas the two following sequences are not represented in OEIS:

$$R_{4,5}(n) = 1,5,30,200,1425,10\,626,81\,900,647\,280,\ldots,$$
$$R_{6,3}(n) = 1,3,21,190,1950,21\,576,250\,971,302\,5308,\ldots.$$

In a recent work Młotkowski [20] has shown that the sequence (17) describes moments of a probability measure $\mu_{p,r}$ with a compact support contained in $[0,\infty)$, if point (r,p) that determines parameters of the Raney numbers belongs to the set Σ defined by inequalities $p \geqslant 1$ and $0 < r \leqslant p$. Note that the point $(1,1)$ implies a constant sequence of moments, $R_{1,1}(n) = 1$, which represents a singular, Dirac delta measure, $\mu_{1,1} = \delta(x-1)$.

In the case where the measure $\mu_{p,r}$ is represented by a density, we will denote it by $W_{p,r}(x)$. Setting r to 1, one gets the Fuss-Catalan numbers, which implies that $W_{2,1}(x)$ represents the Marchenko-Pastur distribution, while $W_{s+1,1}(x)$ reduces to the Fuss-Catalan probability density $P_s(x)$.

In general parameters p and r can be taken to be real, and then the moments of the measure are expressed by the gamma functions:

$$\int x^n \mu_{p,r}(x)\,dx = \frac{r}{np+r}\frac{\Gamma(np+r+1)}{\Gamma(n+1)\Gamma(np+r-n+1)}, \tag{20}$$

where the integration covers entire support of the measure $\mu_{p,r}$. For $1 \leqslant r \leqslant p$ the distribution $W_{p,r}(x)$ is a positive function; see Ref. [20] and Appendix B.

The corresponding distribution $W_{p,r}(x)$ can be written implicitly by its S transform, which allowed Młotkowski to establish relations between various Raney distributions [20], listed in Appendix C. For a precise definition of the S transform (or the free multiplicative convolution), see Eq. (4.9) in Ref. [20]. In spite of these concrete results an explicit form of the Raney distribution $W_{p,r}(x)$ has not appeared in the literature so far.

Making use of the inverse Mellin transform and the Meijer function we can generalize results of the previous section and obtain explicit expressions for the Raney distributions $W_{p,r}(x)$, which correspond to integer values of the parameters p and r.

Repeating steps analogous to Eqs. (7)–(10) we can represent distribution $W_{p,r}(x)$ in terms of the Meijer G function. The explicit expression generalizing Eq. (10) reads

$$W_{p,r}(x) = \frac{r}{\sqrt{2\pi}}\frac{p^{r-p-1/2}}{(p-1)^{r-p+3/2}}G^{p,0}_{p,p}\left(z\Big|^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}\right), \tag{21}$$

where the argument of the function is $z = x(p-1)^{p-1}/p^p$, while its parameters are $\alpha_1 = 0, \alpha_j = (r-p+j)/(p-1)$ for $j = 2,\ldots,p$, and $\beta_j = (r-p-1+j)/p$ for $j = 1,\ldots,p$.

In analogy to (11) this relation can be represented by the following sum consisting of of p terms:

$$W_{p,r}(x) = \sum_{j=1}^{p} \Omega(p,r;j) x^{\frac{r-1+j}{p}-1} {}_pF_{p-1}\left\{ \left[1+\beta_j, (1+\beta_j-\alpha_i)_{i=2}^{p}\right], \left[\left(1+\frac{j-i_2}{p}\right)_{i_2=1}^{j-1}, \left(1+\frac{j-i_3}{p}\right)_{i_3=j+1}^{p}\right], \frac{(p-1)^{p-1}}{p^p}x \right\}, \tag{22}$$

where ${}_pF_{p-1}$ is the hypergeometric function, and the numerical coefficients $\Omega(p,r;j)$, for $j = 1,2,\ldots,p$, read

$$\Omega(p,r;j) := \frac{r}{\sqrt{2\pi}} \frac{p^{r-p-1/2}}{(p-1)^{r-p+3/2}} \left[\frac{(p-1)^{p-1}}{p^p}\right]^{\frac{r-p-1+j}{p}} \times \frac{1}{\Gamma\left(\frac{p-r+1-j}{p}\right)} \frac{\left[\prod_{i_1=1}^{j-1}\Gamma\left(\frac{i_1-j}{p}\right)\right]\left[\prod_{i_2=1}^{p-j}\Gamma\left(\frac{i_2}{p}\right)\right]}{\prod_{i_3=2}^{p}\Gamma\left(\frac{r-p+i_3}{p-1}-\frac{r-p-1+j}{p}\right)}. \tag{23}$$

Convergence properties of the hypergeometric function imply that the Raney distribution $W_{p,r}(x)$ for integer values of its parameters is supported in the interval $[0,K_{p-1}]$, where K_s is given in Eq. (3). Formula (22) implies that for $p > r$ the distribution $W_{p,r}(x)$ displays a singularity for small x of the type $x^{-(p-r)/p}$. For $p = r$ the "diagonal" Raney distributions behave for small arguments as $W_{p,p}(x) \sim x^{1/p}$.

It is helpful to add some clarifying remarks concerning the key formula (22). We draw attention to the fact that some simplifications will always occur for two reasons:

(a) First, one parameter from the "upper" list of the parameters will always be equal to a parameter from the "lower" list. Consider, for instance, the case $p = 4$ and $r = 2$. Then the value of the first parameter in the "upper" list becomes $1 + (j-3)/4$, and it cancels with the value of $i_3 = 3$ of the first sequence of the "lower" parameters. One can demonstrate that a similar cancellation effect takes place for any pair (p,r). Therefore, the hypergeometric function ${}_pF_{p-1}$ in (22) effectively reduces to ${}_{p-1}F_{p-2}$.

(b) Second, we see from Eq. (23) that the coefficient $\Omega(p,r;j)$ vanishes for $j = p+1-r$, due to the presence of the first gamma function in denominator. Therefore the sum in Eq. (22) involves $(p-1)$ terms with different hypergeometric functions ${}_{p-1}F_{p-2}$. Then it can be explicitly verified that from the equality between the numbers $R_{s+1,1}(n) = C_s(n)$, the corresponding equality between the probability distributions

$$W_{s+1,1}(x) = P_s(x) \tag{24}$$

follows. In particular, the following identity between the coefficients holds: $\Omega(s+1,1;j) = \Lambda_{j,s}$, whose demonstration is rather tedious and will not be reproduced here.

With the two provisos explained under items (a) and (b) above, Eq. (22) will be used to obtain explicit forms of distributions $W_{p,r}(x)$ for small values of r and p. In particular, the case $W_{2,2}(x)$ reduces to the celebrated semicircular law [2]:

$$W_{2,2}(x) = \frac{\sqrt{x}}{\pi} {}_1F_0\left(\left[-\frac{1}{2}\right],[\,];\frac{x}{4}\right) = \frac{1}{2\pi}\sqrt{x(4-x)}. \tag{25}$$

The above semicircle is centered at $x = 2$, while the Wigner semicircle centered at $x = 0$ is used in random matrix theory to describe the asymptotic level density of random Hermitian matrices from Gaussian ensembles [2,3]. The Raney distributions $W_{2,r}(x)$ are plotted in Fig. 3 for $r = 1$ and 2. For comparison we have also plotted the function $W_{2,3}(x)$ furnished by Eq. (22). According to the results of Młotkowski [20] and our Appendix B this function is not positive in its domain, so it does not represent a probability distribution. Here we obtain it explicitly.

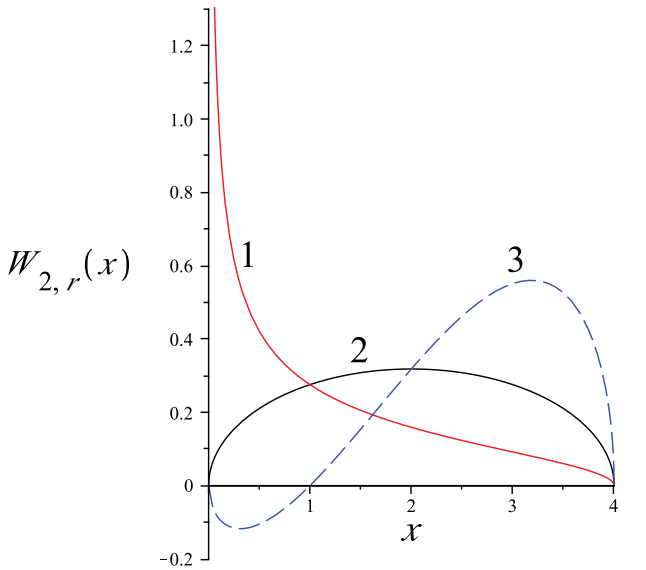

FIG. 3. (Color online) Raney distributions $W_{2,r}(x)$ with values of the parameter r labeling each curve. For $r = 1$ it reduces to the Marchenko-Pastur distribution $P_1(x)$, while a semicircle law is obtained for $r = 2$. For $r = 3$ the function represented by dashed line is not positive, which is implied by $p < r$.

It is easy to observe that the above semicircle distribution is related with the Marchenko-Pastur distributions $P_1(x)$ by a relation $W_{2,2}(x) = xP_1(x)$. This is a special case of a more general relation involving the "diagonal" Raney distributions with $r = p$ and Fuss-Catalan distribution,

$$W_{p,p}(x) = xW_{p,1}(x) = xP_{p-1}(x). \tag{26}$$

This result, established first in Ref. [20], follows also naturally from Eqs. (21) and (22). Thus for $p = 3$ one has

$$W_{3,1}(x) = P_2(x) \quad \text{and} \quad W_{3,3}(x) = xP_2(x), \tag{27}$$

where the Fuss-Catalan distribution $P_2(x)$ is given in (14). Due to an equivalent expression (15) the distribution $W_{3,3}(x)$

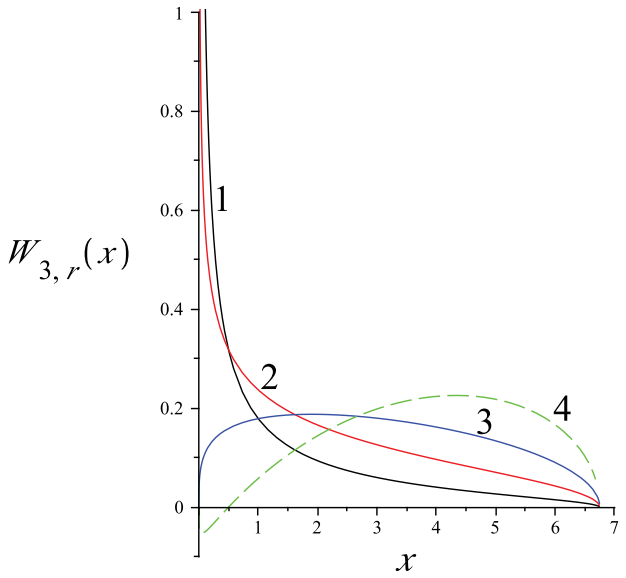

FIG. 4. (Color online) As in Fig. 3 for Raney distributions $W_{3,r}(x)$ supported in $[0,6\frac{3}{4}]$. The case $W_{3,4}(x)$ (dashed line) is not a probability measure. Each curve is labeled by the value of r.

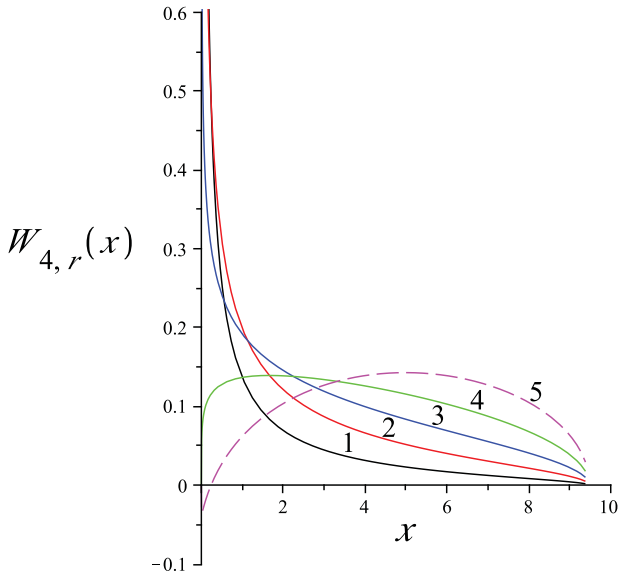

FIG. 5. (Color online) As in Fig. 3 for Raney distributions $W_{4,r}(x)$. Dashed line corresponds to a quasimeasure $W_{4,5}(x)$, which is not positive.

can also be expressed in terms of elementary functions. The intermediate Raney distribution, corresponding to $r = 2$,

$$W_{3,2}(x) = \frac{\sqrt{3}}{2\pi x^{1/3}}\,{}_2F_1\left(\left[-\frac{1}{3},\frac{1}{6}\right],\left[\frac{1}{3}\right];\frac{4}{27}x\right) - \frac{\sqrt{3}x^{1/3}}{18\pi}\,{}_2F_1\left(\left[\frac{1}{3},\frac{5}{6}\right],\left[\frac{5}{3}\right];\frac{4}{27}x\right), \quad (28)$$

is shown in Fig. 4. In a close analogy to Eq. (15) this distribution enjoys a similar representation in terms of elementary functions,

$$W_{3,2}(x) = \frac{\sqrt{3}\sqrt[3]{2}}{36\pi}\,\frac{[(27+3\sqrt{81-12x})^{\frac{4}{3}} - 18\sqrt[3]{2}x^{\frac{2}{3}}]}{x^{\frac{1}{3}}(27+3\sqrt{81-12x})^{\frac{2}{3}}}. \quad (29)$$

Observe that the distribution $W_{3,1}(x)$, $W_{3,2}(x)$, and $W_{3,3}(x)$ behave for small x as $x^{-2/3}$, $x^{-1/3}$, and $x^{1/3}$, respectively.

Let us now discuss the case $p = 4$ illustrated in Fig. 5. Due to relations (26) one has

$$W_{4,1}(x) = P_3(x) \quad \text{and} \quad W_{4,4}(x) = x\,P_3(x), \quad (30)$$

where the Fuss-Catalan distribution $P_3(x)$ is given in (16). Two intermediate Raney distributions have a similar form:

$$W_{4,2}(x) = \frac{1}{\pi x^{1/2}}\,{}_3F_2\left(\left[-\frac{1}{6},\frac{1}{6},\frac{1}{2}\right],\left[\frac{1}{4},\frac{3}{4}\right];\frac{27}{256}x\right) - \frac{\sqrt{2}}{4\pi x^{1/4}}\,{}_3F_2\left(\left[\frac{1}{12},\frac{5}{12},\frac{3}{4}\right],\left[\frac{1}{2},\frac{5}{4}\right];\frac{27}{256}x\right) - \frac{\sqrt{2}x^{1/4}}{128\pi}\,{}_3F_2\left(\left[\frac{7}{12},\frac{11}{12},\frac{5}{4}\right],\left[\frac{3}{2},\frac{7}{4}\right];\frac{27}{256}x\right) \quad (31)$$

and

$$W_{4,3}(x) = \frac{1}{\sqrt{2}\pi x^{1/4}}\,{}_3F_2\left(\left[-\frac{1}{4},\frac{1}{12},\frac{5}{12}\right],\left[\frac{1}{4},\frac{1}{2}\right];\frac{27}{256}x\right) - \frac{3\sqrt{2}x^{1/4}}{64\pi}\,{}_3F_2\left(\left[\frac{1}{4},\frac{7}{12},\frac{11}{12}\right],\left[\frac{3}{4},\frac{3}{2}\right];\frac{27}{256}x\right) - \frac{x^{1/2}}{32\pi}\,{}_3F_2\left(\left[\frac{1}{2},\frac{5}{6},\frac{7}{6}\right],\left[\frac{5}{4},\frac{7}{12}\right];\frac{27}{256}x\right). \quad (32)$$

The general formula (22) allows us to obtain an explicit form of the function $W_{4,5}(x)$, which is not positive, and it does not

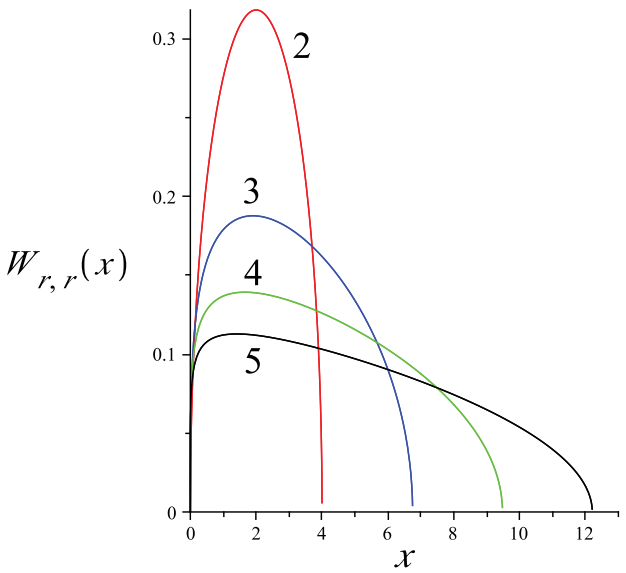

FIG. 6. (Color online) "Diagonal" Raney distributions $W_{r,r}(x)$ form a generalization of the semicircle law $W_{2,2}(x)$. For small x they behave as $x^{1/r}$. Value of r labels each curve.

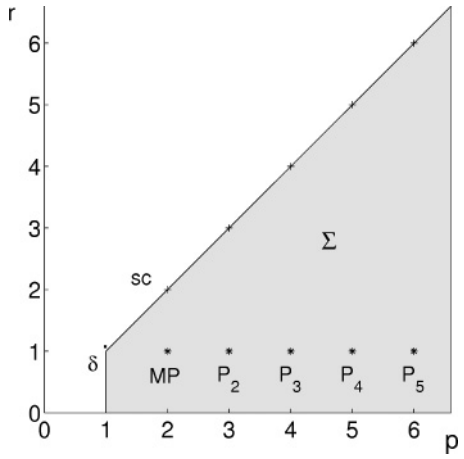

FIG. 7. Raney distributions in the parameter space (p,r): Marchenko-Pastur distribution P_1 is represented by the point $(2,1)$, while Fuss-Catalan distribution P_s is represented by $(*)$ at $(s+1,1)$. The semicircle (sc) law corresponds to $(2,2)$ and the "diagonal" Raney distributions $(+)$ to (p,p). As shown in Ref. [20] any point in the shaded set Σ corresponds to a probability measure with a compact support in $[0,\infty)$.

represent a probability distribution; see the dashed curve in Fig. 5.

Figure 6 presents the Raney distributions $W_{p,r}(x)$ in the "diagonal" case, $r = p$. As the distribution $W_{2,2}(x)$ represents Eq. (25), the diagonal Raney distributions, $W_{r,r}(x)$, can thus by considered as a generalization of the semicircular law: they are defined for $x \in (0, K_{r-1})$ where $K_s = (s+1)^{s+1}/s^s$; they are equal to zero at both ends of the domain, and they are characterized by a single maximum. However, of $r > 2$ the functions $W_{r,r}(x)$ are not symmetric anymore; see Fig. 6. A plot of the parameter space (p,r) in which these distributions are marked together with the Fuss-Catalan distributions is presented in Fig. 7.

IV. CONCLUDING REMARKS

In this work we obtained an explicit form of the Fuss-Catalan distribution $P_s(x)$. The obtained result is exact for an arbitrary s, and it allows for a simple use of these probability distributions. Results derived are relevant from the point of view of statistical physics as they describe asymptotic level density of a normalized positive random matrix of the product form $X = (G_1 \cdots G_s)(G_1 \cdots G_s)^\dagger$, where $G_1, \ldots, G_s$ denote s independent random matrices from the complex Ginibre ensemble. The variable $x = N\lambda$ denotes the rescaled eigenvalue λ of random matrix X of size N.

It should be emphasized here that the Marchenko-Pastur and Fuss-Catalan distributions describe the level density of Wishart-like random matrices in the limiting case $N \to \infty$ only. In practice, for any fixed N the finite size effects occur. As discussed by Blaizot and Nowak [44,45] the finite N effects are related with the diffraction phenomena, while the large N limit of the random matrix theory may be compared with the geometric limit of wave optics or the semiclassical limit of quantum theory. An explicit description of finite N corrections to the spectral density of Wishart matrices obtained from products of Ginibre matrices is provided by Burda *et al.* [32,46].

A simple argument put forward in Ref. [18] (see Appendix B therein) implies that the same FC distributions describe also the spectral density of a product of s random matrices taken from the *real Ginibre* ensemble [47]. The case of a product of two real matrices, recently studied in context of quantum chromodynamics [48], is also important in applications in econophysics, where one uses a product of two real correlation matrices [13].

The Catalan numbers are a special case of a more general, one-parameter family of Fuss-Catalan numbers, which from a subset of two-parameter family of Raney numbers. In the same way, the Marchenko-Pastur distribution $P_1(x)$ is a special case of the Fuss-Catalan distributions, which in turn belong to the two-parameter family of Raney distributions $W_{p,r}(x)$. This wide class of probability distribution includes, e.g., the Dirac delta, $\delta(x-1)$ and the semicircle law, $W_{2,2}(x)$. Applying the inverse Mellin transform for integer parameter values of the parameters we found an explicit exact representation of the Raney distributions in terms of the hypergeometric functions. For any $r = 1,2,\ldots,p$ the Raney distribution $W_{p,r}(x)$ is supported in the interval $[0, p^p/(p-1)^{p-1}]$. For small x the distribution behaves as

$$W_{p,r}(x) \sim \begin{cases} x^{-\frac{p-r}{p}} & \text{if } r < p, \\ x^{\frac{1}{p}} & \text{if } r = p. \end{cases} \tag{33}$$

The Raney numbers (17) imply that the mean value of the Raney distribution $W_{p,r}(x)$ is equal to r, while the second moment reads $r(2p+r-1)/2$.

Let us conclude the paper with the following remark. The Fuss-Catalan distribution describes statistical properties of singular values of products of random matrices of the Ginibre ensemble. It is then natural to ask, whether there exist any ensembles of random matrices, such that their squared singular values can be described by the Raney distributions.

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APPENDIX A: INFORMATION ON SPECIAL FUNCTIONS

Special functions of interest for this note are related to the Euler gamma function, which admits the following integral representation: $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$. Integrating by parts we see that $\Gamma(z+1) = z\Gamma(z)$. For an integer argument the Euler function is given by factorial, $\Gamma(n+1) = n!$. The

Gauss-Legendre formula allows one to compute the Euler gamma function of a multiple of argument [42],

$$\Gamma(kz) = (2\pi)^{(1-k)/2} k^{kz-1/2} \prod_{j=0}^{k-1} \Gamma\left(z + \frac{j}{k}\right), \tag{A1}$$

for $k = 1,2,3,\ldots$ and $z \neq 0, -1, -2, \ldots$.

The generalized hypergeometric series is a series in which the ratio of successive coefficients indexed by n is a rational function of n. It can be defined as

$${}_pF_q\left(\left[\{a_j\}_{j=1}^p\right],\left[\{b_j\}_{j=1}^q\right];z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{A2}$$

where we use the *Pochhammer symbol*, defined by $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$.

The series (A2), if convergent, defines a *generalized hypergeometric function*, which may then be defined over a wider domain of the argument by analytic continuation.

Note that for $s = 2$ expression (11) involves the Gauss (ordinary) hypergeometric function

$${}_2F_1([a,b],[c];z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{A3}$$

which includes many other special functions as special or limiting cases.

APPENDIX B: POSITIVITY OF THE DISTRIBUTIONS $P_s(x)$ AND $W_{p,r}(x)$

For completeness we prove in this appendix that for any integer s the distribution $P_s(x)$ is positive for $x \in (0, K_s)$, where $K_s = (s+1)^{s+1}/s^s$. Equation (6) implies that $P_s(x)$ is given as the inverse Mellin transform of Eq. (8). We are going to use the convolution property for two Mellin transforms, $\mathcal{M}[f(x);\sigma] = f^*(\sigma)$ and $\mathcal{M}[g(x);\sigma] = g^*(\sigma)$, which reads [36,37]

$$\begin{aligned}\mathcal{M}^{-1}[f^*(\sigma)g^*(\sigma);x] &= \int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{dt}{t} \\ &= \int_0^\infty g\left(\frac{x}{t}\right) f(t) \frac{dt}{t}.\end{aligned} \tag{B1}$$

If $x > 0$ and both functions $f(x)$ and $g(x)$ are positive, then their Mellin convolution defined by the integrals (B1) conserves positivity.

Consider now, for a given $j = 0,1,\ldots,(s-1)$, the individual term in the product in Eq. (8). Its Mellin transform will satisfy [due to formula (8.4.2.3), p. 631 in Ref. [40]] the following equality:

$$\mathcal{M}^{-1}\left[\frac{\Gamma\left(\sigma + \frac{j-s}{s+1}\right)}{\Gamma\left(\sigma + \frac{2+j-s}{s}\right)}; x\right] = \frac{x^{j-s}(1-x)^{\frac{1+s(1+j-s)}{s+1}}}{\Gamma\left[\frac{2(s+1)+js-s^2}{s+1}\right]}, \tag{B2}$$

which for all $j = 0,1,\ldots,(s-1)$, is a positive function for $x \in (0,1)$. Then Eq. (8) can be viewed as a $(s-1)$-fold convolution of positive functions, which by (B1) is itself positive for $x \in (0, K_s)$. The upper edge K_s of the support can be read off from the prefactor in Eq. (8).

In a similar way one can prove positivity of Raney distribution $W_{p,r}(x)$ for natural values of the parameters, provided $r \leqslant p$. Here is the streamlined version of the proof. We use the aforementioned formula 8.4.2.3 of Ref. [40] for $b > a$,

$$\mathcal{M}^{-1}\left[\frac{\Gamma(\sigma+a)}{\Gamma(\sigma+b)}; x\right] = [\Gamma(b-a)]^{-1} x^a (1-x)^{b-a-1}, \tag{B3}$$

which describes a positive function for $0 < x < 1$. We quote now the full version of the analog of Eq. (8) for $R_{p,r}(\sigma)$, with $\sigma = n+1$ and $p = 2,3,\ldots$:

$$\begin{aligned}R_{p,r}(\sigma) = \frac{r}{\sqrt{2\pi}} \frac{p^{r-p-1/2}}{(p-1)^{r-p+3/2}} \left[\frac{p^p}{(p-1)^{p-1}}\right]^\sigma \\ \times \frac{\Gamma\left(\sigma + \frac{r-p}{p}\right)}{\Gamma(\sigma)} \prod_{j=1}^{p-1} \left[\frac{\Gamma\left(\sigma + \frac{r-p+j}{p}\right)}{\Gamma\left(\sigma + \frac{r-p+j+1}{p-1}\right)}\right].\end{aligned} \tag{B4}$$

Consider first the case $1 \leqslant r < p$. The weight function $W_{p,r}(x)$ is the inverse Mellin transform, $W_{p,r}(x) = \mathcal{M}^{-1}\left[R_{p,r}(\sigma); x\right]$, and, from (B4) via Eq. (B1), it is the p-fold Mellin convolution of $\Gamma[\sigma + (r-p)/p]/\Gamma(\sigma)$ and of $p-1$ factors in the product in Eq. (B4). For each of these individual ratios of gamma function relation (B3) holds. Therefore, with Eq. (B4), the positivity of the distribution $W_{p,r}(x)$ follows.

In the second case, $r = p$, only $p-1$ factors in Eq. (B4) intervene in the convolution. Hence $W_{p,p}(x)$ is also positive. If $r > p$ the first ratio in Eq. (B4) destroys the positivity, so no further considerations are needed to show that in this case the function $W_{p,r}(x)$ is not a probability distribution.

APPENDIX C: RELATIONS BETWEEN RANEY DISTRIBUTIONS $W_{p,r}(x)$

The Raney distribution $W_{p,r}(x)$ may be implicitly written by its S transform, which allowed Młotkowski [20] to establish a relation between distributions with various values of their parameters in terms of the free multiplicative convolution denoted by $\boxtimes$ (compare Eq. 4.10 in Ref. [20]):

$$W_{1+p,1} \boxtimes W_{1+q,1} = W_{1+p+q,1}. \tag{C1}$$

This result, valid for $p,q > 0$, can be generalized for an arbitrary positive r:

$$W_{p,r} \boxtimes W_{1+q,1} = W_{p+rq,r}. \tag{C2}$$

Moreover, there exists yet another relation:

$$[W_{1+p,1}]^{\boxtimes s} = W_{1+sp,1}, \tag{C3}$$

which holds for $s > 0$ and is equivalent to the free multiplicative convolution property for the Fuss-Catalan distribution.

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