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Effective quantum Brownian dynamics in presence of a rapidly oscillating space-dependent time-periodic field

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We explore the Brownian dynamics in the quantum regime (by investigating the quantum Langevin and Smoluchowski equations) in terms of an effective time-independent Hamiltonian in the presence of a rapidly oscillating field. We achieve this by systematically expanding the time-dependent system-reservoir Hamiltonian in the inverse of driving frequency with a systematic time-scale separation and invoking a quantum gauge transformation within the framework of Floquet theorem.

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External time periodic driving of a Hamiltonian system generates explicit time dependence of the parameters involved in the dynamics. Applications range from the Paul trap [1] to controlling particle bunching and dilution in particle accelerators [2]. The properties of a classical parametric oscillator have been recently investigated experimentally in optically trapped water droplets [3]. When the parameter in question oscillates, certain simplifications occur, as first demonstrated by Kapitza [4]. To analyze Kapitza's pendulum [4], namely, a pendulum where the point of suspension is moved periodically, one basically considers the dynamics of a classical particle moving in one dimension under the influence of a force which is time periodic. Typically, the solutions for time-dependent problems can be attained numerically. However, when the period of the force is small compared to the other time scales of the problem, it is possible to separate the classical motion of the particle into "slow" and "fast" parts. This simplification is due to the fact that the particle does not have the time to react to the periodic force before this force changes its sign, namely, the contribution of the periodic force to the acceleration in one period is negligible. The leading order (with respect to $1/\omega$) of the dynamics was first computed by Kapitza [4]. Kapitza's treatment of the pendulum was generalized by Landau and Lifshitz [5] for any forced bare classical system. The treatment of Landau and Lifshitz was extended to the order of ω^{-4} (where ω is the driving frequency) to demonstrate that for rapidly driven Hamiltonian systems, it is possible to obtain a time-independent Hamiltonian that controls the slow motion [6]. Later, Rahav *et al.* [7] introduced friction phenomenologically to the classical equation of motion and showed that the motion of the slow part can be described by a time-independent equation that is derived as an expansion of the order of ω^{-1}. In the present Rapid Communication, we would like to address this problem for a quantum dissipative system.

We consider a system-reservoir (SR) Hamiltonian where the reservoir is modeled as a set of harmonic oscillators with characteristic frequencies ω_j and masses m_j and the system is acted upon by a periodic field:

$$\begin{aligned}\hat{H} &= \hat{H}_S(\hat{x},\hat{p}) + \hat{H}_B(\{\hat{q}_j\},\{\hat{p}_j\}) + \hat{H}_{SB}(\hat{x},\{\hat{q}_j\})\\ &= \frac{\hat{p}^2}{2m} + \hat{V}_0(\hat{x}) + \hat{V}_1(\hat{x},\omega t)\\ &\quad + \sum_{j=1}^{N}\left\{\frac{\hat{p}_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2\left(\hat{q}_j - \frac{c_j\hat{x}}{m_j\omega_j^2}\right)^2\right\}, \end{aligned} \tag{1}$$

where $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators of the system and $\{\hat{q}_j,\hat{p}_j\}$ are the set of coordinate and momentum operators for the bath oscillators. $\hat{V}_1$ is a periodic function of ωt and its average over a period vanishes. In writing Eq. (1), we have adopted the Zwanzig model [8] of the SR Hamiltonian. The inclusion of coupling of the system with the reservoir results in the irreversibility in the dynamics of the system by introducing damping. The equation of motion for the system that one obtains from the above Hamiltonian Eq. (1) is a quantum Langevin equation [9]:

$$m\ddot{\hat{x}} + \int_0^t dt'\,\gamma(t-t')\dot{\hat{x}}(t') + \frac{\partial\hat{V}(\hat{x},\omega t)}{\partial\hat{x}} = \hat{\xi}(t). \tag{2}$$

Here, $\hat{V} = \hat{V}_0 + \hat{V}_1$ and the damping kernel is given by $\gamma(t-t') = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega[J(\omega)/\omega]\cos\omega(t-t')$, where $J(\omega)$ is the spectral density of the bath. In the sequel, we focus on the Ohmic regime where $J(\omega) = m\gamma\omega$ and the parameter γ denotes the friction coefficient. The fluctuating force operator $\hat{\xi}(t)$ is a zero-centered Gaussian random force and obeys the fluctuation-dissipation relation (FDR)

$$\langle\hat{\xi}(t)\hat{\xi}(t') + \hat{\xi}(t')\hat{\xi}(t)\rangle = \hbar\int_{-\infty}^{+\infty}\frac{d\omega}{\pi}J(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \cos\omega(t-t'), \tag{3}$$

where the average is taken over the initial bath degrees of freedom. Equation (3) is exact and is derived under the assumption that the initial density operator of the system plus the reservoir factorizes. In what follows, we consider that the external force $\hat{F}(\hat{x},\omega t) = -\hat{V}_1'(\hat{x},\omega t)$ varies in time with "high" frequency ω. By high frequency, we mean one such that $\omega \gg \frac{1}{T}$, where T is the order of magnitude of the period of motion which the system would execute in the field

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$\hat{V}_0$ alone. The magnitude of $\hat{F}$ is not assumed to be small in comparison with the forces due to the field $\hat{V}_0$, but we shall assume that the oscillation of the particle as a result of this force is small.

As the Hamiltonian Eq. (1) is periodic in time, $\hat{H}(t+T) = \hat{H}(t)$, one may resort to the Floquet theorem [10] to study the dynamics associated with such a Hamiltonian. The time translation symmetry of the Hamiltonian implies that the solutions of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{4}$$

are linear combinations of Floquet states $\mathbb{U}_\epsilon(x,\omega t)$ of the form $\psi_\epsilon = \exp(-\frac{i}{\hbar}\epsilon t)\mathbb{U}_\epsilon(x,\omega t)$ where ϵ is the Floquet or quasienergy and $\mathbb{U}_\epsilon(x,\omega(t+T)) = \mathbb{U}_\epsilon(x,\omega t)$, with $\omega = \frac{2\pi}{T}$. These states have a natural separation into a slow part $\exp(-i\epsilon t/\hbar)$ and a fast part $\mathbb{U}_\epsilon(x,\omega t)$ that depends only on the fast time $\tau = \omega t$. To have the equation of motion of the slow part, following Ref. [11], we look for a unitary transformation $e^{i\hat{T}(t)}$, where $\hat{T}(t)$ is Hermitian in nature and $\hat{T}(t+T) = \hat{T}(t)$ such that the transformed Hamiltonian does not have explicit time dependence. In terms of the transformed wave function $\chi = e^{i\hat{T}(t)}\psi$, Eq. (4) reads as $i\hbar\frac{\partial\chi}{\partial t} = \hat{H}_{\text{eff}}\chi$, where the time-independent effective Hamiltonian is

$$\hat{H}_{\text{eff}} = e^{i\hat{T}(t)}\hat{H}e^{-i\hat{T}(t)} + i\hbar\left(\frac{\partial e^{i\hat{T}}}{\partial t}\right)e^{-i\hat{T}}, \tag{5}$$

or, in terms of fast time τ,

$$\hat{H}_{\text{eff}} = e^{i\hat{T}}\hat{H}e^{-i\hat{T}} + i\hbar\omega\left(\frac{\partial e^{i\hat{T}}}{\partial \tau}\right)e^{-i\hat{T}}. \tag{6}$$

At high frequency, $\hat{T}$ is assumed to be small, of the order of ω^{-1}. We shall justify this assumption explicitly in the following calculation. We expand $\hat{H}_{\text{eff}}$ and $\hat{T}$ in powers of $\frac{1}{\omega}$: $\hat{H}_{\text{eff}} = \sum_{n=0}^{\infty}\frac{1}{\omega^n}\hat{H}_n^e$ and $\hat{T} = \sum_{n=1}^{\infty}\frac{1}{\omega^n}\hat{T}_n$; and choose $\hat{T}$ such that $\hat{H}_{\text{eff}}$ is time independent in any order. To do so, we compute $\hat{H}_j^e$ in terms of $\hat{T}_1,\hat{T}_2,\ldots,\hat{T}_{j+1}$ and then choose $\hat{T}_{j+1}$, so that $\hat{H}_j^e$ is time independent. In what follows, to calculate the terms in Eq. (5), we use the operator expressions

$$e^{i\hat{T}}\hat{H}e^{-i\hat{T}} = \hat{H} + i[\hat{T},\hat{H}] - \frac{1}{2!}[\hat{T},[\hat{T},\hat{H}]] - \frac{1}{3!}[\hat{T},[\hat{T},[\hat{T},\hat{H}]]] + \cdots \tag{7}$$

and

$$\left(\frac{\partial e^{i\hat{T}}}{\partial \tau}\right)e^{-i\hat{T}} = i\frac{\partial\hat{T}}{\partial\tau} - \frac{1}{2!}\left[\hat{T},\frac{\partial\hat{T}}{\partial\tau}\right] - \frac{1}{3!}\left[\hat{T},\left[\hat{T},\frac{\partial\hat{T}}{\partial\tau}\right]\right] + \cdots. \tag{8}$$

In the leading order $O(\omega^0)$, $\hat{H}_0^e$ is given by

$$\hat{H}_0^e = \frac{\hat{p}^2}{2m} + \hat{V}_0(\hat{x}) + \hat{H}_B + \hat{H}_{SB} + \hat{V}_1(\hat{x},\tau) - \hbar\frac{\partial\hat{T}_1}{\partial\tau}. \tag{9}$$

The terms $\hat{V}_0$, $\hat{V}_1$, $\hat{H}_B$, and $\hat{H}_{SB}$ do not depend on $\hat{p}$. To cancel the time dependence, we choose

$$\hat{T}_1 = \frac{1}{\hbar}\int^{\tau} d\tau'\,\hat{V}_1(\hat{x},\tau'), \tag{10}$$

where the constant of integration has been chosen to be zero to avoid the secular terms. Substituting Eq. (10) into Eq. (9), we have $\hat{H}_0^e = \frac{\hat{p}^2}{2m} + \hat{V}_0(\hat{x}) + \hat{H}_B + \hat{H}_{SB}$. This is the leading order of the effective Hamiltonian. At the order ω^{-1}, from Eq. (6), we have $\hat{H}_1^e = i[\hat{T}_1,\hat{H}] - \hbar\frac{\partial\hat{T}_2}{\partial\tau} - \frac{i\hbar}{2}[\hat{T}_1,\frac{\partial\hat{T}_1}{\partial\tau}]$. We note that $\hat{T}_1$ as given by Eq. (10) depends only on the coordinate $\hat{x}$ and hence it commutes with its time derivative and with $\hat{V}_0$, $\hat{H}_B$, and $\hat{H}_{SB}$. Thus, $\hat{H}_1^e = i[\hat{T}_1,\frac{\hat{p}^2}{2m}] - \hbar\frac{\partial\hat{T}_2}{\partial\tau}$. A periodic $\hat{T}_2$ can always be chosen so that $\frac{\partial\hat{T}_2}{\partial\tau} = \frac{i}{\hbar}[\hat{T}_1,\frac{\hat{p}^2}{2m}]$ and consequently $\hat{H}_1^e$ vanishes. Using Eq. (10), one may easily see that $\hat{T}_2$ can be chosen as

$$\hat{T}_2 = \frac{i}{2m}\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}_1''(\hat{x},\tau) + \frac{i}{m}\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}_1'(\hat{x},\tau)\frac{\partial}{\partial x}, \tag{11}$$

where we have used the coordinate representation of the operator $\hat{p}$. Clearly, this choice makes $\hat{H}_1^e = 0$. At the next order ω^{-2}, $\hat{H}_2^e$ can be calculated from

$$\hat{H}_2^e = i[\hat{T}_2,\hat{H}] - \frac{1}{2}[\hat{T}_1,[\hat{T}_1,\hat{H}]] - \hbar\frac{\partial\hat{T}_3}{\partial\tau} - \frac{i\hbar}{2}\left[\hat{T}_1,\frac{\partial\hat{T}_2}{\partial\tau}\right] - \frac{i\hbar}{2}\left[\hat{T}_2,\frac{\partial\hat{T}_1}{\partial\tau}\right] + \frac{\hbar}{6}\left[\hat{T}_1,\left[\hat{T}_1,\frac{\partial\hat{T}_1}{\partial\tau}\right]\right]. \tag{12}$$

Using $\hat{H} = \hat{H}_0^e + \hbar\frac{\partial\hat{T}_1}{\partial\tau}$ and $\frac{\partial\hat{T}_2}{\partial\tau} = [\hat{T}_1,\frac{\hat{p}^2}{2m}] = [\hat{T}_1,\hat{H}]$, we get

$$\hat{H}_2^e = i\left[\hat{T}_2,\hat{H}_0^e\right] - \hbar\frac{\partial\hat{T}_3}{\partial\tau} + \frac{i\hbar}{2}\left[\hat{T}_2,\frac{\partial\hat{T}_1}{\partial\tau}\right]. \tag{13}$$

We now choose a periodic $\hat{T}_3$ to balance the time dependence of $\hat{H}_2^e$. In doing so, we observe that $\hat{H}_2^e$ has some time-independent part $\frac{i\hbar}{2}[\hat{T}_2,\frac{\partial\hat{T}_1}{\partial\tau}]$. To make $\hat{T}_2$ periodic, $\hat{T}_3$ is chosen so that (the overbar denotes the time average over one period)

$$\frac{\partial\hat{T}_3}{\partial\tau} = \frac{i}{\hbar}\left[\hat{T}_2,\hat{H}_0^e\right] + \frac{i}{2}\left[\hat{T}_2,\frac{\partial\hat{T}_1}{\partial\tau}\right] - \frac{i}{2}\overline{\left[\hat{T}_2,\frac{\partial\hat{T}_1}{\partial\tau}\right]}. \tag{14}$$

Now, using Eqs. (10) and (11), $\hat{T}_3$ is found to be

$$\begin{aligned}\hat{T}_3 = &-\frac{\hbar}{m^2}\int^{\tau} d\tau\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}_1''(\hat{x},\tau)\frac{\partial^2}{\partial x^2} - \frac{\hbar}{m^2}\int^{\tau} d\tau\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}_1'''(\hat{x},\tau)\frac{\partial}{\partial x}\\ &- \frac{\hbar^2}{4m^2}\int^{\tau} d\tau\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}^{(4)}(\hat{x},\tau) - \frac{1}{m\hbar}\hat{V}_0'(\hat{x})\int^{\tau} d\tau\int^{\tau} d\tau\int^{\tau} d\tau\,\hat{V}_1'(\hat{x},\tau)\\ &+ \frac{1}{2m\hbar}\int^{\tau} d\tau\,\hat{Q}(\hat{x},\tau) + \hat{I}(\hat{x},\hat{p}),\end{aligned} \tag{15}$$

where

$$\hat{Q}(\hat{x},\tau) = im\hbar\left[\hat{T}_2,\frac{\partial \hat{T}_1}{d\tau}\right] - im\hbar\overline{\left[\hat{T}_2,\frac{\partial \hat{T}_1}{d\tau}\right]}$$
$$= \overline{\hat{V}_1'(\hat{x},\tau)\int^{\tau} d\tau \int^{\tau} d\tau \hat{V}_1'(\hat{x},\tau)}$$
$$- \hat{V}_1'(\hat{x},\tau)\int^{\tau} d\tau \int^{\tau} d\tau \hat{V}_1'(\hat{x},\tau), \qquad (16)$$

and $\hat{I}(\hat{x},\hat{p})$ is the constant of integration which is a Hermitian operator of $\hat{x}$ and $\hat{p}$ only. In the leading-order correction, one need not calculate $\hat{I}$. For higher-order correction, one requires knowledge of $\hat{I}$. Using Eq. (15) into Eq. (13), the time dependence of $\hat{H}_e^2$ disappears and consequently $\hat{H}_2^e$ is found to be

$$\hat{H}_e^2 = \frac{i\hbar}{2}\overline{\left[\hat{T}_2,\frac{\partial \hat{T}_1}{d\tau}\right]} = \frac{1}{2m}\overline{\left[\int^{\tau} d\tau \hat{V}_1'(\hat{x},\tau)\right]^2}. \qquad (17)$$

Thus, with the nonvanishing leading-order contribution, the time-independent effective Hamiltonian will be given by

$$\hat{H}_{\rm eff} = \frac{p^2}{2m} + \hat{V}_{\rm eff} + \hat{H}_B + \hat{H}_{SB}, \qquad (18)$$

where

$$\hat{V}_{\rm eff} = \hat{V}_0(\hat{x}) + \frac{1}{2m\omega^2}\overline{\left[\int^{\tau} d\tau \hat{V}_1'(\hat{x},\tau)\right]^2}. \qquad (19)$$

Equation (19) generates the equation of motion of the slow variable, and instead of Eq. (2), the Langevin equation for the slow variable reduces to

$$m\ddot{\hat{x}} + \int_0^t dt'\gamma(t-t')\dot{\hat{x}}(t') + \hat{V}_{\rm eff}' = \hat{\xi}(t). \qquad (20)$$

From the very mode of our development it is evident that the SR interaction appears only in the leading-order Hamiltonian $\hat{H}_0^e$ and possesses the same initial or original structure. Consequently, the FDR [Eq. (3)] remains valid. Thus the noise term does not enter into the effective Hamiltonian to the leading order. Here we want to mention the works reported in Ref. [12]. We consider the quantum Langevin equation (2) for a harmonic oscillator with $\hat{V}(\hat{x},\tau) = \frac{1}{2}m\omega^2(t)\hat{x}^2$. The solution of Eq. (2) can be written as $\hat{x}(t) = m\dot{\hat{x}}_0 G_1(t) + m\hat{x}_0 G_2(t) + \hat{X}(t)$, where $\hat{x}_0$ and $\dot{\hat{x}}_0$ are the initial position and velocity of the quantum system and $\hat{x}(t) = \int_0^t dt'\, G(t-t')\hat{\xi}(t')$. By introducing the two solutions $y_1(t)$ and $y_2(t)$ of the equation $\ddot{y}(t) + [\Omega^2(t) - \frac{\gamma}{4}]y(t) = 0$, the functions $G_1(t)$ and $G_2(t)$ are obtained as $G_i(t) = \frac{1}{m}\exp(-\gamma t/2)y_i(t), i = 1,2$. The Green's function $G(t,t')$ is then given by [13]

$$G(t,t') = \frac{1}{m}e^{\frac{\gamma}{2}(t-t')}[y_1(t)y_2(t') - y_1(t')y_2(t)]. \qquad (21)$$

The evolution equation of the Wigner quasiprobability distribution function $W(q,p,t)$ for a harmonic oscillator coupled to a harmonic bath is of the general form [14]

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial q} + m\tilde{\Omega}^2(t)q\frac{\partial W}{\partial p} + 2\Gamma(t)\frac{\partial}{\partial p}(pW)$$
$$+ D_{pp}(t)\frac{\partial^2 W}{\partial p^2} + D_{qp}(t)\frac{\partial^2 W}{\partial q \partial p}. \qquad (22)$$

In the development of Caldeira-Leggett [15], the term containing the mixed diffusion coefficient D_{qp} was absent as they had dealt with high-temperature quantum dynamics. Later, keeping in mind that quantum noise processes are non-Markovian, this term was incorporated [14]. For the expression of the time-dependent parameters $\tilde{\Omega}^2$, $\Gamma(t)$, $D_{pp}(t)$, and $D_{qp}(t)$ for a driven harmonic oscillator, the reader may consult Ref. [16]. The diffusion coefficients $D_{pp}(t)$ and $D_{qp}(t)$ are given by [16] $D_{pp}(t) = \frac{m}{2}\langle\dot{\hat{X}}(t)\hat{\xi}(t) + \hat{\xi}(t)\dot{\hat{X}}(t)\rangle$ and $D_{qp}(t) = \frac{1}{2}\langle\hat{X}(t)\hat{\xi}(t) + \hat{\xi}(t)\hat{X}(t)\rangle$. The above-mentioned diffusion coefficients can be expressed further in terms of the noise correlation function [13] $D_{pp}(t) = \frac{m}{2}\int_0^t dt'\frac{\partial G}{\partial t}\langle\hat{\xi}(t)\hat{\xi}(t') + \hat{\xi}(t')\hat{\xi}(t)\rangle$ and $D_{qp}(t) = \frac{1}{2}\int_0^t dt' G(t,t')\langle\hat{\xi}(t)\hat{\xi}(t') + \hat{\xi}(t')\hat{\xi}(t)\rangle$. A semiclassical approximation of the diffusion coefficients can be obtained by expanding the noise correlation Eq. (3) in powers of $\hbar$ and, in the Ohmic limit, we obtain

$$\langle\hat{\xi}(t)\hat{\xi}(t') + \hat{\xi}(t')\hat{\xi}(t)\rangle = 4\gamma m k_B T\delta(t-t')$$
$$-\frac{\hbar^2}{3}\frac{\gamma m}{k_B T}\delta''(t-t') + O(\hbar^3). \qquad (23)$$

Using Eq. (23), one can eventually arrive at the semiclassical diffusion coefficient up to second order in $\hbar\omega/k_B T$,

$$D_{pp}(t) = m\gamma k_B T + 2\gamma m^2 k_B T\Lambda[\Omega^2(t) - \gamma^2],$$
$$D_{qp}(t) = 2\gamma^2 m k_B T\Lambda, \qquad (24)$$

where $\Lambda = \frac{\hbar^2}{24m(k_B T)^2}$. At the high-temperature limit $D_{qp}(t)$ vanishes and $D_{pp}(t)$ reduces to the original equation of Caldeira-Leggett [15]. The above-mentioned Λ was first introduced by Ankerhold *et al.* [17] to obtain the quantum Smoluchowski equation (QSE) for arbitrary potentials. The full-blown expression for Λ is given by $\Lambda = (\frac{\hbar}{\pi m\gamma})\ln(\frac{\hbar\beta\gamma}{2\pi})$ for $\gamma\hbar\beta \gg 1$ (where $\beta = \frac{1}{k_B T}$), which reduces to $\frac{\hbar^2}{24m(k_B T)^2}$ in the high-temperature domain ($\gamma\hbar\beta \ll 1$) [18]. The mixed diffusion coefficient is independent of frequency of the system and is purely quantum mechanical in nature, i.e., $D_{qp} \longrightarrow 0$ as $\hbar \longrightarrow 0$, while the other diffusion coefficient D_{pp} has a

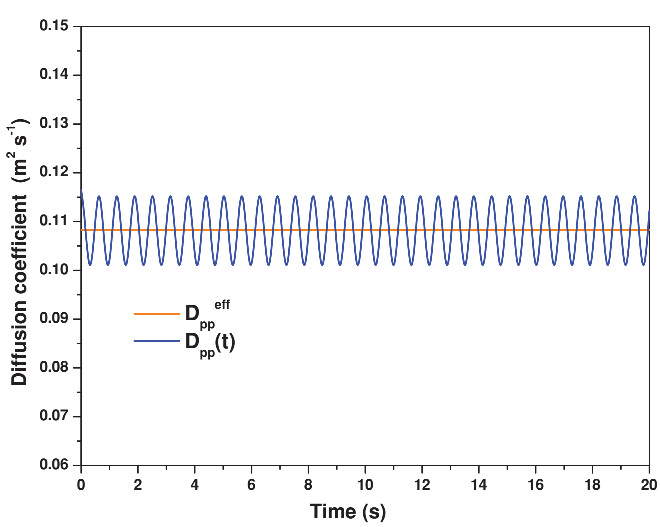

FIG. 1. (Color online) Plot of diffusion coefficient as a function of time with $k_B T = 1.0, \omega_0 = 1.0, \hbar = 1.0, \omega = 10.0$, and $\gamma = 0.1$.

time-independent classical part and a time-dependent quantum part.

If we now consider $\hat{V}_0 = \frac{1}{2}m\omega_0^2\hat{x}^2$ and $\hat{V}_1 = \frac{1}{2}m\hat{x}^2\cos\omega t$ in Eq. (1), we get $\Omega^2(t) = \omega_0^2 + \cos\omega t$. This is an important example of a driven quantum system known as the parametric oscillator with a time-dependent frequency $\Omega(t)$ [19]. It is an experimentally important case in view of the fact that it describes the Paul trap [1]. The effective dynamics of the slow variable, on the other hand, will be governed by $\hat{V}_{\rm eff}$ given by Eq. (19). With the chosen $\hat{V}_0$ and $\hat{V}_1$, one may easily get the effective potential as $\hat{V}_{\rm eff} = \frac{1}{2}m\omega_0^2\hat{x}^2 + \frac{1}{4m\omega^2}\hat{x}^2$. Thus, the time-independent effective frequency would be $\Omega_e^2 = \omega_0^2 + \frac{1}{2m^2\omega^2}$. With this effective frequency, one can calculate the time-independent diffusion coefficient as $D_{pp}^{\rm eff} = m\gamma k_B T + 2\gamma m^2 k_B T \Lambda[\Omega_e^2 - \gamma^2]$. In Fig. 1, we plot the diffusion coefficient as a function of t and observe that $D_{pp}(t)$ has small oscillations around $D_{pp}^{\rm eff}$, which bolsters our belief regarding the validity of our present formulation. Following Ankerhold *et al.* [17], we obtain the QSE in the presence of a highly oscillating periodic force as follows:

$$\frac{\partial P(q,t)}{\partial t} = \frac{1}{\gamma m}\frac{\partial}{\partial q}\left[V'_{\rm eff}(q) + \frac{1}{\beta}\frac{\partial}{\partial q}D_{pp}^{\rm eff}\right]P(q,t). \quad (25)$$

The exact expression of the QSE is a subject of growing interest [18] and a rigorous derivation for a driven quantum system is still lacking. Recently, in Ref. [13], the authors considered a driven quantum harmonic oscillator strongly coupled to a heat bath and obtained the QSE under the condition of high friction, high temperature, and moderate driving. On the other hand, Eq. (25) is applicable for rapid oscillations and being an equation with a time-independent effective potential and diffusion coefficient, it is much easier to handle analytically and numerically.

In summary, starting from a SR model where the system is driven by a rapidly oscillating space-dependent force, we derive the effective quantum Brownian motion in terms of an explicit time-independent effective potential and diffusion coefficient. Consequently, we derive the QSE for a driven harmonic oscillator. We would like to conclude by mentioning that the rapidly oscillating space-dependent fields where the spatial variation of the field is smooth but otherwise arbitrary have been applied successfully to cold atoms, where a very high degree of control is possible. The exploration of the dynamics of cold atoms in strong electromagnetic fields has resulted in many unique and interesting experimental observations [20]. We hope that our present development provides room for explaining such contemporary and futuristic experiments.

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