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Externally driven collisions of domain walls in bistable systems near criticality

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Multidomain solutions to the time-dependent Ginzburg-Landau equation in the presence of an external field are analyzed using the Hirota bilinearization method. Domain-wall collisions are studied in detail considering different regimes of the critical parameter. I show the dynamics of the Ising and Bloch domain walls of the Ginzburg-Landau equation in the bistable regime to be similar to that of the Landau-Lifshitz domain walls. Domain-wall reflections lead to the appearance of bubble and pattern structures. Above the Bloch-Ising transition point, spatial structures are determined by the collisions of fronts propagating into an unstable state. Mutual annihilation of such fronts is described.

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I. INTRODUCTION

Localized and labyrinth structures in excitable media are considered for possible use as carriers of binary information for specific applications [1,2]. Such structures appear in monostable and bistable regimes of magnetic, electrical, chemical, and optical systems. In particular, different realizations of memory and logic elements with bubble-forming bistable media seem to be technologically promising because of the high stability of the bubble structures. Prototype bubble-forming systems are two-dimensional (2D) ferromagnets including ferrofluids [3,4], and 2D ferroelectrics including electrically active liquid crystals [5,6]. In critical parameter ranges, these systems create pattern structures (lamellae, labyrinths) which are not periodic [3,7]. To be precise, I mention that experimental observation of bubbles and lamellar patterns in solid ferroelectric films has not been achieved (unlike in liquid crystals where electroelastic coupling is weaker) [8], although a 2D bubblelike structure has been seen in a system of a finite-size geometry [9]. Unlike patterns in oscillatory critical systems, the above structures are built of domain walls (DWs) connecting stable states of opposite orientation of an order parameter. Similar patterns and bubbles are observed in excitable (reaction-diffusion) chemical systems below the critical (bifurcation) point [10,11]. It is hoped that the difficulties of early magnetic and dipolar bubble technologies reviewed in [12,13] can, nowadays, be overcome thanks to the advance of miniaturization. One observes increased interest in studies of DW complexes with relevance to data storage and processing devices which are based on ferromagnetic and ferroelectric nanoelements. In particular, nanowire structures are under intense investigation because of the possibility of the most dense information packing [14,15]. The importance of critical effects in nanosystems grows with decrease in their diameters because of a significant decrease of the critical (Néel) temperature [16,17].

Generic critical properties of magnetic and electric media are described with the time-dependent Ginzburg-Landau equation which enables qualitative analysis of DW behaviors in all bistable systems. The DW solutions to the 1D Ginzburg-Landau equation are of Ising (named Néel DWs also) or Bloch type. Under external stimulation the DWs move and their collision properties determine the morphology of resulting pattern structures. In the present paper, external-field-induced DW collisions in 1D Ginzburg-Landau systems are studied with reference to bubble formation. I show similarities of the process to the bubble formation in ferromagnetic (or ferroelectric) wires far from the criticality (a system described with the Landau-Lifshitz-Gilbert equation), referring to my previous study of the problem [18] as I throughout the text. The creation of bubbles and DW patterns is described with connection to the property of elastic reflection of DWs observed in excitable media. However, I emphasize that the present study is not related to widely investigated pulses in monostable regimes of excitable systems [19,20].

In Secs. II and III, the field-induced collisions of DWs and phase fronts, respectively, are studied. In Sec. IV, the relevance of predictions on DW and front collisions to pattern formation is discussed.

II. DOMAIN-WALL COLLISIONS

Let us consider the 1D Ginzburg-Landau equation for the ferromagnetic (ferroelectric) wire in a (longitudinal) external field directed along the spontaneous magnetization (polarization),

$$\alpha\frac{\partial m}{\partial t} = J\frac{\partial^2 m}{\partial x^2} + \beta_1 m + \beta_2 m^* - \mu|m|^2 m + \gamma H. \quad (1)$$

Here m denotes a complex order parameter (a two-component magnetization or polarization), β_1 determines the distance from the criticality ($\beta_1 = -\beta_2$ at the phase-transition point), and H denotes an external (magnetic or electric) field intensity (it can take positive or negative real values). Based on arguments related to the time-reversal symmetry, I show that the DWs described with (1) have to reflect upon their collision into the parameter region where they form magnetic bubbles. Under the assumption $\gamma H \ll (\beta_1 + \beta_2)^{3/2}/\mu^{1/2}$, the domains of the stable phase of the system correspond to

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$m \approx \pm\sqrt{\beta_1+\beta_2}/\sqrt{\mu}$, and the single-DW solutions to (1) take the form

$$m = \sqrt{\frac{\beta_1+\beta_2}{\mu}}\frac{1-e^{2\eta}}{1+e^{2\eta}} + i\theta 2\sqrt{\frac{\beta_1-3\beta_2}{\mu}}\frac{e^{\eta}}{1+e^{2\eta}},$$
$$\eta = k(x-x_0) - \frac{\sqrt{\mu}\gamma Ht}{\sqrt{\beta_1+\beta_2}\alpha}, \quad (2)$$

where

$$|k| = \sqrt{\frac{2\beta_2}{J}} \quad \text{for} \quad \theta = \pm 1, \quad (3a)$$

and

$$|k| = \sqrt{\frac{\beta_1+\beta_2}{2J}} \quad \text{for} \quad \theta = 0. \quad (3b)$$

These equations describe the Bloch and the Ising DW, respectively. Only in the limiting case $H=0$, the above solution is strict while, for $H \neq 0$, it neglects the small field-induced deformation of the DW profile (an space asymmetry arising with respect to the DW center; details are given in Appendix A). The DWs move under the condition $H \neq 0$ in a direction dependent on the sign of k. It is seen from (2) and (3a) that Bloch DWs can exist only in the regime $\beta_1 > 3\beta_2$; therefore, at $\beta_1 \to 3\beta_2^+$, the Bloch DWs transform into Ising DWs which are the only known stationary front solutions in the regime $-\beta_2 < \beta_1 < 3\beta_2$ [21]. Stationary two-DW solutions to (1) are given in [22]. In the absence of an external field $H=0$, the double-wall solutions describe localized static bubbles built of pairs of different types (Ising and Bloch) of DWs. The size of the bubble can be changed by the application of a field, $H \neq 0$, because of inducing the motion of its fronts (DWs) and, eventually, their collisions. The collision result (DW reflection or annihilation) determines the final spatial structure created by the DWs, similar to those of 1D ferromagnets far from the critical regime (I). In the present work, I omit the discussion of the formation of bubbles built of two Bloch or two Ising DWs, mentioning the previous studies of Refs. [23]. Such objects are unstable for $H=0$ (initially stationary DWs repel or attract each other); thus, they are less important for potential applications than Ising-Bloch bubbles (although the stability of the Ising-Bloch bubbles against temporal external perturbations is a separate question discussed in [24]).

Obviously, Eq. (1) is not time-reversal invariant, while the reversibility of a dynamical system is a crucial property in terms of the stability of any localized structure (e.g., a bubble) [1,25]. Due to this irreversibility, solitary-wave solutions to (1) are, in the general case, relevant only to the limit of large positive values of time ($t \to \infty$), while they are irrelevant to the opposite limit ($t \to -\infty$), in particular, because of infinite growth of the energy with $t \to -\infty$ which is indicated by nonzero values of the dissipative function (I). With relevance to studying the DW collisions, and thus to investigating the stability of DW complexes, I analyze the dynamics of the bubble in the limit of large negative values of time (in this limit, the colliding DWs are noninteracting objects), modifying the evolution equation (1) by reversing the arrow of time. The inverse-evolution equation takes the form

$$-\alpha\frac{\partial\tilde{m}}{\partial t} = J\frac{\partial^2\tilde{m}}{\partial x^2} + \beta_1\tilde{m} + \beta_2\tilde{m}^* - \mu|\tilde{m}|^2\tilde{m} - \gamma H. \quad (4)$$

Let me mention an analogy to the necessity of doubling the number of degrees of freedom when formulating the dynamics of dissipative (classical or quantum) systems within formalisms relevant to the whole length of the time axis [26,27] (see also I), in particular, the dynamics of essentially dissipative (reaction-diffusion) systems whose excitations are overdamped [28].

Applying Hirota's bilinearization method, following the substitutions $m = g_1/f_1$, $\tilde{m} = -g_2/f_2$ in (1) and (4), where f_1, f_2 take real values, one arrives at the secondary equations of motion

$$\left(-\alpha D_t + J D_x^2\right) g_1 \cdot f_1 + (\beta_1 - \lambda) g_1 f_1 + \beta_2 g_1^* f_1 + \gamma H f_1^2 = 0,$$
$$J D_x^2 f_1 \cdot f_1 + \mu g_1 g_1^* - \lambda f_1^2 = 0, \quad (5a)$$
$$\left(\alpha D_t + J D_x^2\right) g_2 \cdot f_2 + (\beta_1 - \lambda) g_2 f_2 + \beta_2 g_2^* f_2 + \gamma H f_2^2 = 0,$$
$$J D_x^2 f_2 \cdot f_2 + \mu g_2 g_2^* - \lambda f_2^2 = 0. \quad (5b)$$

Here,

$$D_t^m D_x^n b(x,t) \cdot c(x,t) \equiv (\partial/\partial t - \partial/\partial t')^m (\partial/\partial x - \partial/\partial x')^n b(x,t) c(x',t')|_{x=x',t=t'}.$$

The above breaking of (1) and (4) into the pairs of secondary equations (5a) and (5b), respectively, is nonunique but it leads to equations of the lowest possible order in $g_{1(2)}$ and $f_{1(2)}$ (bilinear ones). Upon the inversion of the arrow of time ($t \to -t$), the secondary dynamical variables transform following $g_{1(2)} \to -g_{2(1)}$, $f_{1(2)} \to f_{2(1)}$ while (5a) transform into (5b) and vice versa.

For $\lambda = \beta_1 + \beta_2$ and $\gamma H \ll (\beta_1+\beta_2)^{3/2}/\mu^{1/2}$ (the field is much weaker than the coercivity value; thus, $m \approx \pm\sqrt{\beta_1+\beta_2}/\sqrt{\mu}$ inside the domains and a deformation of the DWs induced by H is negligible), following Appendix A, I find a two-DW (bubble) solution in the form

$$g_1 = a(1 - \nu e^{2\eta_1} - \nu e^{2\eta_2} + e^{2\eta_1+2\eta_2}) + ib\nu^{1/2}e^{\eta_1}(1-e^{2\eta_2}),$$
$$f_1 = 1 + \nu e^{2\eta_1} + \nu e^{2\eta_2} + e^{2\eta_1+2\eta_2}, \quad (6a)$$
$$\eta_j = k_j(x - x_{0j}) - \omega_j t,$$
$$g_2 = a(1 - \nu e^{2\tilde{\eta}_1} - \nu e^{2\tilde{\eta}_2} + e^{2\tilde{\eta}_1+2\tilde{\eta}_2}) + ib\nu^{1/2}e^{\tilde{\eta}_1}(1-e^{2\tilde{\eta}_2}),$$
$$f_2 = 1 + \nu e^{2\tilde{\eta}_1} + \nu e^{2\tilde{\eta}_2} + e^{2\tilde{\eta}_1+2\tilde{\eta}_2}, \quad (6b)$$
$$\tilde{\eta}_j = k_j(x - x_{0j}) + \omega_j t,$$

where

$$a = \sqrt{\frac{\beta_1+\beta_2}{\mu}}, \quad b = 2\sqrt{\frac{\beta_1-3\beta_2}{\mu}},$$
$$k_1 = \mp\sqrt{\frac{\beta_1+\beta_2}{2J}}, \quad k_2 = \pm\sqrt{\frac{2\beta_2}{J}}, \quad \omega_1 = \omega_2$$
$$= \frac{\sqrt{\mu}\gamma H}{\sqrt{\beta_1+\beta_2}\alpha}, \quad \nu = \frac{\beta_1+\beta_2-2\sqrt{\beta_2(\beta_1+\beta_2)}}{\beta_1+\beta_2+2\sqrt{\beta_2(\beta_1+\beta_2)}}. \quad (7)$$

In the limit $H \to 0$, $g_1 = g_2$, $f_1 = f_2$, and both the fields m and $-\tilde{m}$ coincide with the static bubble solution to (1) written in [22].

Studying the DW collision, I analyze the limit $t \to \infty$ of the double-DW solutions to (1), the formula (6a), and the

$t \to -\infty$ limit of the solution to (4), the formula (6b), in the closest vicinity of the ith DW center, noticing that

$$m \approx m^{(i)} = a\frac{1-e^{2\eta_i+\ln(\nu)}}{1+e^{2\eta_i+\ln(\nu)}} + i\delta_{i2}b\frac{e^{\eta_i+\ln(\nu)/2}}{1+e^{2\eta_i+\ln(\nu)}} \quad (8)$$

for $0 \approx \eta_i + \ln(\nu)/2 \gg \eta_k + \ln(\nu)/2$ and $i \neq k$, and

$$\tilde{m} \approx \tilde{m}^{(i)} = -a\frac{1-e^{2\tilde{\eta}_i+\ln(\nu)}}{1+e^{2\tilde{\eta}_i+\ln(\nu)}} - i\delta_{i2}b\frac{e^{\tilde{\eta}_i+\ln(\nu)/2}}{1+e^{2\tilde{\eta}_i+\ln(\nu)}} \quad (9)$$

for $0 \approx \tilde{\eta}_i + \ln(\nu)/2 \gg \tilde{\eta}_k + \ln(\nu)/2$ and $i \neq k$. Note that (8) coincides with the Ising-DW profile (2),(3b) for $i = 1$ and with the Bloch-DW profile (2),(3a) for $i = 2$. In order to establish the magnetization dynamics in the limit of large negative values of time, I invert the motion of $\tilde{m}^{(i)}$, utilizing the property $\tilde{m}^{(i)}(x,0) = -\tilde{m}^{(i)*}(-x + 2x'_{0i},0)$, where $x'_{0i} = x_{0i} - \ln(\nu)/(2k_i)$, and find

$$m \approx -a\frac{1-e^{2\eta_i+\ln(\nu)}}{1+e^{2\eta_i+\ln(\nu)}} + i\delta_{i2}b\frac{e^{\eta_i+\ln(\nu)/2}}{1+e^{2\eta_i+\ln(\nu)}} \quad (10)$$

for $0 \approx \eta_i + \ln(\nu)/2 \ll \eta_k + \ln(\nu)/2$ and $i \neq k$. Comparing (8) and (10), one sees that the result of the collision is a change of the sign of the real part of $m^{(i)}$. Thus, one can think of the colliding DWs that they pass through each other with constant velocity and change their character from the head-to-head into the tail-to-tail structure and vice versa. In fact, the result of the DW collision is their elastic reflection accompanied by change of the Bloch into a Néel DW and vice versa, similarly to the result of field-driven or spontaneous collision of magnetic DWs of the 1D Landau-Lifshitz equation (I). Let me emphasize that taking the $t \to -\infty$ limit of g_1/f_1 at $\eta_{1(2)} + \ln(\nu)/2 \approx 0$ leads to m differing from (10) by the sign of its imaginary part, which confirms the irrelevance of (1) in this limit.

Above, I analyzed an infinite 1D medium. Finite length of the domains leads to additional consequences of the DW collision because of the natural tendency of finite systems toward energy minimization (the energy cannot be defined for infinite systems). The preferred direction of the DW motion before the collision (the growth or decrease of the bubbles) corresponds to a decrease of the Zeeman energy while the collision-induced motion in the reversed direction finishes when the decrease of the interaction energy of the (repulsing) DWs equals the Zeeman-energy increase. Thus, I predict the appearance of localized bubbles due to an external-field application in the parameter range $\beta_1 > 3\beta_2$. Their diameter is determined by the strength of the driving field.

Approaching the Bloch-Ising transition point with β_1 results in a decrease of ν ($\beta_1 \to 3\beta_2^+ \Rightarrow \nu \to 0^+$). Then the imaginary parts of m,$\tilde{m}$ decrease and, eventually, vanish at $\beta_1 = 3\beta_2$. Simultaneously, the centers of both the DWs move away from each other due to the shift of the parameters (7) of (6a) and (6b) ($\nu \to \nu + \Delta\nu$, $k_j \to k_j + \Delta k_j$, $\Delta k_2 = 0$). The shifts of the DW centers are equal to $\Delta x_{0j} = \ln(\nu)/|2k_j| - \ln(\nu + \Delta\nu)/|2k_j + 2\Delta k_j|$. Finally, at $\beta_1 = 3\beta_2$ ($\Delta\nu = -\nu$, $k_j + \Delta k_j = \sqrt{2\beta_2/J}$) the centers of both the DWs diverge. A domain pattern is expected to appear due to the finite density of the mutually interacting bubbles. Further decrease of β_1 (down to $-\beta_2$) results in divergence of the widths of the (Ising) DWs and, thus, in the divergence of the length of the DW interaction. An adiabatic approach with $\beta_1 \to -\beta_2^+$ is impossible in practice due to the instant widening of the DW. Because of the rapid increase of the overlap of DWs, one anticipates the appearance of domains of an unstable phase in our bistable medium in the regime $-\beta_2 < \beta_1 < 3\beta_2$ (above the Bloch-Ising transition and below the bifurcation point). Then, an energy excess is being removed from the system via the propagation of fronts connecting domains of a stable phase with domains of an unstable phase (e.g., fronts between ferromagnetic and paramagnetic domains). They always propagate into an unstable state and below I call them phase fronts [29].

III. PHASE-FRONT COLLISIONS

We can gain insight into the pattern formation near criticality by considering the collision of fronts propagating into unstable states. Let us study front solutions to (1) assuming $\mathrm{Im}(m) = 0$ or, alternatively, to a simplified version of (1),

$$-\alpha\frac{\partial m}{\partial t} + J\frac{\partial^2 m}{\partial x^2} + \theta m - \mu m^3 + \gamma H = 0, \quad (11)$$

where $\theta = \beta_1 + \beta_2$, (the real solutions to both the equations are the same). Equation (11) is relevant to the closest vicinity of the critical point $\theta = 0$, where the order parameter takes real values. On the other hand, $\mathrm{Im}(m)$ vanishes when crossing the Bloch-Ising transition point $\beta_1 = 3\beta_2$; thus, real solutions are expected to qualitatively describe the phase-front motion for the parameter range $-\beta_2 < \beta_1 < 3\beta_2$. Since $m \to 0$ with $\theta \to 0$, I introduce a renormalized field $m' \equiv m/\sqrt{\theta}$ in order to determine dynamical properties of the phase fronts in the vicinity of the critical point. Substituting m' by g/f, one arrives at a trilinear (Hirota) form of (11):

$$\begin{aligned} &f\left[\left(-\alpha D_t + JD_x^2\right)g\cdot f + \gamma H/\sqrt{\theta}f^2\right] \\ &\quad + g\left[\left(-JD_x^2 + \theta\right)f\cdot f - \mu\theta g^2\right] = 0. \end{aligned} \quad (12)$$

For $H = 0$, an exact single-front solution

$$g = \frac{1}{\sqrt{\mu}}e^{k(x-x_{01})+ckt}, \quad f = 1 + e^{k(x-x_{01})+ckt}, \quad (13)$$

where

$$k = \sqrt{\frac{\theta}{2J}}, \quad ck = \frac{3\theta}{2\alpha}, \quad (14)$$

and a double-front solution to (12),

$$\begin{aligned} g &= \frac{1}{\sqrt{\mu}}(e^{k(x-x_{01})+ckt} - e^{-k(x-x_{02})+ckt}), \\ f &= 1 + e^{k(x-x_{01})+ckt} + e^{-k(x-x_{02})+ckt} \end{aligned} \quad (15)$$

were found by Nozaki and Bekki [30]. I mention that these authors have considered a generalization of (11) changing α,J,μ into complex coefficients (and allowing complex m); however, the real parts of these coefficients were found to determine generic dynamical properties of the phase fronts whereas the imaginary parts are responsible mainly for additional periodic structure of the phase domains [30] (1D spiral waves [31]). The solution (15) describes a collision of the phase fronts at the borders of oppositely oriented domains of the stable phase. Upon the collision both the fronts create the so called Nozaki-Bekki hole, a localized object which is similar to the Ising DW.

For $H = 0$, following the derivation in Appendix B, I find an approximate double-front solution to (12) of the form

$$g = \frac{1}{\sqrt{\mu}}(e^{k(x-x_{01})+ckt} + e^{-k(x-x_{02})+ckt} + \theta e^{k(x-x_{01})+ckt} e^{-k(x-x_{02})+ckt}),$$
$$f = \frac{1}{\theta} + e^{k(x-x_{01})+ckt} + e^{-k(x-x_{02})+ckt} + \theta e^{k(x-x_{01})+ckt} e^{-k(x-x_{02})+ckt},$$
$$k = \sqrt{\frac{\theta}{2J}}, \quad ck = \frac{3\theta}{2\alpha}. \tag{16}$$

It describes two phase fronts at the borders of similarly oriented domains of the stable phase since, in contrast to the Nozaki-Bekki hole, all terms of g are of the same sign. They counterpropagate into an unstable area. According to (16), since $t \to \infty \Rightarrow g \to f/\sqrt{\mu}$, these fronts annihilate upon the collision. Above I have neglected the external field because it introduces a big complexity to the solution (Appendix B). However, the asymmetry of the Ginzburg-Landau function due to $H \neq 0$ plays a role in the collision-induced formation of a dissipative structure since it determines the choice of the preferred stable state [32] [though the strength of the field is limited by the condition $\gamma|H| < 2(\theta/3)^{3/2}/\mu^{1/2}$ ensuring that two stable homogeneous solutions to (11) are of opposite signs].

IV. DISCUSSION

Direct observation of the front motion in ferromagnetic and ferroelectric media is difficult because relevant time scales are very narrow. Thus, I refer to an experiment on a (bistable) ferrocyanide-iodate-sulfite chemical reaction performed in a continuous-flow stirred tank reactor [10,11]. When the ferrocyanide concentration is decreased, the morphology of the 2D chemical structure changes. At a relatively low concentration, irregular stationary patterns of DWs (lamellae) appear and change into bubble structures with a further concentration decrease. The lamellae formation is accompanied by the property of DW reflection upon their externally driven collision (which is enforced by an intense irradiation). On increasing the critical parameter (the ferrocyanide concentration), the lamellae disappear while domains become inhomogeneous (spiral structure of the domains appear) and their fronts annihilate upon the collision. I identify the transition between structures analogous to those visualized in Figs. 1(a) and 1(b) with the decrease of the critical parameter β_1, and the transition between structures in Figs. 1(b) and 1(c) with crossing the Bloch-Ising point $\beta_1 = 3\beta_2$. Thus, the predicted dynamical behavior of DWs of a generic (Ginzburg-Landau) model of bistable systems coincides with that observed in a chemical reaction below the critical point. Further increase of the ferrocyanide concentration leads to the appearance of wave-front reflection and self-replication of phase spots. These effects are related to differences in the diffusion coefficients of the chemical species; therefore, they cannot be predicted within the Ginzburg-Landau model. According to calculations using a two-species (Gray-Scott) model of a chemical reaction, the self-replication regime relates to the monostability of the system [33].

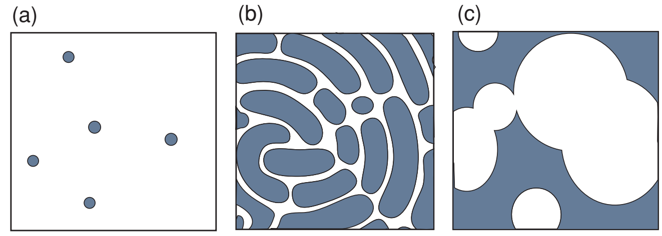

FIG. 1. (Color online) 2D pattern structures in a bistable medium: (a) a bubble structure, (b) a lamellar (labyrinth) pattern, (c) stable (white) domains propagating into (gray) unstable state. In (c) a periodic (spiral) structure of the domains is neglected.

I conclude that the coexistence of subcritical effects of front reflection and annihilation (accompanied by the creation of dissipative structures) can be explained by the appearance of domains of the unstable phase above the Bloch-Ising transition point.

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APPENDIX A: DERIVATION OF DW SOLUTIONS

One manages to find an exact single-DW solution to (1) for $H \neq 0$ in the case $\mathrm{Im}(m) = 0$. It corresponds to an asymmetric Ising DW. In particular, inserting

$$g_1 = a_1 - a_2 e^{2\eta}, \quad f_1 = 1 + e^{2\eta}, \quad \eta = k(x - x_0) - \omega t, \tag{A1}$$

into (12), one establishes the coefficients a_1,a_2,k,ω to be complicated functions of H. For $\gamma H \ll (\beta_1 + \beta_2)^{3/2}/\mu^{1/2}$ (H is much weaker than the coercivity field), the effects of the DW asymmetry $a_1 \neq a_2$ and the dependence of the DW width $1/k$ on H are negligible while (A1) reproduces (2) and (3b). Unfortunately, an exact solution describing a Bloch DW in an external field is unknown. Thus, I search for approximate solutions with $\mathrm{Im}(m) \neq 0$ assuming they satisfy the following requirements: (1) they tend to the exact (single- or double-DW) solutions of Ref. [22] with $H \to 0$, and (2) they satisfy (1) at the centers of the DWs (therefore, I neglect an expected space asymmetry of $\mathrm{Re}(m)$ and $\mathrm{Im}(m)$ with respect to the DW center in the presence of $H \neq 0$). Within this approach, let us consider approximate single-DW solutions to the bilinearized Ginzburg-Landau system (5a) of the form

$$g_1 = a(1 - e^{2\eta}) + ibe^{\eta}, \quad f_1 = 1 + e^{2\eta},$$
$$\eta = k(x - x_0) - \omega t. \tag{A2}$$

Inserting (A2) into (5a), one arrives at

$$a(\beta_1 + \beta_2 - \lambda) + \gamma H + e^{\eta} ib(\beta_1 - \beta_2 - \lambda + Jk^2 + \alpha\omega) + e^{2\eta}2(\gamma H - 2a\alpha\omega) + e^{3\eta} ib(\beta_1 - \beta_2 - \lambda + Jk^2 - \alpha\omega) + e^{4\eta}[-a(\beta_1 + \beta_2 - \lambda) + \gamma H] = 0,$$

$$-a^2\mu+\lambda+e^{2\eta}(2a^2\mu-b^2\mu+2\lambda-8Jk^2)+e^{4\eta}(-a^2\mu+\lambda)=0. \tag{A3}$$

For $\lambda=\beta_1+\beta_2$, $a=\sqrt{\beta_1+\beta_2}/\sqrt{\mu}$, one finds the second equation of (A3) to be satisfied (all coefficients of the LHS expansion in e^η vanish), if $b=2\sqrt{\beta_1-3\beta_2}/\sqrt{\mu}$ and $|k|=\sqrt{2\beta_2}/\sqrt{J}$ (the Bloch DW) or $b=0$ and $|k|=\sqrt{\beta_1+\beta_2}/\sqrt{2J}$ (the Ising DW), whereas, for $\omega=\sqrt{\mu}\gamma H/(\sqrt{\beta_1+\beta_2}\alpha)$, the LHS of the first equation of (A3) simplifies to

$$\gamma H\left(1+e^\eta ib\frac{\sqrt{\mu}}{\sqrt{\beta_1+\beta_2}}-e^{2\eta}2-e^{3\eta}ib\frac{\sqrt{\mu}}{\sqrt{\beta_1+\beta_2}}+e^{4\eta}\right).$$

We see that it vanishes with $H\to0$ or at $\eta=0$ (at the DW center). Therefore, the DW described by (A2) propagates with a constant velocity $c=\omega/k\propto H$.

Let me mention that I do not perform a systematic Hirota construction connected to inserting consecutive elements of the expansion of g_1,f_1 in e^{η_1},e^{η_2} and solving a series of relevant equations one by one. If their solutions were not exact, an error would accumulate in the consecutive steps. In the present approach, the form of the solution (A2) is fixed while I simply verify the coefficients a,b,ω. Similarly, searching for the approximate double-DW solution, I insert (6a) into (5a). The resulting equations take the form

$$\begin{aligned}&(1+e^{4\eta_1+4\eta_2})[a(\beta_1+\beta_2-\lambda)+\gamma H]+e^{\eta_1}ib\nu^{1/2}\big(\beta_1-\beta_2-\lambda+Jk_1^2+\alpha\omega_1\big)+e^{2\eta_1}2\nu(\gamma H-2a\alpha\omega_1)\\&\quad+e^{2\eta_2}2\nu(\gamma H-2a\alpha\omega_2)+e^{3\eta_1}ib\nu^{3/2}\big(\beta_1-\beta_2-\lambda+Jk_1^2-\alpha\omega_1\big)+(e^{4\eta_1}+e^{4\eta_2})\nu^2[-a(\beta_1+\beta_2-\lambda)+\gamma H]\\&\quad+e^{2\eta_1+4\eta_2}2\nu(\gamma H+2a\alpha\omega_1)+e^{4\eta_1+2\eta_2}2\nu(\gamma H+2a\alpha\omega_2)-e^{\eta_1+2\eta_2}ib\nu^{1/2}\big\{(1-\nu)\big[\beta_1-\beta_2-\lambda+Jk_1^2+4Jk_2^2+\alpha\omega_1\big]\\&\quad+(1+\nu)(4Jk_1k_2+2\alpha\omega_2)\big\}+e^{2\eta_1+2\eta_2}2\big[a(1-\nu^2)\big(\beta_1+\beta_2-\lambda+4Jk_1^2+4Jk_2^2\big)+(1+\nu^2)(8aJk_1k_2+\gamma H)\big]\\&\quad+e^{3\eta_1+2\eta_2}ib\nu^{1/2}\big\{(1-\nu)\big[\beta_1-\beta_2-\lambda+Jk_1^2+4Jk_2^2-\alpha\omega_1\big]+(1+\nu)(4Jk_1k_2-2\alpha\omega_2)\big\}\\&\quad-e^{\eta_1+4\eta_2}ib\nu^{3/2}\big(\beta_1-\beta_2-\lambda+Jk_1^2+\alpha\omega_1\big)-e^{3\eta_1+4\eta_2}ib\nu^{1/2}\big(\beta_1-\beta_2-\lambda+Jk_1^2-\alpha\omega_1\big)=0,\\&(1+e^{4\eta_1}\nu^2+e^{4\eta_2}\nu^2+e^{4\eta_1+4\eta_2})(-a^2\mu+\lambda)+(e^{2\eta_1}+e^{2\eta_1+4\eta_2})\nu\big(2a^2\mu+2\lambda-8Jk_1^2-b^2\mu\big)+e^{2\eta_1+2\eta_2}\\&\quad\times2\big[-(1-\nu^2)8Jk_1k_2+(1+\nu^2)\big(-a^2\mu+\lambda-4Jk_1^2-4Jk_2^2\big)+\nu b^2\mu\big]+(e^{2\eta_2}+e^{4\eta_1+2\eta_2})\nu\big(2a^2\mu+2\lambda-8Jk_2^2\big)=0.\end{aligned} \tag{A4}$$

The second equation of (A4) is fulfilled for λ,a,b determined above and for k_1,k_2,ν of (7). When $H=0$, the first equation of (A4) is satisfied for these parameters according to [22]. In the vicinity of the DW centers in a state of well-separated DWs, the regimes $\eta_1+\ln(\nu)/2\approx0$, $e^{\eta_2+\ln(\nu)/2}\approx0$ or $\eta_2+\ln(\nu)/2\approx0$, $e^{\eta_1+\ln(\nu)/2}\approx0$, the system (A4) reproduces (A3); therefore, the centers of both the DWs counterpropagate with constant velocities.

APPENDIX B: DERIVATION OF DOUBLE-FRONT SOLUTION

Looking for the approximate double-front solution (16), I define an error function as the absolute value of the LHS of (12) for $H=0$,

$$\mathrm{Err}(x,t)\equiv\big|f\big[\big(-\alpha D_t+JD_x^2\big)g\cdot f\big]+g\big[\big(-JD_x^2+\theta\big)f\cdot f-\mu\theta g^2\big]\big|. \tag{B1}$$

Inserting

$$\begin{aligned}g&=h(e^{\eta_1}+e^{\eta_2}+\varphi e^{\eta_1+\eta_2}),\\f&=\varphi^{-1}+e^{\eta_1}+e^{\eta_2}+\varphi e^{\eta_1+\eta_2},\end{aligned} \tag{B2}$$

where $\eta_1\equiv k(x-x_{01})+ckt$, $\eta_2\equiv-k(x-x_{02})+ckt$, into (B1), I take $\varphi=\theta$ in order to ensure faster divergence of the front centers in the limit $\theta\to0^+$ than the divergence of the front widths in this limit. We arrive at

$$\begin{aligned}\mathrm{Err}(x,t)=&\,|[e^{3\eta_1+3\eta_2}\theta^4+(e^{2\eta_1+3\eta_2}+e^{3\eta_1+2\eta_2})3\theta^3\\&+(e^{3\eta_1+\eta_2}+e^{\eta_1+3\eta_2})3\theta^2+(e^{3\eta_1}+e^{3\eta_2})\theta]\\&\times b(1-\mu b^2)+(e^{2\eta_1}+e^{2\eta_2})b(2-\alpha ck/\theta-Jk^2/\theta)\\&+e^{2\eta_1+2\eta_2}\theta^2b(8-6\mu b^2-2\alpha ck/\theta)\\&+(e^{\eta_1+2\eta_2}+e^{2\eta_1+\eta_2})\theta b(7-3\mu b^2-3\alpha ck/\theta\\&+Jk^2/\theta)+e^{\eta_1+\eta_2}b(5-4\alpha ck/\theta+6Jk^2/\theta)\\&+(e^{\eta_1}+e^{\eta_2})\theta^{-1}b(1-\alpha ck/\theta+Jk^2/\theta)|.\end{aligned} \tag{B3}$$

For $h=1/\sqrt{\mu}$, $k=\sqrt{\theta/2J}$, $ck=3\theta/(2\alpha)$, all the coordinate-x-dependent terms of (B3) vanish, leading to

$$\begin{aligned}\mathrm{Err}(x,t)=\mathrm{Err}(t)&\sim|e^{2\eta_1+2\eta_2}\theta^2-2e^{\eta_1+\eta_2}|\\&=|e^{4ckt-2k(x_{01}-x_{02})}\theta^2-2e^{2ckt-k(x_{01}-x_{02})}|.\end{aligned} \tag{B4}$$

Before the collision, when the fronts are well separated, Eq. (12) [equivalent to $\mathrm{Err}(x,t)=0$] is approximately fulfilled in the vicinity of the front centers [the regimes $\eta_1+\ln(\theta)\approx0$, $e^{\eta_2+\ln(\theta)}\approx0$ or $\eta_2+\ln(\theta)\approx0$, $e^{\eta_1+\ln(\theta)}\approx0$] as well as in the area of an unstable phase separating the fronts ($e^{\eta_1+\ln(\theta)}\approx0$, $e^{\eta_2+\ln(\theta)}\approx0$). Then

$$\begin{aligned}m'&=\frac{1}{\sqrt{\mu}}\frac{e^{\eta_1}+e^{\eta_2}+\theta e^{\eta_1+\eta_2}}{1/\theta+e^{\eta_1}+e^{\eta_2}+\theta e^{\eta_1+\eta_2}}\\&=\begin{cases}\frac{1}{\sqrt{\mu}}\frac{e^{\eta_1+\ln(\theta)}}{1+e^{\eta_1+\ln(\theta)}}, & \eta_1\sim-\ln(\theta),\quad e^{\eta_2+\ln(\theta)}\sim0,\\\frac{1}{\sqrt{\mu}}\frac{e^{\eta_2+\ln(\theta)}}{1+e^{\eta_2+\ln(\theta)}}, & \eta_2\sim-\ln(\theta),\quad e^{\eta_1+\ln(\theta)}\sim0.\end{cases}\end{aligned} \tag{B5}$$

Assuming the motion to start at $t = 0$, and choosing the reference frame such that $x_{02} = -x_{01}$, the front centers are initially at the points $x = \pm x_1 = \pm[x_{01} - \ln(\theta)\sqrt{2J/\theta}]$ and they meet at the point $x = 0$ after the time $t' = x_1/c$. Note that up to the time $t'' = t' + \frac{\ln(2)}{2ck}$, the function $\mathrm{Err}(t)$ is a decreasing one and $\mathrm{Err}(t'') = 0$. We conclude that the approximate solution (16) satisfies well (12) at the beginning of the motion (at $t = 0$) and the accuracy of approximating the solution by the functions (16) increases with time up to $t = t''$ which corresponds to the moment of finishing the front collision (the moment when the fronts of the width $1/k$ finish passing through each other with the relative velocity $2c$).

The application of a similar analysis to the field-induced collision of DWs below the Bloch-Ising transition, studying (5a) separately from (5b), does not work. Utilizing the argumentation (of Sec. II) based on the time-reversal symmetry is an alternative.

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