Quantized representation of some nonlinear integrable evolution equations on the soliton sector

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The Hirota algorithm for solving several integrable nonlinear evolution equations is suggestive of a simple construction of a quantized representation of these equations and their soliton solutions over a Fock space of bosons or of fermions. The classical nonlinear wave equation becomes a nonlinear equation for an operator. The solution of this equation is constructed through an operator analog of the Hirota transformation. The classical N-soliton solution is the expectation value of the solution operator in an N-particle state in the Fock space. The effect of perturbations that modify soliton identity is demonstrated.

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I. INTRODUCTION

The issue of the extension of quantum mechanics to a nonlinear variety is decades old [1–4]. A subset of the discussion of this topic has evolved in the form of many attempts at rigorous quantization of integrable nonlinear evolution equations, involving powerful tools from canonical procedures, to inverse scattering and symmetry groups (see, e.g., [5–25]). The purpose of this paper is to propose that a simple way exists for obtaining a quantized representation for a family of equations over the soliton sector, when the solution for these equations can be constructed through Forsyth-Hopf-Cole [26–28]-Hirota [29]-type transformations. In the representation discussed here, the coordinates, t and x are mere parameters.

As an example, consider the Korteweg–de Vries (KdV) equation [30],

$$u_t = 6\,u\,u_x + u_{xxx}. \tag{1}$$

Whereas the structure of the single-soliton solutions of Eq. (1) is simple, the structure of its multiple-soliton solution is rather cumbersome. However, using the Hirota algorithm [29] for $u(t,x)$,

$$u(t,x) = 2\,\partial_x^2 \ln[f(t,x)], \tag{2}$$

the function $f(t,x)$ may be given a simple physical interpretation. For an N-soliton solution, with soliton wave numbers k_i $1 \leqslant i \leqslant N$, all different from one another, it is given by

$$f(t,x) = 1 + \sum_{i=1}^{N} \varphi(k_i;t,x) + \sum_{n=2}^{N}\left(\sum_{1\leqslant i_1<\cdots<i_n\leqslant N}\left\{\prod_{j=1}^{n}\varphi\left(k_{i_j};t,x\right)\prod_{i_l<i_m}V\left(k_{i_l},k_{i_m}\right)\right\}\right)$$
$$\left(\varphi(k;t,x) = e^{2k(x+v(k)t)},\quad v(k) = 4k^2,\quad V(k_1,k_2) = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2\right). \tag{3}$$

As one has $V(k,k') \leqslant 1$, $f(t,x)$ is bounded by

$$f(t,x) \leqslant 1 + \sum_{n=1}^{N}\frac{1}{n!}\left(\sum_{i=1}^{N}\varphi(k_i;t,x)\right)^n \leqslant e^{(\sum_{i=1}^{N}\varphi(k_i;t,x))}. \tag{4}$$

Equation (3) looks like a sum of Feynman diagrams containing single- and multiparticle contributions of all possible subsets of $n \leqslant N$ of the particles. The functions $\varphi(k_i;t,x)$ may be viewed as real "plane waves," and $V(k,k')$ may be viewed as a "two-particle coupling coefficient."

II. QUANTIZED REPRESENTATION OF KdV EQUATION

With this observation in mind, it is suggestive of the following simple quantized representation of the solution of Eq. (1) over a Fock space of bosons or of fermions, with creation and annihilation operators $a_k{}^\dagger$ and a_k, respectively, and number operators N_k, defined by

$$N_k = a_k^\dagger a_k,\quad \begin{pmatrix}[a_k,a_{k'}^\dagger] = \delta(k-k') & \text{(Bosons)}\\ \{a_k,a_{k'}^\dagger\} = \delta(k-k') & \text{(Fermions)}\end{pmatrix}. \tag{5}$$

In Eq. (5), [,] stands for the commutator, and {,} for the anticommutator.

Consider the following operator:

$$F(t,x) = 1 + \int_0^\infty \varphi(k;t,x)\,N_k\,dk + \sum_{n=2}^{\infty}\frac{1}{n!}\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left\{\left(\prod_{i=1}^{n}\varphi(k_i;t,x)\,N_{k_i}\right)\times\left(\prod_{1\leqslant l<m\leqslant n}V(k_l,k_m)\right)\right\}dk_1\,dk_2\cdots dk_n. \tag{6}$$

As $V(k_l,k_m) \leqslant 1$, integration down to $k = 0$ does not pose any problem. To improve the convergence properties of the integrals one may multiply $\varphi(k;t,x)$ by a function of k that falls off sufficiently fast as $k \to \infty$, e.g.,

$$\varphi(k;\, t,\, x) \to \varphi(k;\, t,\, x)\, e^{-\alpha k^4} \quad (\alpha > 0)\,. \tag{7}$$

This amounts to a mere phase shift in the trajectory of a soliton. For any state with a finite number of particles, the matrix element of the operator $F(t,x)$ is a finite sum of bounded terms. Hence after calculating matrix elements one may set α to zero.

As $V(k,k') \leqslant 1$, a majorant to the operator $F(t,x)$ is the following operator, the matrix elements of which in states with a finite number of particles are the upper bounds in Eq. (4):

$$M = e^{\int_0^\infty \varphi(k;\, t,\, x)\, N_k\, dk}. \tag{8}$$

Denoting a state with r particles with a given wave number q by $|\{q, r\}\rangle$ (for fermions, obviously, only $r = 1$ is possible), the matrix element of $F(t,x)$ in a single-particle state is

$$\langle \{q,\, 1\}\, |F\,(t,\, x)|\, \{q,\, 1\}\rangle = 1 + \varphi\,(q;\, t,\, x)\,. \tag{9}$$

Equation (9) is identical to the expression for $f(t,x)$ of Eq. (2), when $u(t,x)$ is a single-soliton solution of Eq. (1) [29]. Similarly, the matrix element in a state of two particles with different wave numbers is identical to the expression for $f(t,x)$ when $u(t,x)$ is a two-soliton solution:

$$\begin{aligned} &\langle \{q_1,\, 1\}\,,\{q_2,\, 1\}\; |F\,(t,\, x)|\, \{q_1,\, 1\}\,,\{q_2,\, 1\}\rangle \\ &\quad = 1 + \varphi\,(q_1;\, t,\, x) + \varphi\,(q_2;\, t,\, x) \\ &\qquad + \varphi\,(q_1;\, t,\, x)\, \varphi\,(q_2;\, t,\, x)\, V\,(q_1,\, q_2)\,. \end{aligned} \tag{10}$$

Extension to $N > 2$ is straightforward: $\langle\{q_1, 1\}, \ldots, \{q_N, 1\}\, |F(t,x)|\, \{q_1, 1\}, \ldots, \{q_N, 1\}\rangle$, with $q_i \neq q_j$ $(1 \leqslant i, j \leqslant N,\ i \neq j)$, is the expression for $f(t,x)$ corresponding to an N-soliton solution of Eq. (1).

If the particles are bosons, then a given momentum state may be occupied by more than one particle. A matrix element in a state, in which a given wave number is occupied by several bosons, yields a soliton solution with a simple phase shift. For example, the matrix element

$$\begin{aligned} &\langle \{q,\, n_q\}\, |F\,(t,\, x)|\, \{q,\, n_q\}\rangle \\ &\quad = 1 + n_q \varphi\,(q;\, t,\, x) \\ &\quad = 1 + \varphi\,(q;\, t,\, x + \delta)\, \{\delta = \ln[n_q]/(2q)\} \end{aligned} \tag{11}$$

is equal to $f(t,x)$ for a single-soliton solution with a phase shift δ in the soliton trajectory. The same applies to a state with several distinct wave numbers. For every wave number that is occupied by more than one boson, the corresponding soliton is subjected to a similar phase shift.

The fact that the expression for $f(t,x)$ in the classical case is obtained as the expectation value of a quantum-mechanical operator leads directly to an operator version of Eq. (1). To this end, consider the operator analog of Eq. (2):

$$U\,(t,\, x) = 2\partial_x[F\,(t,\, x)_x\; F(t,\, x)^{-1}]. \tag{12}$$

As $F(t,x)$ is a diagonal operator, the order of multiplication in Eq. (12) is unimportant, and $U(t,x)$ obeys Eq. (1) on any state with a finite number of particles. Finally, the N-soliton solution of Eq. (1) is equal to the expectation value

$$u\,(t,\, x) = \langle \{q_1, 1\}\,, \ldots, \{q_N, 1\}|U\,(t,\, x)|\{q_1,\, 1\}\,, \ldots, \{q_N,\, 1\}\rangle\,. \tag{13}$$

Using Eq. (12), one can construct a Hamiltonian operator from the classical Hamiltonian, from which Eq. (1) can be derived [31,32], by replacing the classical solution u by the operator U:

$$\begin{aligned} H\,[u] &= \int_{-\infty}^{\infty} \left(u^3 - \frac{1}{2}\,(u_x)^2 \right) dx \\ &\Rightarrow H\,[U] = \int_{-\infty}^{\infty} \left(U^3 - \frac{1}{2}\,(U_x)^2 \right) dx. \end{aligned} \tag{14}$$

Similarly, the infinite sequence of conserved quantities that characterize the soliton solutions of Eq. (1) [33,34] corresponds to an infinite sequence of operators. When u is an N-soliton solution, with soliton wave numbers $q_1, \ldots, q_N$, the conserved quantities $c_n(q_1, \ldots, q_N)$ are integrals of known differential polynomials in u, denoted by $h_n[u]$. The corresponding operators are obtained, again, by replacing u by the operator U:

$$\begin{aligned} c_n\,(q_1, q_2,\, \ldots q_N) &= \int_{-\infty}^{\infty} h_n\,[u_{N-\text{solitons}}]\; dx \\ &\Rightarrow C_n = \int_{-\infty}^{\infty} h_n\,[U]\; dx. \end{aligned} \tag{15}$$

For any multiparticle state in the Fock space, one has

$$\begin{aligned} &C_n\, |\{q_1,\, 1\}\,, \ldots, \{q_N,\, 1\}\rangle \\ &\quad = c_n\,(q_1, q_2, \ldots q_N)\, |\{q_1,\, 1\}\,, \ldots, \{q_N,\, 1\}\rangle\,. \end{aligned} \tag{16}$$

For example, the first conserved quantity

$$c_1 = \int_{-\infty}^{+\infty} u\,(t,\, x)\; dx \tag{17}$$

is now replaced by the operator

$$C_1 = \int_{-\infty}^{+\infty} U\,(t,\, x)\; dx. \tag{18}$$

Its action on any state yields

$$\begin{aligned} &C_1\, |\{q_1,\, 1\}\,, \ldots, \{q_N,\, 1\}\rangle \\ &\quad = c_1\,(q_1,\, \ldots, q_N)\, |\{q_1,\, 1\}\,, \ldots, \{q_N,\, 1\}\rangle\,, \end{aligned} \tag{19}$$

where $c_1(q_1, \ldots, q_N)$ is the value of c_1 for the corresponding N-soliton solution.

III. MULTIPLE-SOLITON PROJECTION OPERATORS: KdV CASE

In classical soliton dynamics, the single-soliton solution plays a unique role. There is an infinite hierarchy of differential polynomials in u, the solution of an evolution equation, which vanish identically when u is a single-soliton solution ("special polynomials" [35,36]). In the quantized representation, these polynomials correspond to an infinite hierarchy of commuting projection operators. As an example, consider the case of the KdV equation. The lowest scaling weight, in which special polynomials exist, is 3. There are only two of them, given by

$$R^{(3,1)}[u] = u_x + q^{(1,1)} u, \quad R^{(3,2)} = \frac{3}{2}\left(\int_{-\infty}^{x} q^{(1,1)} R^{(3,1)}[u]\,dx - \int_{x}^{\infty} q^{(1,1)} R^{(3,1)}[u]\,dx\right)$$
$$\left(q^{(1,1)} = \frac{1}{2}\left(\int_{-\infty}^{x} u(t,x)\,dx - \int_{x}^{\infty} u(t,x)\,dx\right)\right). \tag{20}$$

[In each superscript (W,i), W is the scaling weight, and i counts the polynomials with this scaling weight.] Replacing in Eq. (20) the function $u(t,x)$ by the operator $U(t,x)$ of Eq. (12), both special polynomials become operators, which project the full Fock space into its multiparticle subspace.

The polynomials in Eq. (20) contain nonlocal functionals of u. (They are all bounded.) A local special polynomial (containing only powers of u and of its spatial derivatives) first appears at scaling weight 6. It is given by [35,36]

$$R^{(6,1)}[u] = u^3 + u\,u_{xx} - (u_x)^2. \tag{21}$$

Using Eq. (12), one can construct the corresponding projection operator:

$$R^{(6,1)}[U] = U(t,x)^3 + U(t,x)\,\partial_x^2 U(t,x) - [\partial_x U(t,x)]^2. \tag{22}$$

Again, the action of this operator on any single-particle state is readily found to vanish:

$$R^{(6,1)}[U]\,|\{q,1\}\rangle = 0. \tag{23}$$

IV. OTHER INTEGRABLE EVOLUTION EQUATIONS

The same ideas apply to several other integrable equations.

Sawada-Kotera equation [37,38].

$$u_t = 45\,u^2 u_x + 15\,u\,u_{xxx} + 15\,u_x u_{xx} + u_{xxxxx}. \tag{24}$$

Equation (24) is integrable [37,38]. Its soliton solutions are also given by Eqs. (2) and (3), with

$$V(k,k') = \left(\frac{k - k'}{k + k'}\right)^2 \left(\frac{k^2 - k k' + k'^2}{k^2 + k k' + k'^2}\right). \tag{25}$$

Hence the quantization procedure described in the case of the KdV equation applies.

mKdV equation [39,40].

$$u_t = 6\,u^2 u_x + u_{xxx}. \tag{26}$$

Equation (26) is integrable [39,40]. Its soliton solutions are given by

$$u(t,x) = 2\,\partial_x \tan^{-1}(g(t,x)/f(t,x)). \tag{27}$$

In Eq. (27),

$$g(t,x) = \sum_{i=1}^{N} \varphi(k_i;t,x) + \sum_{\substack{n=3\\ n\ \mathrm{odd}}}^{N}\left(\sum_{1\leqslant i_1<\cdots<i_n\leqslant N}\left\{\prod_{j=1}^{n}\varphi(k_{i_j};t,x)\prod_{i_l<i_m} V(k_{i_l},k_{i_m})\right\}\right), \tag{28}$$

$$f(t,x) = 1 + \sum_{\substack{n=2\\ n\ \mathrm{even}}}^{N}\left(\sum_{1\leqslant i_1<\cdots<i_n\leqslant N}\left\{\prod_{j=1}^{n}\varphi(k_{i_j};t,x)\prod_{i_l<i_m} V(k_{i_l},k_{i_m})\right\}\right), \tag{29}$$

$$V(k_1,k_2) = -\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{30}$$

In this case, corresponding to the functions $f(t,x)$ and $g(t,x)$, there are two operators, $F(t,x)$ and $G(t,x)$, which contain terms with, respectively, even and odd n in Eq. (6).

Bidirectional KdV equation [41,38].

$$u_{tt} - u_{xx} - \partial_x(6uu_x + u_{xxx}) = 0. \tag{31}$$

Equation (31) is integrable [41,38]. Its soliton solutions are given by Eqs. (2) and (3). The solitons may move in either direction along the x axis. Hence their velocities are given by

$$v(k,\sigma) = \sigma 4k^2, \quad \sigma = \pm 1. \tag{32}$$

In addition, the "coupling coefficients" $V(k,k')$ are replaced by ones that depend on the wave numbers, as well as on the velocities. For the scaling employed in Eq. (31), they are given by

$$V(k,\sigma,k',\sigma') = \frac{12(k-k')^2 + [v(k,\sigma) - v(k',\sigma')]^2}{12(k+k')^2 + [v(k,\sigma) - v(k',\sigma')]^2}. \tag{33}$$

These coefficients vanish in the single-particle limit ($k' = k, \sigma = \sigma'$). Therefore the quantized representation described above can be constructed, with particle states characterized by two "quantum numbers": k and σ. The fundamental operators are denoted by $a^{\dagger}_{k,\sigma}$, $a_{k,\sigma}$, and $N_{k,\sigma}$, and an N-particle state by$|\{q_1,\sigma_1,1\},\ldots,\{q_N,\sigma_N,1\}\rangle$. The operator in Eq. (6) is replaced by

$$F(t,x) = 1 + \sum_{\sigma=\pm 1}\int_0^{\infty} \varphi(k,\sigma;t,x)\,N_{k,\sigma}\,dk + \sum_{n=2}^{\infty}\frac{1}{n!}\sum_{i=1}^{n}\sum_{\sigma_i=\pm 1}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\left\{\left(\prod_{i=1}^{n}\varphi(k_i,\sigma_i;t,x)\,N_{k_i,\sigma_i}\right)\left(\prod_{\substack{1\leqslant l<m\leqslant n\\ \sigma_l,\sigma_m=\pm 1}} V(k_l,\sigma_l,k_m,\sigma_m)\right)\right\} dk_1\,dk_2\cdots dk_n. \tag{34}$$

The fact that σ has two values is suggestive of a formulation in terms of spin-1/2 fermions.

The quantized representation depends crucially on the fact that the "coupling coefficients" in Eq. (3) vanish in the limit $k_i = k_j$, $i \neq j$. Hence such a representation is not possible if this requirement is not satisfied. Two examples are given in the following.

Kaup-Kupershmidt equation [42,43].

$$u_t = 180\,u^2\,u_x + 30\,u\,u_{xxx} + 75\,u_x\,u_{xx} + u_{xxxxx}. \tag{35}$$

Equation (35) is integrable [42–49]. Its multiple-soliton solutions are given by

$$u(t\,x) = \tfrac{1}{2}\partial_x^2 \ln[f(t,x)]. \tag{36}$$

The "plane waves," $\varphi(k;t,x)$, are defined as in Eq. (3), with soliton velocities given by

$$v(k) = 16k^4. \tag{37}$$

However, the structure of $f(t,x)$ does not follow the pattern of Eq. (3). For the single-soliton solution one has

$$f(t,x) = 1 + \varphi(q\,,t,x) + \tfrac{1}{16}\,\varphi(q\,,t,x)^2. \tag{38}$$

For the two-soliton solution, the expression for $f(t,x)$ is:

$$\begin{aligned} f(t,x) = {}& 1 + \varphi(q_1;\,,t,x) + \varphi(q_2;\,,t,x) + \frac{1}{16}\varphi(q_1;\,,t,x)^2 \\ &+ \frac{2q_1^4 - q_1^2q_2^2 + 2q_2^4}{2(q_1+q_2)^2\left(q_1^2+q_1q_2+q_2^2\right)} \\ &\times \varphi(q_1;\,,t,x)\,\varphi(q_2;\,,t,x) + \frac{1}{16}\varphi(q_2;\,,t,x)^2 \\ &+ V(q_1,q_2)\,[\varphi(q_1;\,,t,x)^2\,\varphi(q_2;\,,t,x) \\ &+ \varphi(q_1;\,,t,x)\,\varphi(q_2;\,,t,x)^2] \\ &+ V(q_1,q_2)^2\,\varphi(q_1;\,,t,x)^2\,\varphi(q_2;\,,t,x)^2 \\ &\left(V(q_1,q_2) = \frac{(q_1-q_2)^2\left(q_1^2-q_1q_2+q_2^2\right)}{16(q_1+q_2)^2\left(q_1^2+q_1q_2+q_2^2\right)}\right). \end{aligned} \tag{39}$$

Obviously, not all two-wave "coupling coefficients" vanish in the limit $q_1 = q_2$.

Caudrey-Dodd-Gibbon equation [38].

$$\begin{aligned} u_t = {}& 420\,u^3\,u_x + 210\,u^2\,u_{xxx} + 420\,u\,u_x\,u_{xx} + 28\,u\,u_{xxxxx} \\ &+ 28\,u_x\,u_{xxxx} + 70\,u_{xx}\,u_{xxx} + u_{xxxxxxx}. \end{aligned} \tag{40}$$

The integrability of Eq. (40) is still an open question. The single- and two-soliton solutions do follow the Hirota structure of Eqs. (2) and (3). The two-particle "coupling coefficient" is [38]

$$V(k,k') = \left(\frac{k-k'}{k+k'}\right)^2\left(\frac{k^2-kk'+k'^2}{k^2+kk'+k'^2}\right)^2. \tag{41}$$

Attempting to construct a three-soliton solution of Eq. (40), one finds that the coefficients of the second-order terms, $g(k_i;t,x)\cdot g(k_j;t,x)$ ($1 \leqslant i,j \leqslant 3$, $i \neq j$), are of the Hirota form with $V(k_i,k_j)$ of Eq. (41). However, although the coefficient of the third-order term, $g(k_1;t,x)\cdot g(k_2;t,x)\cdot g(k_3;t,x)$, does vanish if any of the two wave numbers are equal, it cannot be factorized into a product of the two-particle coefficients $V(k_i,k_j)$. The same applies to the (necessary) fourth-order terms.

Burgers equation [50]. The Burgers equation

$$u_t = 2u\,u_x + u_{xx} \tag{42}$$

is not integrable. Yet, a very simple quantized representation exists in that case as well. Shock-front solutions of Eq. (42) are obtained through the Forsyth-Hopf-Cole transformation [26–28]

$$u(t,x) = \partial_x \ln[f(t,x)]. \tag{43}$$

Here, $f(t,x)$ has the following simple structure:

$$\begin{aligned} f(t,x) &= 1 + \sum_{i=1}^{N}\varphi(k_i;t,x) \\ &\left(\varphi(k;t,x) = e^{k(x+v(k)t)}, v(k) = k\right). \end{aligned} \tag{44}$$

Consequently, the operator $F(t,x)$ is given by

$$F\left(t,x\right)=1+\int_{-\infty}^{\infty}\varphi\left(k;\,t,x\right)N_{k}\,dk. \tag{45}$$

V. NONDIAGONAL PERTURBATIONS

In classical systems, the perturbation added to an integrable evolution equation is a functional of the unknown solution, typically, a differential polynomial in the latter. A common way for analyzing the effect of the perturbation is through a normal form expansion [51–56,35,36]. In this approach, the zero-order approximation is a single-soliton or a multiple-soliton solution of the normal form. To this solution, one may apply the quantization procedure delineated above. However, the classical perturbation, as well as the higher-order corrections to the solution in the normal form expansion, cannot lead to a change in soliton parameters, or in the number of solitons. The reason is that, in the quantized version presented here, as in the case of the unperturbed evolution equations, they are diagonal operators, functions of the number operator N_k.

However, the proposed quantum-mechanical representation opens a new vista for adding perturbations to a nonlinear wave equation. One may add nondiagonal perturbations, containing terms that destroy one soliton, and generate another one instead, e.g., $a_{k_2}^{\dagger}a_{k_1}$, or terms that change the number of solitons, such as $a_{k_3}^{\dagger}a_{k_2}^{\dagger}a_{k_1}$.

As an example, consider the case of the perturbed KdV equation, and focus on the following perturbed operator equation, which contains the operator $R^{(6,1)}$ [see Eq. (22)]:

$$U_t=6UU_x+U_{xxx}+\varepsilon\partial_x R^{(6,1)}\left[U\right]\int r(k,k')a_k^{\dagger}a_{k'}\,dk\,dk',$$
$$\varepsilon\ll 1,\qquad \int|r(k,k')^2|\,dk\,dk'<\infty. \tag{46}$$

To understand the motivation for this choice, let us begin with the classical equation:

$$u_t=6uu_x+u_{xxx}+\varepsilon R^{(6,1)}\left[u\right]. \tag{47}$$

$R^{(6,1)}$ is given in Eq. (21). To solve Eq. (47) through $O(\varepsilon)$, one writes

$$u\left(t,x\right)=u^{(0)}\left(t,x\right)+\varepsilon u^{(1)}\left(t,x\right). \tag{48}$$

The first reason for the choice of the perturbation in Eqs. (46) and (47) is that $R^{(6,1)}$ is localized around the soliton-collision region, and falls off exponentially away from that region [35,36]. The second reason is the simplicity of the equation obeyed by the zero-order approximation $u^{(0)}$. As $R^{(6,1)}$ does not contain linear, higher-derivative terms, such as u_xxxxx, $u^{(0)}$ obeys the unperturbed equation [51–56,35,36]:

$$u_t^{(0)}=6\,u^{(0)}\,u_x^{(0)}+u_{xxx}^{(0)}. \tag{49}$$

The third reason is the simplicity of the solution for the first-order correction $u^{(1)}$, which is given in terms of $R^{(3,1)}$ [see Eq. (20)] by [35,36]

$$u^{(1)}=-\tfrac{1}{3}\,\partial_x R^{(3,1)}. \tag{50}$$

The fourth reason is that $R^{(6,1)}$ generates a genuinely inelastic, multiple-soliton effect. The asymptotic form of $u^{(1)}$, away from the soliton-collision region, has a simple structure. For example, when $u^{(0)}$ is a two-soliton solution, one has [35,36]

$$u^{(1)}\left(t,x\right)\underset{|t|\to\infty}{\to}\partial_x\left[2k_2u_{\text{single}}\left(t,x;\,k_1\right)\right.$$
$$\left.-2k_1u_{\text{single}}\left(t,x;\,k_2\right)\right]\operatorname{sgn}\left(t\right),\qquad\left(k_2>k_1\right). \tag{51}$$

In Eq. (51), $u_{\text{single}}(t,x;k)$ is the single-soliton solution with wave number k. Thus the asymptotic form preserves the profiles of the zero-order solitons, but modifies their amplitude in an inelastic manner [57,35,36]; each soliton profile is affected by the wave numbers of the *other* solitons, and the whole contribution changes sign after the collision.

Turning to the operator version, Eq. (46), its solution is, again, expanded through $O(\varepsilon)$:

$$U\left(t,x\right)=U^{(0)}\left(t,x\right)+\varepsilon U^{(1)}\left(t,x\right). \tag{52}$$

Again, the zero-order term is the solution of the operator version of Eq. (1):

$$U_t^{(0)}=6U^{(0)}U_x^{(0)}+U_{xxx}^{(0)}. \tag{53}$$

The first-order correction is

$$U^{(1)}=-\frac{1}{3}\partial_x R^{(3,1)}[U^{(0)}]\int r(k,k')a_k^{\dagger}a_{k'}\,dk\,dk'. \tag{54}$$

Here $R^{(3,1)}[U^{(0)}]$ is the operator version of the special polynomial $R^{(3,1)}[u]$, given in Eq. (20).

The diagonal matrix element of this correction is

$$\langle\{q_1,1\},\{q_2,1\}|U^{(1)}\left(t,x\right)|\{q_1,1\},\{q_2,1\}\rangle$$
$$=-\frac{\left[r\left(q_1,q_1\right)+r\left(q_2,q_2\right)\right]}{3}\partial_x R^{(3,1)}[u^{(0)}\left(q_1,q_2\right)], \tag{55}$$

corresponding to the classical correction discussed above. In Eq. (55), $u^{(0)}(q_1,q_2)$ is a two-soliton solution of the KdV equation, with wave numbers q_1 and q_2.

On the other hand, off-diagonal matrix elements correspond to the change in the identity of one of the particles. For example,

$$\langle\{q_3,1\},\{q_2,1\}\,|U^{(1)}\left(t,x\right)|\{q_1,1\},\{q_2,1\}\rangle$$
$$=-\frac{r\left(q_1,q_3\right)}{3}\partial_x R^{(3,1)}[u^{(0)}\left(q_2,q_3\right)]. \tag{56}$$

VI. CONCLUDING COMMENTS

The quantized representation proffered here is a simple consequence of the Hirota transformation [29]. It avoids the need to resort to the powerful tools usually employed in the formal quantization of a classical dynamical system. In fact, it is a very trivial extension of the solution method applied to the classical equations that have been discussed. The reason is that the representation, although formulated in quantum-mechanical terms, involves only diagonal, commuting operators. In particular, in the representation discussed here, the coordinates t and x are mere parameters.

This quantized representation has two attributes. First, the solitons are not the solutions of a classical dynamical equation, but the expectation values of the solution of the dynamical equations obeyed by quantum-mechanical operators. Second, this approach allows for the introduction, in a very simple manner, of perturbations of a type that does not have a classical analog—interactions that lead to the "destruction" of solitons and the "creation" of others instead, or to a change in the number of solitons in a physical state. This opens the road to using classical nonlinear evolution equations, the solutions of which are known, as starting points for the study of toy field theories.

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