Frequency dependence of the subharmonic Shapiro steps

Jasmina Tekić[1,2] and Zoran Ivić[1]

[1]*Theoretical Physics Department 020, "Vinča" Institute of Nuclear Sciences, University of Belgrade, P. O. Box 522, 11001 Belgrade, Serbia*

[2]*Centre for Nonlinear Studies and The Beijing-Hong Kong-Singapore Joint Centre for Nonlinear and Complex Systems (Hong Kong), Hong Kong Baptist University, Hong Kong, China*

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Frequency dependence of the subharmonic Shapiro steps has been studied in the ac driven overdamped Frenkel-Kontorova model with deformable substrate potential. As potential gets deformed, in addition to the harmonic steps, subharmonic steps appear in the number and size that increase as the frequency of the external force increases. It was found that size of both harmonic and subharmonic steps strongly depend on the frequency where in the high-amplitude limit oscillatory dependence appears. When expressed as a function of period these oscillations of the step size with frequency have the same form as the oscillations of the step size with amplitude. Deformation of the potential has strong influence on these oscillations, and as in the case of amplitude dependence, with the increase of deformation, the same three distinctive types of behavior have been classified.

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I. INTRODUCTION

Due to the great significance for technical applications of interference phenomena, properties of the Shapiro steps, particularly their amplitude and frequency dependence have been matter of many theoretical and experimental studies in systems such as charge-density wave conductors [1,2], systems of Josephson-junction arrays biased by external currents [3–8], and, in recent years, superconducting nanowires [9,10]. In order to gain some insight into the physics behind interference phenomena, the attention has been always focused on the simple many-body models among which the dissipative (overdamped) Frenkel-Kontorova (FK) model is one of the simplest but still capable of exhibiting all complexities of interference effects [11].

The standard Frenkel-Kontorova model represents a chain of harmonically interacting particles subjected to a sinusoidal substrate potential. In the presence of an external ac+dc driving force, the dynamics is characterized by staircase macroscopic response or the appearance of the Shapiro steps in the response function $\bar{v}(\bar{F})$ of the system [12–14]. These steps are due to the dynamical mode-locking (interference or synchronization) of the internal frequency that comes from the motion of particles over the periodic substrate potential with the frequency of an external ac force. If the locking appears for integer values of frequency the steps are called harmonic, while for the locking at noninteger rational values they are called subharmonic. Though, the standard FK model has been very successful in the studies of harmonic steps, it could not be used for the studies of any phenomena related to the behavior of subharmonic steps. In the standard FK model, for commensurate structures with integer values of winding number only harmonic steps exist [12], while in the commensurate structures with noninteger values of winding number, subharmonic steps also appear; however, their size is so small that they are invisible on the regular plot of the response function what makes analysis of their properties very difficult.

The large subharmonic steps can appear in the presence of nonsinusoidal or deformable substrate potential [15]. In the real physical systems, the shape of the substrate potential can deviate from the standard (sinusoidal) one, and the application of standard FK model could be very restricted. Introducing a family of nonlinear periodic deformable potentials, Remoissenet and Peyrard [16] obtained in a controlled manner by an adequate choice of parameters rich variety of deformable potentials that allow the modeling of many specific physical situations without employing perturbation methods and are related to the physical systems such as charge-density wave condensates, Josephson junctions, and crystals with dislocations. They have shown that the shape of the substrate potential was of great importance for the modeling of discrete systems.

In our previous works on the FK model with deformable potentials, we have studied the amplitude dependence of harmonic and subharmonic steps [17]. We have shown that there is a strong correlation between harmonic and subharmonic steps, where, in their amplitude dependence, three differen types of behavior have been classified. In the present paper, following the similar procedure, we will examine frequency dependence of both harmonic and subharmonic Shapiro steps in the ac+dc-driven FK model with deformable substrate potential.

In the theoretical and experimental studies of the interference phenomena, contrary to the very extensive studies of amplitude dependence, a relatively small number of studies have been devoted to the dependence of frequency. Frequency dependence of the Shapiro steps has been the subject of many controversies. In the theoretical studies, two very competing and fundamentally different groups of theories have been proposed. In charge density wave systems, according to the classical approach [18,19], which considers a deformable charge elastic medium, with the internal degrees of freedom, the step width and the critical depinning force should be strongly frequency dependent and, after an initial increase, decreases to zero at the high frequencies. In contrast, the other theoretical approach based on single coordinate model results in a frequency-independent mode locking at the high frequencies [2]. In the same way, in the systems of Josephson-junction arrays, according to the single junction model [3–5,20], the widths of harmonic steps after a gradual increase at the high frequencies, saturate to the frequency independent value. On the other side, according to many degree of freedom theories, they are strongly frequency dependent and disappear at the high frequencies [21–23].

In our previous works, frequency dependence of harmonic Shapiro steps has been studied only in the standard FK model where an interesting phenomenon, frequency oscillations of the step size in the high amplitude limit have been observed [24,25]. These oscillations appear in the high amplitude limit, irrespectively of the number of degrees of freedom (they have been also observed in the commensurate structure $\omega = 1$ when FK model reduces to a single-particle case). When expressed as a function of period, these oscillations have the Bessel-like form, the same as in the case of amplitude dependence. This clearly proves that there is an analogy between the role of the amplitude and the period in the dynamical-mode-locking phenomena. Due to significance for technical application of interference phenomena and experiments, our observation of the oscillatory dependence in the standard FK model, therefore, raises a question whether these oscillations could appear in more realistic systems. Working in a model with deformable potential gives us possibility not only to study more realistic model but also to examine properties of subharmonic Shapiro steps. In this work, we have shown that as deformation of potential changes, frequency dependence of the harmonic and subharmonic steps exhibits the same three different types of behavior that we have observed in the studies of amplitude dependence [17].

The paper is organized as follows. The model is introduced in Sec. II. Simulation results are presented and analyzed in Sec. III. Finally, Sec. IV concludes the paper.

II. MODEL

We consider the dissipative (overdamped) dynamics of series of harmonically interacting particles subjected in one from the family of parameterized deformable periodic potentials, the asymmetric deformable potential (ASDP) [16]:

$$V(u) = \frac{K}{(2\pi)^2} \frac{(1-r^2)^2[1-\cos(2\pi u)]}{[1+r^2+2r\cos(\pi u)]^2}, \tag{1}$$

where K is the pinning strength and r is the shape parameter ($-1 < r < 1$). In Fig. 1, the ASDP is presented for different values of the shape parameter r. This potential refers to the same physical systems as the overdamped FK model [16], and by an appropriate choice of the shape parameter, it can be tuned in a controlled manner from the simply sinusoidal (standard) potential for $r = 0$ to an asymmetric periodic one for $0 < |r| < 1$ with a constant barrier height and two inequivalent successive wells with a flat and sharp bottom, respectively. The position u_b of the potential barrier is determined by the relation $\cos(\pi u_b) = 2r/(1+r^2)$. Precisely, here the asymmetry means that the pinning in the two successive potential minima differs. This type of potential is considered as a natural way to describe lattice with diatomic basis or dual lattices by generalizing the standard model that assumes simple sinusoidal potential [16]. In this model, particles during their motion interpolate between two media with different physical properties. The model has two energetically equivalent ground states, but these two states are not physically equivalent; in particular, they do not have the same dynamical properties [16]. The pinning of the particles strongly depends on the shape of the potential well, and as it was shown previously [16], in the potential with sharp maxima and wide minima, even the very large kinks can be pinned.

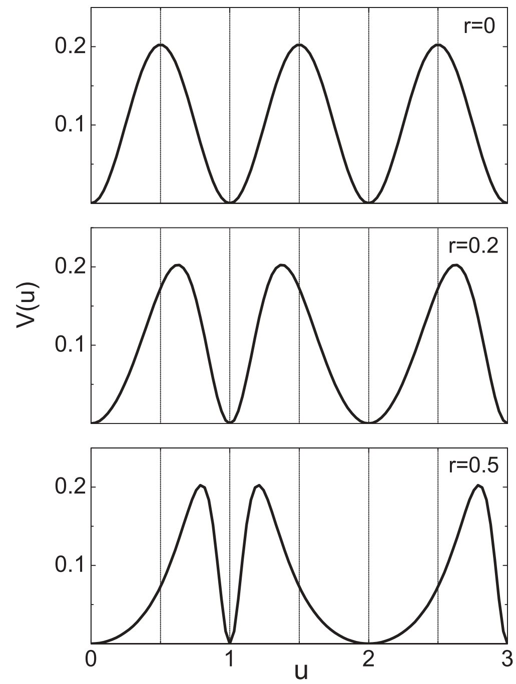

FIG. 1. Asymmetric deformable potential for $K = 4$, and different values of the shape parameters r.

The system is driven by dc and ac forces:

$$F(t) = \bar{F} + F_{ac}\cos(2\pi\nu_0 t), \tag{2}$$

where $\bar{F}$ is the dc force while F_{ac} and $2\pi\nu_0$ are the amplitude and frequency of the ac force, respectively. The equation of motion is

$$\dot{u}_l = u_{l+1} + u_{l-1} - 2u_l - V'(u_l) + F(t), \tag{3}$$

where $l = -\frac{N}{2},\ldots,\frac{N}{2}$. In the ac driven systems, the competition between two frequency scales (the frequency ν_0 of the external periodic force and the characteristic frequency of the motion over the periodic substrate potential driven by the average force $\bar{F}$) can result in the appearance of the synchronization phenomena (resonance). The ac force induces additional polarization energy into the system that differs from zero (less than zero) only when the velocity reaches the resonant values [12]:

$$\bar{v} = \frac{i\omega + j}{m}\nu_0, \tag{4}$$

where ω represents the interparticle average distance (winding number) and i,j and m are integers ($m = 1$ for harmonic, and $m > 1$ for subharmonic steps). The winding number ω is rational for commensurate and irrational for the incommensurate structures. The system will get locked since the average pinning energy of the locked state (on the step) is lower than of the unlocked state. As $\bar{F}$ increases, the particles will stay locked until the pinning force can cancel the increase in $\bar{F}$.

Equations (3) have been integrated using the fourth-order Runge-Kutta method with the periodic boundary conditions for commensurate structure with $\omega = \frac{1}{2}$. The time step used in the simulations was 0.02τ for lower values of r and 0.0002τ for $r > 0.8$ (τ was the period of ac force). The force was varied with the step 10^{-4} (10^{-5} or 10^{-6} is used for the studies of very small subharmonic steps) and a time interval of 100τ was used as a relaxation time to allow the system to reach the steady state. The appearance and frequency dependence of the steps are analyzed in the different amplitude regimes and for different shapes of the substrate potential.

III. RESULTS

When potential gets deformed large subharmonic steps appear in the response function $\bar{v}(\bar{F})$ of the system. In our previous works on amplitude dependence of the subharmonic Shapiro steps [17], we have noticed that the number and size of subharmonic steps significantly increase if the frequency of applied ac force increases. The response function $\bar{v}(\bar{F})$ of the system for two different values of frequency is presented in Fig. 2. We can clearly see, particularly in Figs. 2(b) and 2(d), which represent enlarged curves in Figs. 2(a) and 2(c), the significant increase in the step number and size with the increase in frequency. The increase of the fractional Shapiro steps in order and size with the frequency has been recently observed in superconducting nanowires [9]. While at the low frequencies only harmonic steps appear, at the higher frequencies, steps of order 1/2, 1/3, 1/4, even 1/6 appear. This effect is attributed to the nonsinusoidal and multivalued current phase relation [9]. In the same way, in our case, the subharmonic steps appear due to deformed and, therefore, nonsinusoidal substrate potential.

In Fig. 3, frequency dependence of the step width for the first harmonic, half-integer, the subharmonic step $\frac{1}{3}$, and the critical depinning force is presented. As frequency increases, harmonic, half-integer and subharmonic steps increase, reaching their maximum values, and then slowly decrease toward zero at the high frequencies. Meanwhile, critical depinning force increases and then saturates to the frequency independent threshold value for dc driven system F_{c0} for $r = 0.2$. When $\nu_0 \to 0$, it reaches the value $F_{c0} - F_{ac}$. What is particularly interesting is that with the increase of frequency, the half-integer step not only increases but, around the value that corresponds to the maximum of first harmonic step, becomes even larger than the harmonic step.

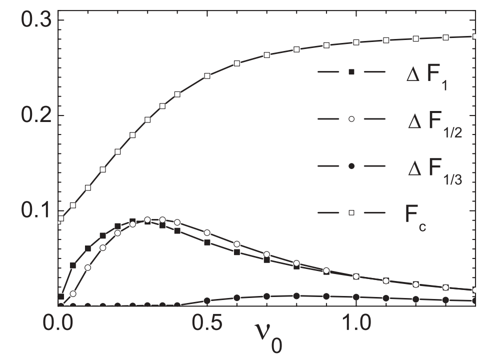

FIG. 3. Frequency dependence of the step width for the first harmonic ΔF_1, half-integer $\Delta F_{\frac{1}{2}}$, subharmonic $\Delta F_{\frac{1}{3}}$ steps, and the critical depinning force F_c for $\omega = \frac{1}{2}$, $K = 4$, $F_{ac} = 0.2$, and $r = 0.2$.

It was shown in our previous work [17] that dynamical dc threshold increases as deformation of potential increases and diverges when $r \to 1$. For the deformable substrate potential with the deformation parameter $r = 0.2$, the dynamical dc threshold is $F_{c0} = 0.289$. If we apply the ac force with the amplitude larger than the dynamical dc threshold, the oscillatory dependence appears at the low frequencies [24]. In Fig. 4, the frequency dependence of the step width for the first harmonic, half-integer, and subharmonic step $\frac{1}{3}$, and the critical depinning force is presented for $F_{ac} = 0.3$. Contrary to the results in Fig. 3, where $\frac{F_{ac}}{F_{c0}} < 1$, in Fig 4, the system is in the high-amplitude regime $\frac{F_{ac}}{F_{c0}} > 1$, and we can clearly see the change in the behavior at the low frequencies.

If we increase the ac amplitude even more, these oscillations become more pronounced. In Fig. 5, the frequency dependence

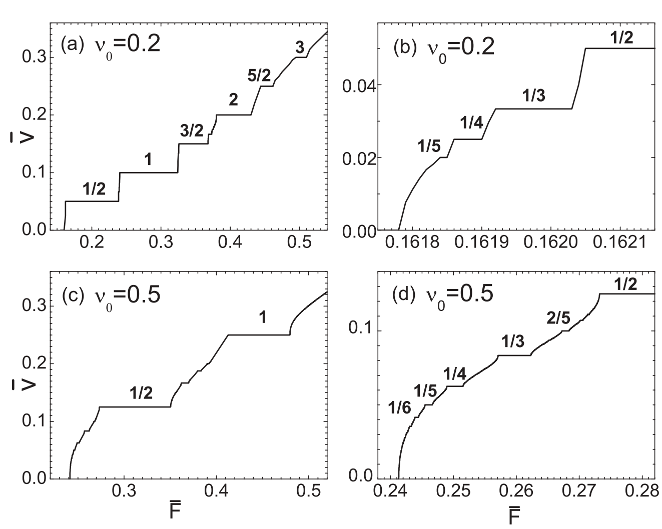

FIG. 2. Average velocity as a function of the average driving force for $\omega = \frac{1}{2}$, $K = 4$, $F_{ac} = 0.2$, $r = 0.2$, and two different values of the frequency: $\nu_0 = 0.2$ in (a) and (b) and 0.5 in (c) and (d).

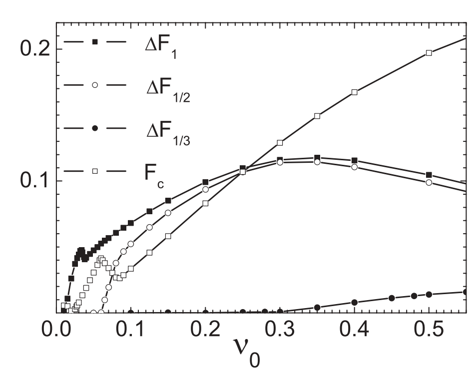

FIG. 4. Frequency dependence of the step width for the first harmonic ΔF_1, half-integer $\Delta F_{\frac{1}{2}}$, subharmonic $\Delta F_{\frac{1}{3}}$ steps, and the critical depinning force F_c for $\omega = \frac{1}{2}$, $K = 4$, $F_{ac} = 0.3$, and $r = 0.2$.

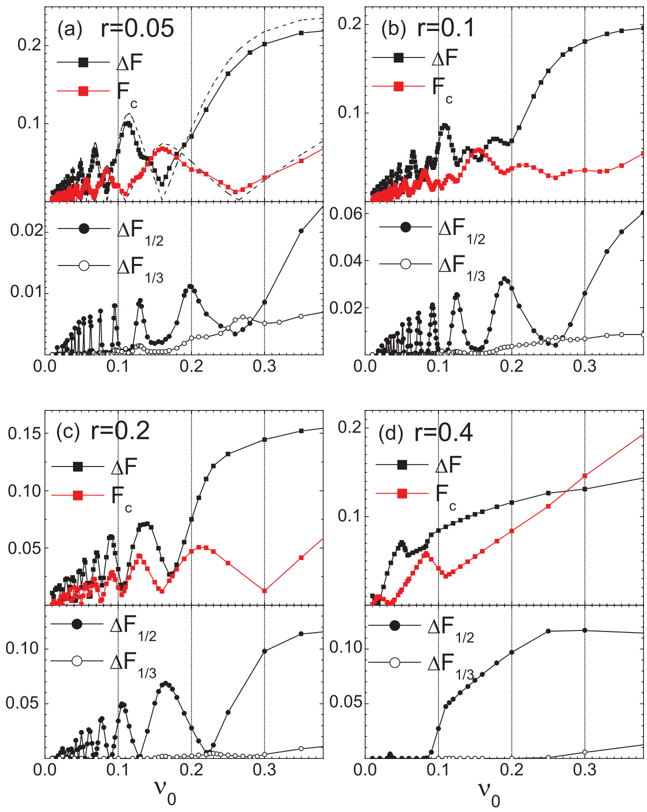

FIG. 5. (Color online) The width of the first harmonic ΔF_1, halfinteger step $\Delta F_{1/2}$, subharmonic $\Delta F_{1/3}$ steps and the critical depinning force F_c as a function of the ac frequency for $\omega = \frac{1}{2}$, $K = 4$, $F_{ac} = 0.5$, and $r = 0.05, 0.1, 0.2,$ and 0.4 in (a-d) respectively. Dashed line in (a) corresponds to the case $r = 0$.

of the critical depinning force, the first harmonic, half-integer, and $\frac{1}{3}$ subharmonic steps is presented for different values of the shape parameter. Frequency dependence for the standard case ($r = 0$) is presented by dashed line in Fig. 5(a). These results clearly show how the frequency dependence evolves as the size of subharmonic steps increases with the deformation. For the small deformation $r = 0.05$ and very small subharmonic steps in Fig. 5(a), the behavior of the system is still like in the standard FK model [24], where the maxima (minima) of the oscillations for the step size correspond to the minima (maxima) of the critical depinning force. As the deformation increases for $r = 0.1$ in Fig. 5(b), and large half-integer steps start to appear at the minima of the critical depinning force, the form of oscillations starts to change, and new lobes in curves for ΔF and F_c to develop. For $r = 0.2$ and the very large half-integer step (they even oversize the harmonic step) in Fig. 5(c), the behavior of ΔF and F_c is completely changed; now the maxima of one curve corresponds to the maxima of another. However, the increase of deformation will also affect the dynamical dc threshold F_{c0} and therefore the ratio $\frac{F_{ac}}{F_{c0}}$. Since F_{c0} starts to increase, at some point, for a given ac amplitude, the system will transfer from the high-amplitude limit $\frac{F_{ac}}{F_{c0}} > 1$ to the low-amplitude limit $\frac{F_{ac}}{F_{c0}} \leqslant 1$ when this oscillatory behavior disappears. In Fig. 5(d), the oscillations start to change and disappear as the ratio $\frac{F_{ac}}{F_{c0}}$ changes, and the system is approaching to the low amplitude limit. For $r = 0.2$, $\frac{F_{ac}}{F_{c0}} = 1.731$, while for $r = 0.4$, $\frac{F_{ac}}{F_{c0}} = 1.031$, and the behavior in Fig. 5(d) is similar to the behavior in Fig. 4, where $\frac{F_{ac}}{F_{c0}} = 1.038$. In the limit of very large deformation, when $r \to 1$, Shapiro steps disappear. For the commensurate structure $\omega = \frac{1}{2}$, we have two particles per one potential minima, and for small values of r, the half of particles are still in sharp minima and the half in the wide minima. With the further increase of r, as the sharp minima become more and more narrow, there will be one particle in sharp and three in wide minima. For very large deformations, when $r \to 1$, the sharp minima practically disappear, and all particles are strongly pinned in the wide minima with very sharp potential maxima. In this limit, the critical depinning force diverges and dynamical-mode locking disappears.

If the results in Fig. 5 are expressed as a function of period $\frac{1}{\nu_0}$, the physical origins of these oscillations can be understood and an analogy with the amplitude dependence revealed. In Fig. 6, the critical depinning force, the first harmonic, half-integer, and $\frac{1}{3}$ subharmonic steps are presented as a function of period $\frac{1}{\nu_0}$ for different values of the shape parameter. The appearance and the physical origin of these oscillations have been discussed in detail in our previous works [24]. These oscillations of the step size with frequency (period) are the result of the simultaneous competition and contributions of the dc and ac components of the force $F(t)$ to

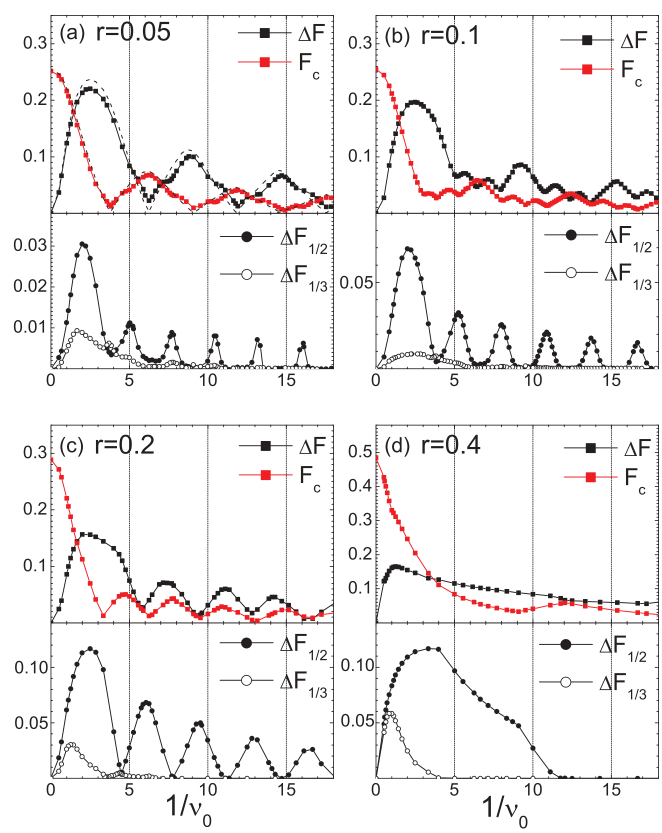

FIG. 6. (Color online) The width of the first harmonic ΔF_1, half-integer $\Delta F_{1/2}$, subharmonic $\Delta F_{1/3}$ steps, and the critical depinning force F_c as a function of period $\frac{1}{\nu_0}$ for $\omega = \frac{1}{2}$, $K = 4$, $F_{ac} = 0.5$, and $r = 0.05, 0.1, 0.2,$ and 0.4 in (a)–(d) respectively. The dashed line in (a) corresponds to the case $r = 0$.

the pinning energy. When $\frac{F_{ac}}{F_{c0}} > 1$, the ac contribution, which is responsible for the appearance of these oscillations, will dominate in the pinning energy. The oscillations appear due to the backward and forward motions of particles induced by the ac force. Namely, in the presence of the ac+dc driving forces, dynamics of particles is characterized by two types of motion: linear motion in the direction of the dc force and the back and forward jumps due to the presence of the ac force. Therefore, during one period, particle at site i will jump n sites back, reach the $i - n$ site, and then hop again $n + 1$ sites forward to the site $i + 1$. In that way, by repeating these back-forward jumps with every period of the ac force it will move. It was already known that the amplitude determines how much this motion is retarded [2]. However, our previous results [24] have shown that the distance over which particle moves was determined not only by the ac amplitude but also by the period or frequency. Particles will jump between more distant sites not only if the amplitude is large enough but also if they have enough time, which means if the period is long enough [24]. Therefore, for the values of period that correspond to the first maximum in Fig. 6, particles will spend most of the time pinned, and then hop to the next well, while for the values at the second maximum, particles will jump one site back and two forward. As the period increases, the particles will hop between the wells that are more and more distant while staying less and less time pinned and, consequently, the step width will decrease.

As in the case of the amplitude dependence [17], the deformation will affect in the same way the frequency dependence. As deformation increases in Fig. 6, we can distinguish three types of behavior:

(i) Standard behavior for small half-integer steps when $r = 0.05$ in Fig. 6(a). Though the shape and the size of maxima start to change, the oscillations are still almost as in the case of the pure sinusoidal potential ($r = 0$), where the maxima (minima) of ΔF_1 correspond to the minima (maxima) of F_c.

(ii) Behavior for intermediate half-integer steps when $r = 0.1$ in Fig. 6(b). As half-integer steps increase, the Bessel-like form of ΔF_1 and F_c is completely deformed and new maxima start to develop.

(iii) Behavior in the presence of large half-integer steps for $r = 0.2$ in Fig. 6(c). The harmonic step and critical depinning force oscillate, however, contrary to the standard case ($r = 0$), the maxima (minima) of one curve corresponds to the maxima (minima) of another [in Fig. 6(d) oscillations start to disappear as system is approaching to the low-amplitude limit].

These three types of behavior were first observed and classified in the experimental studies of amplitude dependence of the Shapiro steps in the high-T_c grain-boundary junctions [6]. Dissipative dynamics of the Frenkel-Kontorova model is closely related to the many phenomena observed in the Josephson-junction arrays. These systems are often described by the system of equations usually referred to as a discrete sine-Gordon model that are actually equations of motion of the driven FK model [12]. In our previous works on the amplitude dependence of the subharmonic Shapiro steps in the ac-driven FK model [24], we not only have observed the same types of behavior but also have given a detailed explanation of the physical processes behind the results. The great analogy between results in Fig. 6 and our previous results for amplitude dependence [17] as well as those experiments [6] clearly proves that the amplitude and the frequency play the same role in the ac-driven dynamics. This work together with our works on amplitude dependence [17] provides a detail presentation of the dynamical mode locking phenomena in the FK model with asymmetric deformable substrate potential. Since we have considered only one particular type of potential, in order to get a complete answer about behavior of the Shapiro steps in realistic systems, other types of substrate potentials also have to be studied. These problems will be part of our future examinations.

IV. CONCLUSION

In this paper we have presented the studies of the frequency dependence of the Shapiro steps in the ac-driven Frenkel-Kontorova model with a deformable substrate potential. As the potential becomes deformed, with the appearance of subharmonic steps the behavior of harmonic steps begins to change. While both harmonic and subharmonic steps increase and after reaching their maxima disappear at the high frequencies, in the high-amplitude limit they exhibit oscillatory dependence at the low frequencies. When expressed as a function of period these oscillations have the same form as in the case of the amplitude dependence. The steps are strongly correlated and the appearance of large half-integer steps will cause deviation from the standard behavior of harmonic steps. As deformation changes, we could observe how the frequency (period) dependence of the harmonic steps and the critical depinning force evolves and classify three distinctive types of behavior. In the presence of large half-integer steps, in the oscillatory dependence of harmonic steps and the critical depinning force, new maxima develop where, contrary to the standard case, now the maxima (minima) of one curve correspond to the maxima (minima) of another. The great analogy of the above results with the results obtained in the theoretical [17] and experimental [6] studies of the amplitude dependence clearly prove that the period (frequency) of the ac force plays the same role as the amplitude in the dynamical mode locking phenomena.

Presented results could be important for studies of charge- or spin-density waves systems and systems of Josephson-junction arrays that are particularly motivated by technical applications of the interference phenomena [1,12]. Any fabrication of synchronization and superconducting Shapiro step devices requires a theoretical guideline for the observation of Shapiro steps. Our aim was not to favor or criticize any of the existing theoretical approaches. The results that we have obtained are universal and should be observed in any system and irrespectively of the number of degrees freedom. Our studies of frequency dependence of the Shapiro steps in realistic models could not only contribute to the understanding of their behavior in real systems but also bring insight to the theory of Shapiro steps and the continuing debate about their frequency dependence.

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