Solitonic propagation and interaction for a generalized variable-coefficient forced Korteweg–de Vries equation in fluids

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Under investigation is a generalized variable-coefficient forced Korteweg–de Vries equation in fluids and other fields. From the bilinear form of such equation, the N-soliton solution and a type of analytic solution are constructed with symbolic computation. Analytic analysis indicates that: (1) dispersive and dissipative coefficients affect the solitonic velocity; (2) external-force term affects the solitonic velocity and background; (3) line-damping coefficient and some parameters affect the solitonic velocity, background, and amplitude. Solitonic propagation and interaction can be regarded as the combination of the effects of various variable coefficients. According to a constraint among the nonlinear, dispersive, and line-damping coefficients in this paper, the possible applications of our results in the real world are also discussed in three aspects, i.e., solution with the constraint, solution without the constraint, and approximate solution.

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I. INTRODUCTION

With the inhomogeneities of media and boundaries taken into account, the variable-coefficient Korteweg–de Vries (KdV)-typed equations can be obtained [1–15]. Hereby, with the aid of symbolic computation [14–17], we will investigate the following generalized variable-coefficient forced KdV equation [2–11]:

$$u_t + a(t)\,u\,u_x + b(t)\,u_{xxx} + c(t)\,u + d(t)\,u_x = f(t) \qquad (1)$$

where u is a function of the scaled "space" coordinate x and "time" coordinate t, and $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $f(t)$ are the analytic functions of t, representing the nonlinear, dispersive, line-damping, dissipative, and external-force coefficients, respectively [2–11].

With different forms of the coefficients, Eq. (1) has been seen to describe the nonlinear waves in a fluid-filled tube [2–4], weakly nonlinear waves in the water of variable depth [5,6], trapped quasi-one-dimensional Bose-Einstein condensates [7], internal gravity waves in lakes with changing cross sections [8], the formation of a trailing shelf behind a slowly-varying solitary wave [9], dynamics of a circular rod composed of a general compressible hyperelastic material with the variable cross sections and material density [10], and atmospheric and oceanic dynamical systems [11]. Variable coefficients of Eq. (1) are caused by the geometrical and physical inhomogeneities, e.g., varying radius, material density, and so on [2–11]. Some special cases of Eq. (1) in fluids will be discussed later, which are the one from the nonlinear inviscid barotropic nondivergent vorticity equation [11], Eq. (24) from an inviscid fluid in a fluid-filled tube [3,4], and Eq. (26) from the water of variable depth [6].

In Ref. [18], Eq. (1) has been transformed into a variable-coefficient KdV model just with the nonlinear and dispersive terms, based on which the Bäcklund transformation and soliton solution in the Wronskian form have been derived. In Ref. [19], Eq. (1) has been transformed into the cylindrical and standard KdV equations with symbolic computation. In Ref. [20], two Miura transformations from Eq. (1) to a modified KdV equation have been constructed, through which the auto-Bäcklund transformations, nonlinear superposition formulas, Lax pairs, and soliton solutions have been presented. In Ref. [21], a special case of Eq. (1), i.e., Eq. (24), has been transformed into a potential KdV equation and solved.

However, to our knowledge, the N-soliton solutions for Eq. (1) in the explicit bilinear forms have not been obtained directly and the features of the solitonic propagation and collision, caused by the variable coefficients, have not been discussed. In this paper, we will investigate Eq. (1) under the constraint

$$a(t) = \frac{6\,b(t)}{\rho}\, e^{\int c(t)dt}, \qquad (2)$$

where ρ is a nonzero constant. In Sec. II, a dependent variable transformation will be proposed, Eq. (1) will be transformed into its bilinear form, and the N-soliton solutions in the explicit forms will be constructed, from which the characteristic-line method [22,23] will be employed to investigate the effects of the variable coefficients on the solitonic velocity, amplitude and background. In Sec. III, a type of analytic solution will be obtained when another kind of solution form is substituted into the bilinear equation. Section IV will show the originality of our results and the possible applications in the real world. Finally, Sec. V will present the conclusions.

II. SOLITON SOLUTIONS

Through the dependent variable transformation

$$u = 2\rho e^{-\int c(t)dt}\left[(\log\Phi)_{xx} + \frac{1}{2\rho}\int e^{\int c(t)dt} f(t)dt + \chi\right], \quad (3)$$

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where Φ is a real function of x and t, and χ is an arbitrary constant, the bilinear equation of Eq. (1) turns out to be the following form:

$$\left\{ D_x D_t + b(t)\, D_x^4 + \left[d(t) + \frac{6b(t)}{\rho} \int e^{\int c(t)dt} f(t)dt \right.\right.$$
$$\left.\left. +12\chi\, b(t) \right] D_x^2 \right\} \Phi \cdot \Phi = 0\,, \tag{4}$$

where $D_x^m D_t^n$ is the Hirota bilinear derivative operator [24,25] defined by

$$D_x^m D_t^n a \cdot b$$
$$\equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x,t)\, b(x',t') \bigg|_{x'=x,\, t'=t}. \tag{5}$$

We expand Φ in the power series of a small parameter ϵ as

$$\Phi = 1 + \epsilon\, \Phi_1 + \epsilon^2\, \Phi_2 + \cdots . \tag{6}$$

Substituting Expansion (6) into Eq. (4) and collecting the coefficients of the same power of ϵ, through the standard process of the Hirota bilinear method, we can derive the N-soliton solutions for Eq. (1), which can be denoted as

$$u = 2\,\rho\, e^{-\int c(t)dt} \left\{ \frac{\partial^2}{\partial x^2} \left\{ \log \left[\sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j\, \xi_j \right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \sum_{1\leqslant j<l}^{N} \mu_j\, \mu_l\, A_{jl} \right) \right] \right\} + \frac{1}{2\,\rho} \int e^{\int c(t)dt} f(t)dt + \chi \right\}, \tag{7}$$

with

$$\xi_j = k_j\, x + \omega_j(t) + \xi_j^0\,, \tag{8}$$

$$\omega_j(t) = -k_j^3 \int b(t)dt - k_j$$
$$\times \int \left[d(t) + \frac{6b(t)}{\rho} \int e^{\int c(t)dt} f(t)dt + 12\chi\, b(t) \right] dt, \tag{9}$$

$$e^{A_{jl}} = \frac{(k_j - k_l)^2}{(k_j + k_l)^2}\,, \tag{10}$$

where k_j and ξ_j^0 $(j = 1,2,\ldots,N)$ are arbitrary real constants, $\sum_{\mu=0,1}$ is a summation over all possible combinations of $\mu_1 = 0,1$, $\mu_2 = 0,1$, $\ldots$, $\mu_N = 0,1$, and $\sum_{1\leqslant j<l}^{N}$ means a summation over all possible pairs (j,l) chosen from the set $(1,2,\ldots,N)$, with the condition that $1 \leqslant j < l$ [25].

Next, we will apply the characteristic-line method [22] to discuss the effects of the coefficients on the solitonic propagation and interaction, and express the velocity v_j of each solitary wave as

$$v_j = k_j^2\, b(t) + d(t) + \frac{6b(t)}{\rho} \int e^{\int c(t)dt} f(t)dt$$
$$+ 12\chi b(t)\, (j = 1,2,\ldots,N)\,, \tag{11}$$

the sign and absolute value of which control the solitary moving direction and speed, respectively [23]. As shown in Figs. 1(a) and 1(b), the initial superposed solitons with the different velocities are separated over a period of time for different $b(t)$ and $d(t)$, i.e., the dispersive and dissipative coefficients affecting the solitonic velocity. As shown in Fig. 1(c), within the same time, the initial superposed solitons with different $f(t)$ finally travel different distances along x direction and rise to different levels, i.e., the external-force term affecting the solitonic velocity and the position of background. For the concept of background, the similar terminology has appeared in Ref. [26].

Effect of the line-damping coefficient is illustrated in Figs. 2(a) and 2(b). When $f(t) = 0$, from Fig. 2(a), we can observe that $c(t)$ has only influence on the solitonic amplitude. However, when $f(t) \neq 0$, as shown in Fig. 2(b), the solitonic background and amplitude can be affected obviously by $c(t)$. Expression (11) can further show that the velocity has also changed. In conclusion, the effects of the various variable coefficients have been listed in Table I.

Besides the variable coefficients discussed above, there also exist two parameters ρ and χ in Expression (7), the effects of which can be discussed in the similar manner. Relation (2) indicates that the influence of ρ can be considered as one part of the nonlinear coefficient. Moreover, as shown in Ref. [25], there only exist the overtaking solitons for the constant-coefficient KdV equation,

$$u_t + 6\, u\, u_x + u_{xxx} = 0\,. \tag{12}$$

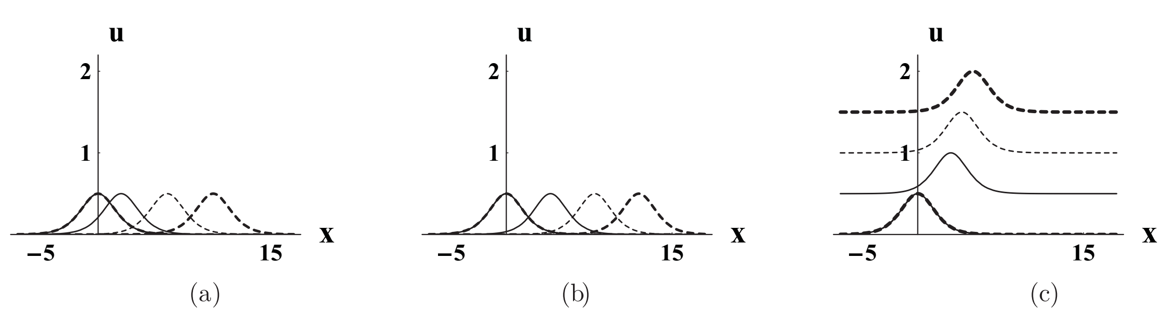


FIG. 1. Evolution plots of the one-soliton solution given by Expression (7) with parameters $\rho = 1$, $\chi = 0$, $k_1 = 1$, $c(t) = 0$, $\xi_1^0 = 0$, and (a) $d(t) = 0$, $f(t) = 0$, $b(t) = \tau\, t$, $\tau = 1,3,5$, respectively, $t = 0$ (superposed line), $\tau = 1, t = 2$ (solid line), $\tau = 3, t = 2$ (dashed line), $\tau = 5, t = 2$ (bold dashed line); (b) $b(t) = 1$, $f(t) = 0$, $d(t) = \tau\, t$, $\tau = 1,3,5$, respectively, $t = 0$ (superposed line), $\tau = 1, t = 2$ (solid line), $\tau = 3, t = 2$ (dashed line), $\tau = 5, t = 2$ (bold dashed line); (c) $b(t) = d(t) = 1$, $f(t) = \tau\, t$, $\tau = 1,2,3$, respectively, $t = 0$ (superposed line), $\tau = 1, t = 1$ (solid line), $\tau = 2, t = 1$ (dashed line), $\tau = 3, t = 1$ (bold dashed line).

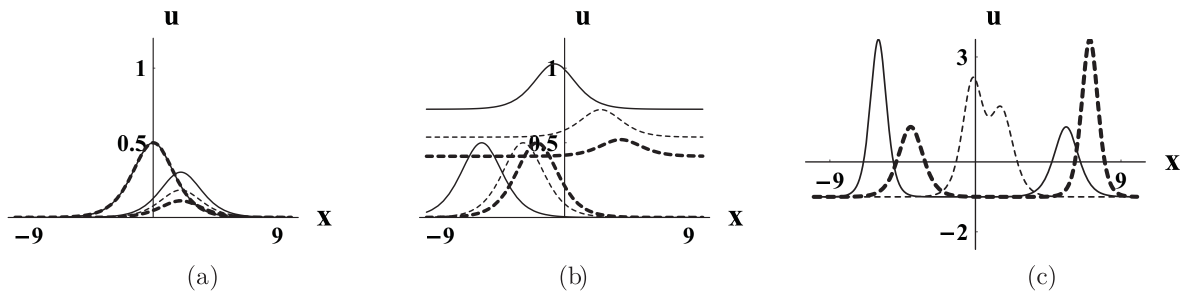


FIG. 2. Evolution plots of the soliton solution given by Expression (7) with parameters (a) $\rho = 1$, $\chi = 0$, $k_1 = 1$, $\xi_1^0 = 0$, $b(t) = d(t) = 1$, $f(t) = 0$, $c(t) = \tau\, t$, $\tau = 1,2,3$, respectively, $t = 0$ (superposed line), $\tau = 1$, $t = 1$ (solid line), $\tau = 3$, $t = 1$ (dashed line), $\tau = 5$, $t = 1$ (bold dashed line); (b) $\rho = 1$, $\chi = 0$, $k_1 = 1$, $\xi_1^0 = 0$, $b(t) = d(t) = 1$, $f(t) = 1$, $c(t) = \tau\, t$, $\tau = 1$, $t = 0$ (left solid line), $\tau = 2$, $t = 0$ (left dashed line), $\tau = 3$, $t = 0$ (left bold dashed line), $\tau = 1$, $t = 1$ (right solid line), $\tau = 2$, $t = 1$ (right dashed line), $\tau = 3$, $t = 1$ (right bold dashed line); (c) $\rho = 1$, $\chi = -0.5$, $\xi_1^0 = \xi_2^0 = 0$, $k_1 = 2$, $k_2 = 3$, $b(t) = 1$, $c(t) = d(t) = f(t) = 0$, $t = -2$ (solid line), $t = 0$ (dashed line), $t = 2$ (bold dashed line).

However, without consideration on the influences of the variable coefficients, i.e., $b(t) = 1$ and $c(t) = d(t) = f(t) = 0$, and for the existence of parameter χ in Expression (7), the head-on solitons for Eq. (12) could also be derived, which has been illustrated in Fig. 2(c).

In our opinion, the precise appearance of solitons for Eq. (1) includes velocity, amplitude, and background. The total height of solitons can be seen as the superposition of the amplitude and background, just like the case that one stands on the floor. In Fig. 3, three types of two-soliton interactions are illustrated. Figure 3(a) shows that the solitonic characteristic line, amplitude, and background are all periodic. However, the forms of background in Figs. 3(b) and 3(c), i.e., the forms of *floor*, are taken as $0.1t^2e^{-\sin(2t)}$ and shockwave-like shape, respectively. We can present more solitonic structures with the different velocity, amplitude, and background, by choosing the different variable coefficients.

III. ANOTHER KIND OF ANALYTIC SOLUTIONS

Solutions for Eq. (1) can be also assumed to be in the form [27]

$$\Phi = \sigma_1\, e^{Qx+\beta(t)} + \sigma_2\, \cos[Px - \alpha(t)] + \sigma_3\, e^{-Qx-\beta(t)}, \quad (13)$$

where P, Q, σ_1, σ_2, and σ_3 are arbitrary constants, and $\alpha(t)$ and $\beta(t)$ are the functions of t. Taking

$$n(t) = d(t) + \frac{6b(t)}{\rho} \int e^{\int c(t)dt} f(t)dt + 12\chi\, b(t), \quad (14)$$

substituting Expression (13) into Eq. (4) and collecting the coefficients of different terms, we can obtain three ordinary differential equations:

$$(P^4 - 6P^2Q^2 + Q^4)\, b(t) + (-P^2 + Q^2)\, n(t) + P\,\alpha'(t) + Q\,\beta'(t) = 0, \quad (15)$$

$$(-4P^3Q + 4PQ^3)\, b(t) + 2PQ\, n(t) + P\,\beta'(t) - Q\,\alpha'(t) = 0, \quad (16)$$

$$\left(-2P^2\sigma_2^2 + 8Q^2\sigma_1\sigma_3\right) n(t) + 8\left(P^4\sigma_2^2 + 4Q^4\sigma_1\sigma_3\right) b(t) + 8Q\sigma_1\sigma_3\beta'(t) + 2P\sigma_2^2\alpha'(t) = 0. \quad (17)$$

Solving Eqs. (15)–(17) gives the solution of Eq. (1):

$$\alpha(t) = \int [-P^3\, b(t) + 3P\, Q^2\, b(t) + P\, n(t)]dt, \quad (18)$$

$$\beta(t) = \int [-Q^3\, b(t) + 3P^2\, Q\, b(t) - Q\, n(t)]dt, \quad (19)$$

$$\sigma_2 = 2\,i\,\frac{Q}{P}\sqrt{\sigma_1\sigma_3}. \quad (20)$$

For such types of solutions, effects of the variable coefficients can be discussed similarly to those in Sec. II. Moreover, based on Expressions (18)–(20), we will take different values of P, Q, σ_1, and σ_3 to obtain different structures. As shown in Figs. 4 and 5, three types of structures are illustrated, and the function Φ in Expression (13) has the following form, correspondingly:

$$\Phi = \gamma_1 \cosh[\alpha_1 x + \beta_1(t)] + \gamma_2 \cosh[\alpha_2 x + \beta_2(t)], \quad (21)$$

$$\Phi = \gamma_3 \sinh[\alpha_3 x + \beta_3(t)] + \gamma_4 \cos[\alpha_4 x + \beta_4(t)], \quad (22)$$

and

$$\Phi = \gamma_5 \cos[\alpha_5 x + \beta_5(t)] + \gamma_6 \sin[\alpha_6 x + \beta_6(t)], \quad (23)$$

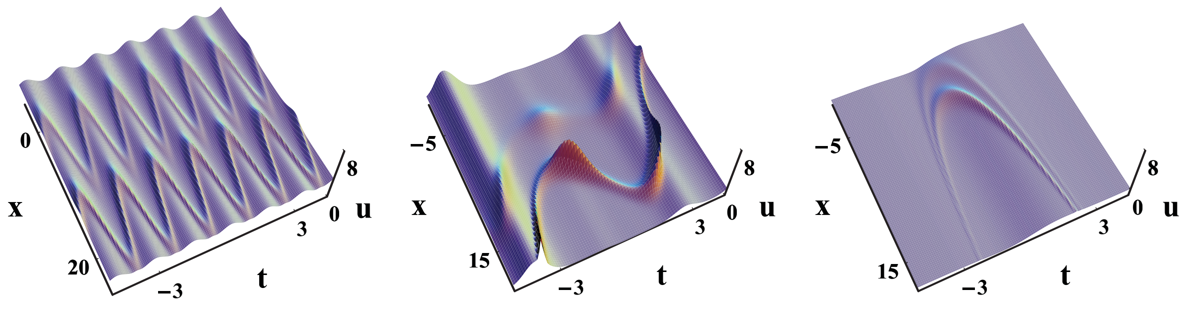
where α_j, β_j, and γ_j $(j = 1,2,\ldots,6)$ can be calculated by Expressions (18)–(20).

IV. DISCUSSIONS

In this section, we will show the originality of our results compared with Refs. [18–21,28], and possible applications in the real world.

TABLE I. Effects of the various variable coefficients.

Dispersive term	Line-damping term	Dissipative term	External-force term
Velocity	Velocity, amplitude, background	Velocity	Velocity, background



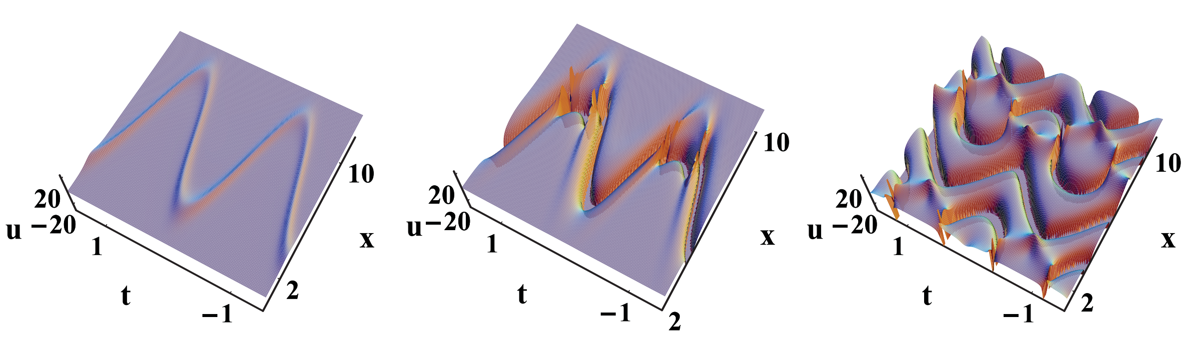
(a) (b) (c)

FIG. 3. (Color online) Evolution plots of the two-soliton solution given by Expression (7) with parameters (a) $\rho = 1$, $\chi = 1$, $k_1 = 1$, $k_2 = 1.2$, $\xi_1^0 = 0$, $\xi_2^0 = -10$, $b(t) = \sin(4t)$, $c(t) = \sin(4t)$, $d(t) = 0$, and $f(t) = \sin(4t)$; (b) $\rho = 1$, $\chi = 0$, $k_1 = 1$, $k_2 = 2$, $\xi_1^0 = 0$, $\xi_2^0 = -20$, $b(t) = \cos(t)$, $c(t) = 2\cos(2t)$, $d(t) = \sin(2t)$, and $f(t) = 0.2te^{-\sin(2t)}$; (c) $\rho = 1$, $\chi = 0$, $k_1 = 1$, $k_2 = -1.5$, $\xi_1^0 = 0$, $\xi_2^0 = 0$, $b(t) = 1$, $c(t) = t$, $d(t) = 1$, and $f(t) = 1$.

From the viewpoint of mathematics, the main difficulty of directly solving Eq. (1) comes from the external-force term [18–21]. Reference [28] has transformed a forced and damped KdV equation into its bilinear form under the logarithmic transformation $u = 2(\log\Phi)_{xx}$. In our opinion, under the logarithmic transformation, although there exists the bilinear "form," models like Eq. (1) are difficult to be solved analytically. Actually, Ref. [28] does not solve the bilinear "form" directly, but gives the Bäcklund transformation and Lax pair only. To avoid such problem, Refs. [18–20] have transformed Eq. (1) into its simpler counterparts to investigate its properties indirectly. Recently, Ref. [21] has solved a special case of Eq. (1), i.e., Eq. (24), by transforming it into a potential KdV equation without the external-force term, compared with which our results are more generalized. Therefore, only under the transformation with the line-damping and external-force terms considered, e.g., Transformation (3), Eq. (1) can be transformed into a directly solvable bilinear form. Moreover, compared with Refs. [18–21], besides the mathematical calculation, we have also made a discussion on the features of the solitonic propagation and collision. Effects of each coefficient are concluded as: (i) dispersive coefficient $b(t)$ and dissipative coefficient $d(t)$ affect the solitonic velocity; (ii) external-force term $f(t)$ affects the solitonic velocity and background; (iii) line-damping coefficient $c(t)$ affects the solitonic velocity, background and amplitude.

However, from the viewpoint of physics, Constraint (2) among the nonlinear, dispersive, and line-damping coefficients mainly limits the applications of solutions. Constant coefficients in Eq. (12) satisfy Constraint (2) spontaneously. Note that the "time"-dependent dissipative coefficient has no influence on Constraint (2) or Transformation (3), which indicates that it is easier to be dealt with than other ones. Constraint (2) is necessary to balance the nonlinearity and dispersion to generate solitons, and the similar conditions exist in Refs. [18–20], also for the variable-coefficient modified KdV equation [29,30] and Kadomtsev-Petviashvili equation [31,32]. Based on Constraint (2), we will discuss the possible applications in the following three aspects:

Case A. Models with the variable nonlinear, dispersive, and line-damping coefficients [8–11] can be solved under Constraint (2). Realization of Constraint (2) depends on the geometrical and/or physical balance [8–11]. Thereinto, Ref. [11] has derived a special case of Eq. (1) from the nonlinear inviscid barotropic nondivergent vorticity equation in a β plane by means of the multiscale expansion method in



(a) (b) (c)

FIG. 4. (Color online) Evolution plots of the analytic solution given by Expression (13) with parameters $\rho = 1$, $\chi = 0$, $b(t) = \sin(4t)$, $c(t) = \sin(4t)$, $d(t) = 0$, $f(t) = \sin(4t)$, and (a) $P = i$, $Q = 2$, $\sigma_1 = 1$, and $\sigma_3 = 1$; (b) $P = 1$, $Q = 2$, $\sigma_1 = 1$, and $\sigma_3 = -1$; (c) $P = 1$, $Q = 2i$, $\sigma_1 = 1$, and $\sigma_3 = 1$.

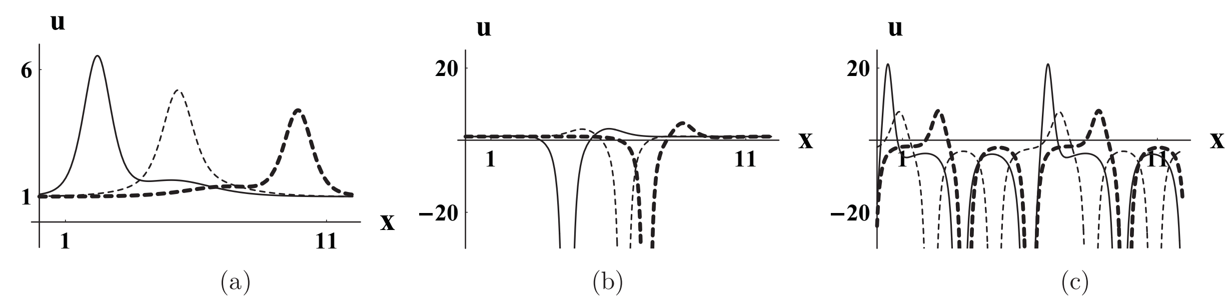


FIG. 5. (Color online) Profiles of the waves shown in Fig. 4 at $t = 0$ (solid line), $t = 0.35$ (dashed line), $t = 0.7$ (bold dashed line). (a) shows the two-soliton structure, while Figs. 5(b) and 5(c) have some singularities.

two different ways, with and without the so-called y-average trick. That case presents some special forms of all the five variable coefficients investigated in this paper, and might generate the similar solitonic structures to those shown in Secs. II and III. For example, to generate the structure similar to that presented in Fig. 3(b), we predict that the line damping $c(t)$ there should be periodic. The external-force term in Fig. 3(b) is assumed to have a special form. Actually, the form of the external-force term has no influence on the integrability, but leads to the form of solitonic background, which means that any form of the external-force term is allowed. Therefore, with the parameters in a real system to satisfy Constraint (2), the structure similar to that presented in Fig. 3(b) might exist.

Case B. Sometimes, Constraint (2) can be satisfied spontaneously. For example, if $c(t) = 0$ and the coefficients of the nonlinear and dispersive terms are constants, the following special case of Eq. (1) [3,4,21]:

$$u_t + a\, u\, u_x + b\, u_{xxx} + d(t)\, u_x = f(t)\,, \tag{24}$$

is completely integrable and can be solved directly without any extra condition. Equation (24) describes an inviscid fluid in a prestressed thin walled elastic tube in arterial mechanics [3]. Equation (24) also describes an incompressible inviscid fluid in an incompressible, isotropic thin elastic tube subjected to a variable initial stretches both in the axial and radial directions [4]. In such a case, for Solution (7), the solitonic amplitude keeps invariable, while the solitonic velocity and background depend on the forms of $d(t)$ and $f(t)$.

Case C. From the above analysis, we see that the line-damping coefficient $c(t)$ limits the more applications of solutions compared with the nonlinear coefficient $a(t)$ and dispersive coefficient $b(t)$. For expanding the application, we will also discuss the approximate solutions and consider the following special case of Eq. (1):

$$u_t + a\, u\, u_x + b\, u_{xxx} + \cos(\varphi\, t)\, u + d(t)\, u_x = f(t)\,, \tag{25}$$

where φ is a nonzero constant. The coefficients of Eq. (25) do not satisfy Constraint (2) strictly. But, if φ is big enough, we can find that $\frac{\sin(\varphi t)}{\varphi} \approx 0$ and $e^{\frac{\sin(\varphi t)}{\varphi}} \approx 1$. In such a case, Constraint (2) can be satisfied approximately and the similar structure shown in Fig. 2(b) might be observed. More approximate solutions could be obtained under other assumptions.

Reference [6] has presented another special case of Eq. (1) for the propagation of weakly nonlinear waves in the water of variable depth, with its variable coefficients not satisfying Constraint (2) in general, as below,

$$u_t + \frac{3}{2}\psi^3(t)\, u\, u_x + \frac{1}{6\psi(t)}\, u_{xxx} - \frac{1}{2\psi(t)}\frac{d\psi(t)}{dt}\, u = 0, \tag{26}$$

where $\psi(t)$ is a quantity connected with the profile at the bottom of the channel [6]. To satisfy Constraint (2), $\psi(t)$ must be a constant and Eq. (26) will not be a variable-coefficient one any more.

V. CONCLUSIONS

As a generalized variable-coefficient model in fluids and other fields [2–11], Eq. (1) has been investigated with symbolic computation. Under Constraint (2), Eq. (1) has been transformed into Bilinear Form (4) directly, based on which N-soliton Solution (7) has been constructed. In Sec. III, a type of analytic solution has also been obtained.

Solitonic propagation and interaction for Eq. (1) can be regarded as the combination of the effects of various variable coefficients, as shown in Figs. 1–5. Effects of the dispersive, line-damping, dissipative, and external-force terms on the solitonic velocity, amplitude and background have been summarized in Table I. Finally, according to Constraint (2), the possible applications of our results in the real world have been discussed.

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