Classical relativistic model for spin dependence in a magnetized electron gas

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The response of a cold electron gas is generalized to include the spin of the electron described by the relativistically correct quasiclassical Bargmann-Michel-Telegdi (BMT) equation. The magnetization of the electron gas is assumed to be along the background magnetic field **B** and the spin-dependent contribution to the response tensor is proportional to the magnitude of the magnetization. The dispersion equation is shown to be quadratic in the refractive index squared, and dispersion curves for the two wave modes are plotted for cases where the magnetic field associated with magnetization is comparable with **B**. Two intrinsically spin-dependent wave modes are identified: one bounded by two resonances and the other by two cutoffs. The counterpart of the *z* mode can escape without encountering a resonance or a cutoff.

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I. INTRODUCTION

There is extensive recent literature on spin dependence in plasmas, motivated partly by application to neutrino emission from the interiors of dense stars [1–6], partly by potential applications of nonlinear waves in quantum plasmas [7–9], and partly by more speculative suggested applications [10–14]. Spin dependence in a plasma introduces a new source of dispersion that is intrinsically quantum mechanical. Generally, inclusion of any additional source of dispersion in a medium leads to modification of the wave properties and may lead to some intrinsically new modes. Our purpose in this paper is to discuss how the properties of waves in a cold electron gas (the "magnetoionic" modes) are modified by the inclusion of spin dependence.

Several different quasiclassical models have been used to calculate the spin-dependent contribution to wave dispersion. In a quasiclassical treatment the spin, denoted by vector **s**, is treated as an intrinsic angular momentum associated with an electron. An equation of motion for **s** is introduced, the magnetic moment of the electron is related to **s** by the Bohr magneton, and the current density induced by an electromagnetic perturbation on **s** is included in calculating the response of the plasma, described by the dielectric tensor for example. In a nonrelativistic treatment, a perturbation in the magnetic field leads to a perturbation in the magnetization **M** (where **M** is the magnetic moment per unit volume) and the current density is identified as curl **M**. After Fourier transforming (introducing the frequency ω and wave vector $\mathbf{k}$ corresponding to the 4 vector $k^{\mu} = [\omega,\mathbf{k}]$), this leads to a spin-dependent contribution $\propto |\mathbf{k}|^2$ to the dielectric tensor. Some details are given in the Appendix. A relativistic treatment introduces additional terms $\propto (\omega/c)^2$, and these terms need to be included (even in a cold plasma) to treat the wave dispersion correctly, where c is the speed of light.

In this paper we describe the spin of the electrons using the relativistically correct Bargmann-Michel-Telegdi (BMT) equation [15]. The use of the BMT equation potentially resolves a difficulty in the comparison with a fully relativistic quantum treatment: the proper choice of spin operator. Unlike the Schrödinger-Pauli theory, where the spin is independent of the dynamics, in Dirac's theory the spin operator needs to be identified. Sokolov and Ternov [16,17] showed that there is only one choice of spin operator whose eigenvalues do not precess in a magnetic field (when radiative corrections are included), and this is the component of the magnetic moment operator along the magnetic field. We show that the BMT equation (with the radiative corrections included through $g - 2 \neq 0$) satisfies this requirement. This adds plausibility to the interpretation of the spin vector in the BMT equation as a classical counterpart of the spin operator identified by Sokolov and Ternov.

In the absence of spin (and of collisions), the magnetoionic plasma (a cold electron gas) is characterized by the electron density n_e and magnetic field B. These are incorporated into two frequencies, the plasma frequency $\omega_p \propto n_e^{1/2}$ and the electron cyclotron frequency $\Omega_e = eB/m_e$, where e is the fundamental charge and m_e is the rest mass of the electron. Magnetization of the plasma introduces a contribution $\mu_0\mathbf{M}$ to the magnetic field, where μ_0 is the permeability of free space. This introduces an additional frequency $\Omega_m = e\mu_0 M/m_e$ into the theory. We are interested in strongly magnetized plasmas $\omega_p \ll \Omega_e$, and find that the inclusion of the spin leads to interesting new effects when Ω_m/Ω_e is of order unity or greater. We note that this requires extreme conditions, due to the maximum value of Ω_m, when all the spins are aligned, being of order $\hbar\omega_p^2/m_e c^2$.

In Sec. II we write down the BMT equation and discuss its relevance in the present context. In Sec. III we derive the covariant form for the response tensor for cold electrons described by the BMT equation, and use it to write down the spin-dependent contribution to the dielectric tensor. In Sec. IV we extend the magnetoionic theory to include $\Omega_m \neq 0$. Examples of dispersion curves are plotted and discussed in Sec. V. Although there are only two wave modes at any frequency, as in the magnetoionic theory, two new branches of the wave modes appear, one limited at both low and high frequencies by resonances and the other limited at at both low and high frequencies by cutoffs.

The 4-tensor notation used here has greek indices with values $\mu = 0, 1, 2, 3,$ the metric tensor is diagonal with

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components 1, -1, -1, -1, and the inner product of two 4 vectors $a^\mu = [a^0, \mathbf{a}]$, $b^\mu = [b^0, \mathbf{b}]$ written $ab = a^0 b^0 - \mathbf{a}\cdot\mathbf{b}$. An electron is described by its 4 velocity $u^\mu = [\gamma, \gamma\mathbf{v}]$ in units with $c = 1$. The background magnetic field is described by the Maxwell tensor $F^{\mu\nu} = Bf^{\mu\nu}$, where B is the magnetic field in the rest frame of the plasma.

II. QUASICLASSICAL MODEL FOR SPIN: BMT EQUATION

In the simplest approach the magnetic moment of the electron is identified as $\mathbf{m} = g\mu_B\mathbf{s}$, with

$$\mu_B = \frac{e\hbar}{2m_e} = 9.274\times 10^{-24}\,\mathrm{J\,T^{-1}}, \qquad g = 2.00\,232, \tag{1}$$

where μ_B is the Bohr magneton and the gyromagnetic ratio g differs from 2 due to radiative corrections in quantum electrodynamics (QED). For an electron at rest, a classical form for the equation of motion of the spin is

$$\frac{d\mathbf{s}}{dt} = \frac{ge}{2m_e}\mathbf{s}\times\mathbf{B}. \tag{2}$$

A covariant generalization of the equation of motion (2) for the spin leads to the Bargmann-Michel-Telegdi (BMT) equation [15]. The spin vector $\mathbf{s}$ is interpreted as the space components of a 4 vector in the frame in which the electron is at rest. Writing $s^\mu = [s^0, \mathbf{s}]$ in an arbitrary frame, one assumes $s^0 = 0$ in the rest frame, and then $su = 0$ in the rest frame implies $\gamma(s^0 - \mathbf{s}\cdot\mathbf{v}) = 0$, and hence $s^0 = \mathbf{s}\cdot\mathbf{v}$ in an arbitrary frame. For an accelerated particle, in its instantaneous rest frame, one has $ds^0/dt = \mathbf{s}\cdot d\mathbf{v}/dt$, and together with (2), this determines ds^μ/dt in the instantaneous rest frame. This results in the BMT equation

$$\begin{aligned}\frac{ds^\mu}{d\tau} &= -\frac{e}{m_e}\left[\tfrac{1}{2}gF^{\mu\nu}s_\nu + \left(\tfrac{1}{2}g - 1\right)s_\alpha F^{\alpha\beta}u_\beta u^\mu\right],\\ \frac{du^\mu}{d\tau} &= -\frac{e}{m_e}F^{\mu\nu}u_\nu,\end{aligned} \tag{3}$$

with $d\tau = dt/\gamma$, where τ is the proper time. In this model the spin does not affect the dynamics, in the sense that there is no term corresponding to the force associated with the gradient of the magnetic energy $-\frac{1}{2}g\mu_B\mathbf{s}\cdot\mathbf{B}$.

A covariant form of the magnetic moment is in terms of the second rank 4 tensor

$$m^{\mu\nu} = -\tfrac{1}{2}g\mu_B\epsilon^{\mu\nu\alpha\beta}s_\alpha u_\beta, \tag{4}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric pseudotensor with $\epsilon^{0123} = 1$. We choose the 3 axis along $\mathbf{B}$. The rest frame corresponds to $u^\beta = [1, \mathbf{0}]$, and with the spin along the direction of $\mathbf{B}$, the only nonzero components of $m^{\mu\nu}$ in this frame are $m^{12} = -m^{21} = \frac{1}{2}g\mu_B s$. Equations (3) and (4) imply $dm^{\mu\nu}/d\tau = 0$. This conservation law also applies when the radiative correction $g - 2 \neq 0$ is included. This suggests that $m^{\mu\nu}$ may be interpreted as a classical counterpart of the magnetic moment operator introduced by Sokolov and Ternov [16,17].

The 4 magnetization of the electron gas is $M^{\mu\nu} = n_e m^{\mu\nu}$. The assumption that the electron gas is magnetized implies that there is a nonzero mean spin, denoted $\bar{s}^\mu$. Let the average magnetization be $M^{\mu\nu} = n_e\bar{m}^{\mu\nu}$, with $\bar{m}^{\mu\nu} = g\mu_B\epsilon^{\mu\nu\alpha\beta}\bar{s}_\alpha\bar{u}_\beta$, where an overbar denotes an average value. In the rest frame of the cold electron gas one has $\bar{u}^\mu = [1, \mathbf{0}]$, $\bar{s}^\mu = [0, \mathbf{s}]$, implying a 3 magnetization $\mathbf{M} = g\mu_B n_e\bar{\mathbf{s}}$ and zero polarization $\mathbf{P}$ which is the induced electric dipole moment per unit volume. A notable feature of our treatment based on the BMT equation is that there is a perturbation in $\mathbf{P}$ in the BMT treatment, implying that the medium is magnetoelectric, whereas there is no such effect in a nonrelativistic treatment.

III. LINEAR RESPONSE 4 TENSOR

In this section we use a covariant formalism to derive the response tensor for a cold distribution of electrons that obey the BMT equation. The linear response tensor in the absence of spin may be found by linearizing and Fourier transforming the second of Eqs. (3) to find the perturbation $u^{(1)}_\mu(k)$ in the 4 velocity and applying the same procedure to the continuity equation to find the perturbation in the number density $n^{(1)}_e(k)$ with the 4 current given by $J^{(1)}_\mu(k) = -e[n_e u^{(1)}_\mu + n^{(1)}_e(k)\bar{u}_\mu]$. We do not write down the resulting covariant form for the cold plasma response tensor, but we do include its contribution in deriving the dispersion relations in the next section. Here we apply the same procedure to the first of Eqs. (3) to find the perturbation $s^{(1)}_\mu(k)$ in the spin. In the model used here, the spin does not affect the dynamics, and hence there is no spin-related perturbation in the electron number density n_e.

The perturbations in the spin and 4 velocity lead to a perturbation

$$M^{(1)\mu\nu}(k) = -\mu_B n_e\epsilon^{\mu\nu\alpha\beta}\left[s^{(1)}_\alpha(k)\bar{u}_\beta + \bar{s}_\alpha u^{(1)}_\beta(k)\right] \tag{5}$$

in the magnetization. The associated 4 current is $J^{(1)\mu}(k) = ik_\nu M^{(1)\mu\nu}(k)$, and writing this in the form

$$ik_\rho M^{(1)\mu\rho}(k) = \Pi^{\mu\nu}_m(k)A_\nu(k) \tag{6}$$

defines the spin-dependent contribution $\Pi^{\mu\nu}_m(k)$ to the response tensor.

The conventional cold-plasma response can be derived in covariant form using fluid theory. The perturbation in the electron 4 velocity follows from the equation of motion and is

$$u^{(1)\mu}(k) = \frac{e}{m_e k\bar{u}}[k\bar{u}\,\tau^{\mu\nu}(k\bar{u}) - k_\rho\tau^{\mu\rho}(k\bar{u})\bar{u}^\nu]A_\nu(k), \tag{7}$$

where the fluctuating electromagnetic field is described by its 4 potential $A^\mu(k)$, and with

$$\tau^{\mu\nu}(\omega) = g^{\mu\nu}_\parallel + \frac{\omega}{\omega^2 - \Omega_e^2}(\omega g^{\mu\nu}_\perp - i\Omega_e f^{\mu\nu}), \tag{8}$$

with $g^{\mu\nu}_\parallel$ and $g^{\mu\nu}_\perp$ diagonal with components 1,0,0,-1 and 0, -1,-1,0 respectively. The analogous perturbation in the spin 4 vector follows from (3). For simplicity we neglect the term due to the radiative correction, which involves setting $g - 2 \to 0$. To be consistent the frequency of precession of the spin is then not distinguished from the cyclotron frequency Ω_e. The perturbation in the spin becomes

$$s^{(1)\mu}(k) = \frac{e}{m_e k\bar{u}}[k\bar{s}\,\tau^{\mu\nu}(k\bar{u}) - k_\rho\tau^{\mu\rho}(k\bar{u})\bar{s}^\nu]A_\nu(k). \tag{9}$$

Explicit evaluation of the contribution to the response tensor gives

$$\Pi_m^{\mu\nu}(k) = -\frac{ien_e}{m_e k\bar{u}} k_\rho \epsilon^{\mu\rho}{}_{\alpha\beta}\{[k\bar{s}\,\tau^{\alpha\nu}(k\bar{u}) - k_\tau \tau^{\alpha\tau}(k\bar{u})\bar{s}^\nu]\bar{u}^\beta + [k\bar{u}\,\tau^{\beta\nu}(k\bar{u}) - k_\tau \tau^{\beta\tau}(k\bar{u})\bar{u}^\nu]\bar{s}^\alpha\}, \qquad (10)$$

with $\tau^{\mu\nu}$ given by (8).

The covariant form (10) applies to a collection of electrons at rest in the frame moving with 4 velocity $\bar{u}^\mu$. One can reinterpret (10) in a way that allows one to include an arbitrary distribution of particles in parallel velocity v_z. One replaces $\bar{u}$ by u, with $u^\mu = \gamma[1,0,0,v_z]$, $\gamma = 1/(1-v_z^2)^{1/2}$, and replaces n_e by the differential proper number density $dv_z\, g(v_z)/\gamma$, where $g(v_z)$ is the distribution function. After integrating over v_z, this generalization of (10) gives the magnetic moment contribution to the response tensor for the distribution of electrons. This model does not include the spiraling motion of the electrons, and the resulting response tensor applies in the small-gyroradius limit. We do not discuss this generalization further in the present paper.

A. Spin-dependent response in the rest frame

The spin-dependent contribution (10) to the response tensor simplifies considerably in the rest frame of the (cold) electron gas, when one has $\bar{u}^\mu = [1,\mathbf{0}]$, $\bar{s}^\mu = [0,\bar{s}\mathbf{b}]$, where $\mathbf{b} = (0,0,1)$ is a unit vector along the magnetic field. One then has $k\bar{u} = \omega$, $k\bar{s} = -k_z\bar{s}$, and the spin-dependent contribution to the response tensor is proportional to the magnetization $M = \mu_B n_e \bar{s}$.

In the rest frame (10) reduces to

$$\Pi_m^{\mu\nu}(k) = -\frac{eM}{m_e(\omega^2 - \Omega_e^2)}\begin{pmatrix} k_\perp^2\Omega_e & \omega k_\perp\Omega_e & i\omega^2 k_\perp & 0 \\ \omega k_\perp \Omega_e & (\omega^2 - k_z^2)\Omega_e & i(\omega^2 - k_z^2)\omega & k_\perp k_z \Omega_e \\ -i\omega^2 k_\perp & -i(\omega^2-k_z^2)\omega & (\omega^2 - k_z^2)\Omega_e & -i\omega k_\perp k_z \\ 0 & k_\perp k_z\Omega_e & i\omega k_\perp k_z & -k_\perp^2\Omega_e \end{pmatrix}. \qquad (11)$$

The spin-dependent contribution to the dielectric tensor is identified by noting that the dielectric tensor is the sum of the unit 3 tensor and the susceptibility 3 tensor. In translating between the 4-tensor components and the 3-tensor components, we note that the contravariant component Π_m^{ij} is equal to $-1/\omega^2\varepsilon_0$ times the ij component of the susceptibility 3 tensor. Thus the additional contribution to the ij component of the dielectric tensor due to the spin dependence follows from (11) by deleting the leading row and column and multiplying by $-1/\omega^2\varepsilon_0 = -\mu_0 c^2/\omega^2$. Reverting to ordinary units, this gives

$$[K_m]^i{}_j(k) = \frac{\Omega_m c^2}{\omega^2(\omega^2 - \Omega_e^2)}\begin{pmatrix} (\omega^2/c^2 - k_z^2)\Omega_e & i(\omega^2/c^2 - k_z^2)\omega & k_\perp k_z\Omega_e \\ -i(\omega^2/c^2 - k_z^2)\omega & (\omega^2/c^2 - k_z^2)\Omega_e & -ik_\perp k_z\omega \\ k_\perp k_z \Omega_e & ik_\perp k_z\omega & -k_\perp^2\Omega_e \end{pmatrix}, \qquad (12)$$

where the frequency associated with the magnetization is

$$\Omega_m = \frac{\mu_0 M}{B}\Omega_e = \frac{\hbar\bar{s}\omega_p^2}{m_e c^2}. \qquad (13)$$

The ratio Ω_m/ω_p is small except in dense, strongly magnetized plasmas, where the plasmon energy $\hbar\omega_p$ is of order the rest energy $m_e c^2$, and $\bar{s}$ is of order unity.

A strictly nonrelativistic version of relation (12) is given in the Appendix. It corresponds to replacing $\omega^2/c^2 - k_z^2$ with $-k_z^2$ in (12). This implies that the terms ω^2/c^2 are intrinsically relativistic.

IV. SPIN-DEPENDENT WAVE MODES

The addition of the spin-dependent contribution to the dielectric tensor (12) leads to a generalization of the magnetoionic modes. For arbitrary values of the ratios of ω_p, Ω_e, Ω_m we find that the dispersion equation is a quadratic equation for the square of the refractive index n^2, which is unexpected due to the components of the response tensor all depending on $\mathbf{k}$. This allows one to solve for the dispersion relations and to identify the cutoffs and resonances for arbitrary angles of propagation without making simplifying assumptions such as parallel [18] or perpendicular [19] propagation.

A. Dispersion equation

The dispersion equation, in the general case of oblique propagation for arbitrary values of ω_p, Ω_e, Ω_m, can written in a form similar to that used by Stix [20] for a cold plasma:

$$A'n^4 - B'n^2 + C' = 0, \qquad (14)$$

with the coefficients given by

$$\begin{aligned} A' &= (P\cos^2\theta + S'\sin^2\theta)\big[1 - A_1(1+\cos^2\theta) + \big(A_1^2 - A_2^2\big)\cos^2\theta\big], \\ B' &= PS'(1+\cos^2\theta) + (S'^2 - D'^2)(1-A_1)\sin^2\theta - 2P\cos^2\theta(S'A_1 - D'A_2), \\ C' &= P(S'^2 - D'^2). \end{aligned} \qquad (15)$$

The cold-plasma quantities S, D, P defined by Stix [20] are replaced by

$$S' = 1 - \frac{\omega_p^2 + \Omega_m\Omega_e}{\omega^2 - \Omega_e^2},$$
$$D' = -\frac{\omega_p^2\Omega_e + \Omega_m\omega^2}{\omega(\omega^2 - \Omega_e^2)},$$
$$P = 1 - \frac{\omega_p^2}{\omega^2}, \tag{16}$$

with the spin-dependent terms appearing through Ω_m in (16) and in

$$A_1 = \frac{\Omega_m\Omega_e}{\omega^2 - \Omega_e^2}, \qquad A_2 = \frac{\Omega_m\omega}{\omega^2 - \Omega_e^2}. \tag{17}$$

The solutions of the quadratic equation (14) give two modes,

$$n^2 = n_\pm^2, \qquad n_\pm^2 = \frac{B' \pm F'}{2A'}, \qquad F'^2 = B'^2 - 4A'C'. \tag{18}$$

These modes reduce to the magnetoionic modes for $\Omega_m \to 0$.

B. Cutoffs and resonances

Cutoffs ($n^2 = 0$) and resonances ($n^2 \to \infty$) occur at $C'/A' = 0$ and $A'/C' = 0$, respectively. For $\Omega_m \to 0$ there is one cutoff in the ordinary mode, at $\omega = \omega_p$, and two cutoffs in the extraordinary mode, at $\omega = \pm\frac{1}{2}\Omega_e + \frac{1}{2}(4\omega_p^2 + \Omega_e^2)^{1/2}$. There are resonances at $\omega^2 = \frac{1}{2}(\omega_p^2 + \Omega_e^2) \pm \frac{1}{2}[(\omega_p^2 + \Omega_e^2)^2 - 4\omega_p^2\Omega_e^2\cos^2\theta]^{1/2}$. These separate the magnetoionic modes into four branches, two for each of the ordinary and extraordinary modes, with stop bands (with $n^2 < 0$) between the resonance in the lower branch and the cutoff for the upper branch.

The cutoffs at $C' = 0$ correspond to $P = 0$ or $S'^2 - D'^2 = 0$. The cutoff at $P = 0$ is that in the ordinary mode at $\omega = \omega_p$, and is unaffected by $\Omega_m \neq 0$. The solutions of $S'^2 - D'^2 = 0$ are either $S' = \pm D'$, and these give

$$(\omega \mp \Omega_e)\left[\omega^2 \pm \omega(\Omega_e + \Omega_m) - \omega_p^2\right] = 0, \tag{19}$$

respectively. Although there is a solution at $\omega = \Omega_e$, the coefficients in (14) all diverge at $\omega = \Omega_e$, and the behavior of the dispersion curves near $\omega = \Omega_e$ needs further consideration. The other two cutoffs, at

$$\omega = \tfrac{1}{2}\left[(\Omega_e + \Omega_m)^2 + 4\omega_p^2\right]^{1/2} \pm \tfrac{1}{2}(\Omega_e + \Omega_m), \tag{20}$$

reduce to the cutoffs in the extraordinary mode for $\Omega_m \to 0$.

The resonances at $A' = 0$ satisfy

$$(P\cos^2\theta + S'\sin^2\theta)[1 - A_1(1 + \cos^2\theta) + (A_1^2 - A_2^2)\cos^2\theta] = 0. \tag{21}$$

The first factor in (21) implies resonances at

$$\omega^2 = \tfrac{1}{2}\left(\Omega_e^2 + \omega_p^2 + \Omega_e\Omega_m\sin^2\theta\right) \pm \tfrac{1}{2}\left[\left(\Omega_e^2 + \omega_p^2 + \Omega_e\Omega_m\sin^2\theta\right)^2 - 4\Omega_e^2\Omega_m^2\cos^2\theta\right]^{1/2}, \tag{22}$$

which reproduce the magnetoionic resonances for $\Omega_m = 0$. The second factor in (21) reduces to

$$(\omega^2 - \Omega_e^2)\left[\omega^2 - \Omega_e^2 - \Omega_e\Omega_m(1 + \cos^2\theta) - \Omega_m^2\cos^2\theta\right] = 0. \tag{23}$$

As already mentioned, the behavior near $\omega = \Omega_e$ needs to be treated separately, and the first factor in (23) is ignored for the present. The other solution of (23),

$$\omega^2 = \Omega_e^2 + \Omega_e\Omega_m(1 + \cos^2\theta) + \Omega_m^2\cos^2\theta, \tag{24}$$

is an intrinsically new resonance associated with $\Omega_m \neq 0$.

C. Dispersion near the cyclotron resonance

The refractive indices at the cyclotron frequency may be found by retaining only the terms $\propto 1/(\omega^2 - \Omega_e^2)^2$ in (14) with (15). The dispersion equation reduces to

$$n^2(n^2 - n_0^2)\sin^2\theta = 0,$$
$$n_0^2 = \frac{\Omega_m\Omega_e - \omega_p^2}{\Omega_e[\Omega_e(1 + \cos^2\theta) + \Omega_m\cos^2\theta]}. \tag{25}$$

It follows that, except for $\sin^2\theta = 0$, there is a cutoff at $\omega = \Omega_e$ in one mode, with the other mode satisfying $n^2 = n_0^2$, with $n_0^2 < 0$ for $\Omega_m\Omega_e < \omega_p^2$ and $n_0^2 > 0$ for $\Omega_m\Omega_e > \omega_p^2$. The result (25) has no obvious counterpart for $\Omega_m = 0$.

V. DISPERSION CURVES

Dispersion in a cold electron gas is traditionally represented by plots of the refractive indices for the two magnetoionic modes as a function of frequency at fixed angle θ or as a function of angle for fixed frequency (CMA diagram). The inclusion of spin dependence introduces an additional frequency, denoted as Ω_m here, and two parameters need to be specified to define the cold electron gas. We assume a strong magnetic field $\omega_p/\Omega_e = 0.2$ and plot dispersion curves for $\Omega_m/\Omega_e = 1.5$ for a range of angles θ. At low frequencies $\omega < \omega_p$ the magnetoionic modes corresponds to the whistler branch and the z-mode branch, and their properties are not changed substantially by inclusion of spin dependence. We concentrate here on the properties of the modes for $\omega > \omega_p$.

Dispersion curves for parallel propagation are plotted in Fig. 1. The mode corresponding to the solid curve has a cutoff, corresponding to the + sign in (20), and a resonance, given by (24), with a small stop band between them. The other mode has no cutoff or resonance for $\omega > \omega_p$. The case of parallel propagation is special and how the mode structure changes for a small but nonzero angle is illustrated in Fig. 2. The notable change is the addition of a resonance and cutoff, separated by a stop band, in the mode corresponding to the solid curve. The curves for $\theta = \pi/4$ are plotted in Fig. 3. The mode described by the solid curve has three branches. The low-frequency branch, at $\omega < \omega_p$, corresponds to the whistler mode. A second branch in bounded at both low and high frequencies by cutoffs, and a third branch in bounded at both low and high frequencies by resonances. The highest frequency branch extends from a cutoff to an asymptotic value $n^2 \to 1$ at $\omega \to \infty$. The other mode, described by the dashed curve, has a cutoff at $\omega < \omega_p$ and asymptotes more rapidly to $n^2 \to 1$ at $\omega \to \infty$ without encountering any resonance. The special case of perpendicular propagation is illustrated in Fig. 4. Dispersion curves for two additional angles are plotted in Fig. 5 to show

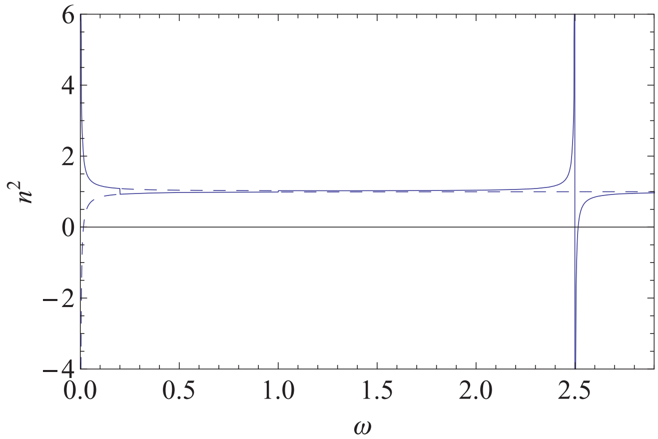

FIG. 1. (Color online) Dispersion curves are plotted for parallel propagation $\theta = 0$ for $\omega_p/\Omega_e = 0.2$ and $\Omega_m/\Omega_e = 1.5$. The two solutions of the dispersion equation are shown as the solid and dashed curves.

how the oblique case approaches the perpendicular case: the mode bounded by two resonances gets squeezed as the two resonances approach each other, and the curve disappears off the top of the diagram. In addition, the mode that experiences no resonances for oblique propagation develops a peak, which becomes resonant-like for $\theta \to \pi/2$.

We identify three intrinsically new features compared with the magnetoionic theory. Two of these are intrinsically new branches in the mode that corresponds to the whistler mode at low frequencies: a branch bounded by cutoffs at both low and high frequencies, and a branch bounded by resonances at both low and high frequencies. The third new feature is that the mode that corresponds to the z mode at low frequencies extends to high frequencies without encountering a cutoff or a resonance. (For $\Omega_m = 0$, the z mode encounters a resonance and a stop band before continuing to higher frequencies as the x mode.)

Care is required in labeling these modes. In the magnetoionic theory the whistler mode and z mode correspond to the low-frequency branches of the ordinary and extraordinary magnetoionic modes, respectively, and the high-frequency branches, labeled the o mode and x mode say, have $n_o^2 > n_x^2$.

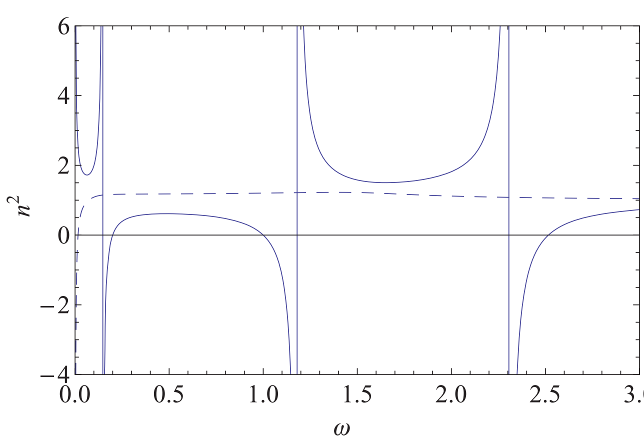

FIG. 2. (Color online) Same as Fig. 1 but for $\theta = \pi/6$.

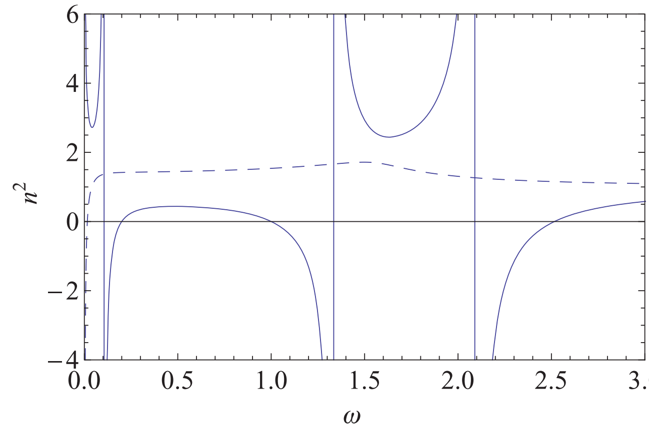

FIG. 3. (Color online) Same as Fig. 1 but for $\theta = \pi/4$.

The inclusion of Ω_m affects this labeling. The high-frequency branch that joins on continuously to the z mode for $\omega_p < \Omega_e < \Omega_m$ has the larger refractive index, and so would correspond to the o mode by analogy with the conventional labeling of the magnetoionic modes. The labeling of the $\pm$ solutions of a quadratic equation is changed by inclusion of an additional zero of the discriminant, and here this leads to the impossibility of making the labeling as "ordinary" and "extraordinary" consistent with magnetoionic theory at both high and low frequencies.

A. Comparison with existing results

The properties of waves in a spin-dependent plasma were discussed for the case of parallel propagation in Ref. [18]. Comparison with the results derived here is complicated by the neglect of relativistic effects in Ref. [18]. The nonrelativistic theory, as outlined in the Appendix, differs from the relativistically correct theory in that the factors $\omega^2/c^2 - k_z^2$ in the 11, 12, 21, and 22 components of the response tensor (12) being replaced by $-k_z^2$. Thus, the nonrelativistic theory is valid for parallel propagation only for $n^2 \gg 1$. Thus the nonrelativistic theory treats the resonances correctly, and the discussion in [18,21] of absorption at the cyclotron resonance

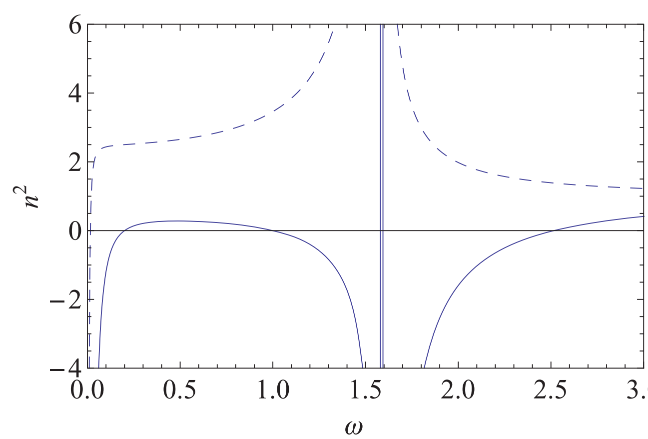

FIG. 4. (Color online) Same as Fig. 1 but for perpendicular propagation, i.e., $\theta = \pi/2$.

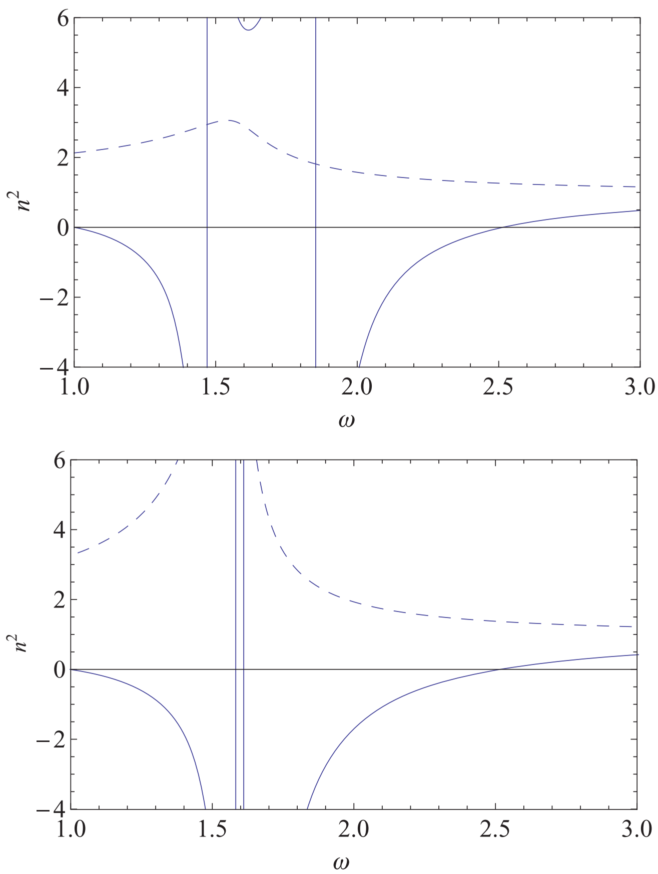

FIG. 5. (Color online) Same as Fig. 1 but for two angles approaching perpendicular propagation: upper $\theta = \pi/2 - \pi/6$, lower $\theta = \pi/2 - \pi/20$.

needs no change in the relativistic case. The nonrelativistic treatment is incorrect near cutoffs, and the cutoff frequencies (20) are different from those derived in Ref. [18].

The case of perpendicular propagation was discussed in Ref. [22], where use of a kinetic theory allowed the effect of nonzero gyroradii to be taken into account. An approximate dispersion relation was derived in the limit of zero gyroradii {Eq. (16) of [22]}, and this can be compared with the result derived here. For perpendicular propagation, the two solutions (18) reduce to $n^2 = P/(1 - A_1),(S'^2 - D'^2)/S'$. The former of these reproduces the result derived in [22] to lowest order in an expansion in $(\omega - \Omega_e)/\Omega_e$. Our results show that the mode structure for perpendicular propagation is special, and does not reflect the three new features discussed above for oblique propagation.

VI. DISCUSSION AND CONCLUSIONS

Our main objective in this paper is to generalize the theory of wave dispersion in a cold electron gas (the magnetoionic theory) to include spin dependence. The spin is treated quasiclassically using the (relativistically correct) BMT equation. The spin-dependent part of the dielectric tensor is proportional to the magnetization **M** of the electron gas, and this is incorporated into a natural frequency $\Omega_m = e\mu_0 M/m_e$. The only difference between the results derived for a cold plasma using the BMT equation, compared with a strictly nonrelativistic treatment, is that the perpendicular components of the response tensor are proportional to $\omega^2/c^2 - k_z^2$, rather than $-k_z^2$ in the strictly nonrelativistic limit.

Inclusion of the spin-dependent contribution to the dielectric tensor modifies the dispersion equation for magnetoionic-like waves, but the dispersion equation remains a quadratic equation for n^2. We concentrate on the case $\omega_p \ll \Omega_e \approx \Omega_m$. Intrinsically new features of the dispersion curves are: one mode, which has a low-frequency branch that is the whistler mode, has two intermediate frequency branches, one bounded by two cutoffs and the other by two resonances, and it becomes the branch with the lower refractive index at high frequency; the other mode, which corresponds to the z mode at low frequency joins on continuously (no cutoffs or resonances) to the branch with the higher refractive index at high frequency. These properties apply for arbitrary angles of propagation, and generalize some known results for parallel [18] and perpendicular [22] propagation.

Spin is an intrinsically quantum effect, and a rigorous treatment of the effects of spin requires use of relativistic quantum mechanics. Although a completely general result for the response tensor of a magnetized quantum electron gas has long been available [23,24], it has only recently been reduced to a more convenient (but still cumbersome) form for a spin-independent electron gas [25]. It is desirable to repeat this calculation for a spin-dependent occupation number, and explore how the general result reduces to the quasiclassical counterpart derived in the present paper. Comparison of the results will then determine the limits of validity of quasiclassical model.

Finally, we remark that the name "spin waves" has been used by some authors to denote the spin-dependent wave modes whose properties are discussed here. We avoid this name because it is used widely to refer to waves, sometimes called Bloch spin waves, that are associated with the spin-spin interaction, that is central to understanding of ferromagnetism. The spin waves of interest in the present paper are due to long-range order, in the ferromagnetic context, and the spin-spin interaction between nearest neighbors is not included.

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APPENDIX: NONRELATIVISTIC DERIVATION OF RESPONSE TENSOR

The solutions of (2) of a mean spin and associated mean magnetization, 3 magnetization $\mathbf{M} = g\mu_B n_e \bar{\mathbf{s}}$, along the directions of the background magnetic field can be written in the matrix form

$$\begin{pmatrix} \delta s_x \\ \delta s_y \\ \delta s_z \end{pmatrix} = i\frac{2\mu_B \bar{s}}{\hbar(\omega^2 - \Omega_e^2)} \begin{pmatrix} \omega & i\Omega_e & 0 \\ -i\Omega_e & \omega & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\delta B_y \\ \delta B_x \\ 0 \end{pmatrix}. \quad \text{(A1)}$$

The perturbation in the magnetization $\delta\mathbf{M} = g n_e \mu_B \delta\mathbf{s}$ leads to a magnetization current $\mathbf{J}_M = i\mathbf{k}\times\delta\mathbf{M}$. On writing $J_{Mi} = \sigma_{ij}E_j$ and noting that the dielectric tensor is related to the conductivity tensor by $K_{ij} = \delta_{ij} + i\sigma_{ij}/\varepsilon_0\omega$, one finds that the spin contribution to the dielectric tensor is

$$\mathbf{K}_m = \frac{\Omega_m c^2}{\omega^2(\omega^2-\Omega_e^2)}\begin{pmatrix} -k_z^2\Omega_e & -ik_z^2\omega & k_\perp k_z\Omega_e \\ ik_z^2\omega & -k_z^2\Omega_e & -ik_\perp k_z\omega \\ k_\perp k_z\Omega_e & ik_\perp k_z\omega & -k_\perp^2\Omega_e \end{pmatrix}. \tag{A2}$$

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