Self-propulsion of a planar electric or magnetic microbot immersed in a polar viscous fluid

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A planar sheet immersed in an electrically polar liquid like water can propel itself by means of a plane wave charge density propagating in the sheet. The corresponding running electric wave polarizes the fluid and causes an electrical torque density to act on the fluid. The sheet is convected by the fluid motion resulting from the conversion of rotational particle motion, generated by the torque density, into translational fluid motion by the mechanism of friction and spin diffusion. Similarly, a planar sheet immersed in a magnetic ferrofluid can propel itself by means of a plane wave current density in the sheet and the torque density acting on the fluid corresponding to the running wave magnetic field and magnetization. The effect is studied on the basis of the micropolar fluid equations of motion and Maxwell's equations of electrostatics or magnetostatics, respectively. An analytic expression is derived for the velocity of the sheet by perturbation theory to second order in powers of the amplitude of the driving charge or current density. Under the assumption that the equilibrium magnetic equation of state may be used in linearized form and that higher harmonics than the first may be neglected, a set of self-consistent integral equations is derived which can be solved numerically by iteration. In typical situations the second-order perturbation theory turns out to be quite accurate.

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I. INTRODUCTION

It is known that a ferrofluid can be pumped by the application of a running magnetic wave [1,2]. Similarly, a neutral electrically polar liquid such as water can be pumped by the application of a running electric wave, or even by a rotating electric field if advantage is taken of spatial asymmetry [3]. Earlier work on pumping in electrohydrodynamics or magnetohydrodynamics involved fluids with free ions or electrons [4].

Previously we have studied ferrohydrodynamic pumping of a ferrofluid through a planar duct by means of a running magnetic wave [5]. The theory of electrohydrodynamic pumping of a polar liquid by means of a running electric wave is quite analogous [5]. The possibility of pumping implies that self-propulsion of a rigid body by means of a running electric or magnetic wave, generated by a plane wave charge or current density on its surface, should also be feasible. In the following we study the phenomenon for the simple geometry of a planar sheet immersed in infinite fluid.

In practice, it is not necessary to consider an infinite sheet in order to achieve propulsion. An electrical circuit located on a finite-sized body, generating a running electric or magnetic wave in the surrounding fluid, will be sufficient to make the body move. The plane wave character of the excitation may be achieved by geometric arrangement of conductors, as in the case of magnetic wave pumping [1,2].

For definiteness we discuss only the magnetic case in detail. The problem is nonlinear, since the magnetic torque density acting on the fluid is bilinear in magnetic field and magnetization, and moreover the flow velocity of the fluid and the rotational velocity of suspended magnetic particles couple convectively to the magnetization in the magnetic relaxation equation. On a slow time scale the inertial terms in the equations of motion for flow velocity and particle rotational velocity may be neglected. In analogy to the theory of ferrohydrodynamic pumping [5] we calculate the propulsion velocity of the sheet first in perturbation theory to second order in powers of the amplitude of the exciting current density.

The perturbation calculation has the advantage of simplicity. It leads to an explicit expression for the propulsion velocity and hence allows insight into its dependence on the system parameters. Moreover, one obtains a picture of the flow and the magnetic field and magnetization.

With the additional assumptions that the equilibrium magnetic equation of state may be used in linearized form and that harmonics higher than the first may be neglected, a more complete solution may be obtained. With these assumptions the coupled differential equations for flow velocity and particle rotational velocity can be integrated. This leads to a self-consistent set of integral equations which may be solved by iteration, with the perturbation solution as a starting point. We call the solution thus obtained the primary solution. In both the magnetic and the electric case the primary solution hardly differs numerically from the perturbation solution for typical situations under consideration.

II. EQUATIONS OF MOTION

We consider a planar sheet immersed in an incompressible polar viscous fluid with shear viscosity η. We use Cartesian coordinates such that the sheet is located at $x = 0$. The fluid can be either electrically or magnetically polar. For definiteness we use language appropriate to a magnetic ferrofluid. With minor changes the same equations apply in the electrical case.

Due to incompressibility of the fluid the divergence of the flow velocity $\boldsymbol{v}(\boldsymbol{r},t)$ vanishes, $\nabla \cdot \boldsymbol{v} = 0$. The flow velocity satisfies the momentum balance equation

$$\rho \frac{d\boldsymbol{v}}{dt} = \nabla \cdot (\boldsymbol{\sigma}_{hyd} + \boldsymbol{\sigma}_m), \tag{2.1}$$

where $d/dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla$ is the substantial derivative, $\boldsymbol{\sigma}_{hyd}$ is the hydrodynamic stress tensor, and $\boldsymbol{\sigma}_m$ is the Maxwell

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stress tensor. The hydrodynamic stress tensor has Cartesian components [6,7]

$$\sigma_{hyd,\alpha\beta} = -p\delta_{\alpha\beta} + \eta(\partial_\alpha v_\beta + \partial_\beta v_\alpha) + \zeta\epsilon_{\alpha\beta\gamma}(\nabla\times\boldsymbol{v} - 2\boldsymbol{\omega}_p)_\gamma, \tag{2.2}$$

where p is the pressure, η is the shear viscosity, ζ is the vortex viscosity [8], and $\boldsymbol{\omega}_p$ is the rate of rotation of suspended particles. In SI units the Maxwell stress tensor has the form [8]

$$\boldsymbol{\sigma}_m = \boldsymbol{B}\boldsymbol{H} - \frac{\mu_0}{2}H^2\mathbf{1}, \tag{2.3}$$

where $\boldsymbol{B}(\boldsymbol{r},t)$ is the magnetic induction, $\boldsymbol{H}(\boldsymbol{r},t)$ is the magnetic field, μ_0 is the magnetic permeability of vacuum, $H^2 = \boldsymbol{H}\cdot\boldsymbol{H}$, and $\mathbf{1}$ is the unit tensor. The fields are related by

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}), \tag{2.4}$$

where $\boldsymbol{M}(\boldsymbol{r},t)$ is the magnetization. The fields satisfy Maxwell's equations of magnetostatics

$$\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{H} = \boldsymbol{j}, \tag{2.5}$$

where $\boldsymbol{j}(\boldsymbol{r},t)$ is the electrical current density located in the planar sheet. The latter acts as a source of the fields, and is assumed to be known. The current density is taken to be given by

$$\boldsymbol{j}(\boldsymbol{r},t) = K(z,t)\delta(x)\boldsymbol{e}_y \tag{2.6}$$

in the rest frame of the sheet, where $K(z,t)$ has the plane wave form

$$K(z,t) = K_0\cos(kz - \omega t), \tag{2.7}$$

with amplitude K_0, positive wave number k, and positive frequency ω.

The relaxation of magnetization is assumed to be governed by the constitutive equation [8]

$$\frac{\partial\boldsymbol{M}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{M} - \boldsymbol{\omega}_p\times\boldsymbol{M} = -\gamma[\boldsymbol{M} - \boldsymbol{M}_{eq}(\boldsymbol{H})], \tag{2.8}$$

where $\boldsymbol{M}_{eq}(\boldsymbol{H})$ is given by the equilibrium equation of state, and the relaxation rate γ is the inverse of the relaxation time τ. The rotation rate $\boldsymbol{\omega}_p$ is related to the spin $\boldsymbol{S}$ per unit mass by $\boldsymbol{S} = I\boldsymbol{\omega}_p$, where I is an average moment of inertia per unit mass. The equation of motion for the spin per unit mass is taken as

$$\rho\frac{d\boldsymbol{S}}{dt} = 2\zeta(\nabla\times\boldsymbol{v} - 2\boldsymbol{\omega}_p) + \mu_0\boldsymbol{M}\times\boldsymbol{H} + \eta'\nabla^2\boldsymbol{\omega}_p, \tag{2.9}$$

where η' is the spin viscosity [8]. The first term on the right is the hydrodynamic torque density, and the second term is the magnetic torque density. In the situations considered in the following, $\nabla\cdot\boldsymbol{\omega}_p = 0$ due to spatial symmetry, so that there is no need to introduce a bulk spin viscosity [9].

We neglect the inertial term on the left-hand side in Eqs. (2.2) and (2.9). Then Eq. (2.9) reduces to

$$2\zeta(\nabla\times\boldsymbol{v} - 2\boldsymbol{\omega}_p) = -\mu_0\boldsymbol{M}\times\boldsymbol{H} - \eta'\nabla^2\boldsymbol{\omega}_p. \tag{2.10}$$

Substituting this into Eq. (2.2) we find from Eq. (2.1)

$$\eta\nabla^2\boldsymbol{v} - \nabla p + \nabla\cdot\boldsymbol{\sigma}_m^S + \frac{1}{2}\eta'\nabla\times\nabla^2\boldsymbol{\omega}_p = 0, \tag{2.11}$$

where $\boldsymbol{\sigma}_m^S$ is the symmetric part of the Maxwell stress tensor,

$$\boldsymbol{\sigma}_m^S = \frac{1}{2}(\boldsymbol{B}\boldsymbol{H} + \boldsymbol{H}\boldsymbol{B}) - \frac{\mu_0}{2}H^2\mathbf{1}. \tag{2.12}$$

Using Maxwell's equations of magnetostatics one may express the divergence of this tensor as [10]

$$\boldsymbol{F} = \nabla\cdot\boldsymbol{\sigma}_m^S = \mu_0\boldsymbol{M}\cdot(\nabla\boldsymbol{H}) + \frac{\mu_0}{2}\nabla\times(\boldsymbol{M}\times\boldsymbol{H}). \tag{2.13}$$

The first term on the right is the Kelvin force density. The second term may be expressed as the divergence of an antisymmetric tensor. For our purposes the alternative expression [11]

$$\boldsymbol{F} = \frac{\mu_0}{2}\nabla(\boldsymbol{M}\cdot\boldsymbol{H}) - \frac{\mu_0}{2}\boldsymbol{H}\times(\nabla\times\boldsymbol{M}) - \frac{1}{2}\boldsymbol{B}(\nabla\cdot\boldsymbol{M}) \tag{2.14}$$

is also useful.

The reduced equations of motion (2.10) and (2.11) must be supplemented with boundary conditions for $\boldsymbol{v}$ and $\boldsymbol{\omega}_p$ at the plane $x = 0$. We assume that $\boldsymbol{v}$ satisfies the no-slip condition $\boldsymbol{v}|_{x=\pm b} = 0$ and that $\boldsymbol{\omega}_p$ satisfies the mixed boundary condition

$$\mp\lambda_s\frac{\partial\boldsymbol{\omega}_p}{\partial x}\bigg|_{x=\pm0} + \boldsymbol{\omega}_p|_{x=\pm0} = 0, \tag{2.15}$$

with slip length λ_s. The field $\boldsymbol{H}$ is assumed to vanish for $x\to\pm\infty$. Together with Maxwell's equations of magnetostatics (2.5) and the magnetization relaxation equation (2.8) the equations constitute a nonlinear set. We first solve the equations by formal perturbation expansion in powers of the amplitude K_0 of the exciting current density, putting

$$\begin{aligned}\boldsymbol{H} &= \boldsymbol{H}_1 + \boldsymbol{H}_3 + \cdots, \quad \boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_3 + \cdots,\\ \boldsymbol{v} &= \boldsymbol{v}_2 + \boldsymbol{v}_4 + \cdots, \quad p = p_0 + p_2 + p_4 + \cdots,\\ \boldsymbol{\omega}_p &= \boldsymbol{\omega}_{p2} + \boldsymbol{\omega}_{p4} + \cdots,\end{aligned} \tag{2.16}$$

where p_0 is the static equilibrium pressure and the subscripts denote the power of K_0. We perform the calculation to second order in K_0. In a later section we present a calculation valid to all orders.

III. FIRST-ORDER FIELDS AND SECOND-ORDER FLOW AND DISSIPATION

As a first step in the perturbation calculation we must calculate the first-order fields $\boldsymbol{H}_1$ and $\boldsymbol{M}_1$ in the absence of flow and particle rotation. To first order in the current amplitude K_0 we need to deal only with the set of Eqs. (2.4)–(2.7) and the linearized version of Eq. (2.8). It is clear by symmetry that the components H_{1y} and M_{1y} vanish and that the x and z components of the fields depend on z and t in plane wave fashion. It is convenient to decompose as

$$\begin{aligned}H_{1\alpha}(x,z,t) &= H_{1\alpha c}(x)\cos(kz-\omega t) + H_{1\alpha s}(x)\sin(kz-\omega t),\\ M_{1\alpha}(x,z,t) &= M_{1\alpha c}(x)\cos(kz-\omega t) + M_{1\alpha s}(x)\sin(kz-\omega t),\end{aligned} \tag{3.1}$$

and to use complex notation. Then with linear susceptibility $\chi = \chi' + i\chi''$ the field and magnetization components are related by

$$\boldsymbol{M}_{1c} = \chi'\boldsymbol{H}_{1c} + \chi''\boldsymbol{H}_{1s}, \quad \boldsymbol{M}_{1s} = \chi'\boldsymbol{H}_{1s} - \chi''\boldsymbol{H}_{1c}. \tag{3.2}$$

In the present case we find from Eq. (2.8)

$$\chi' = \chi_0\frac{\gamma^2}{\omega^2+\gamma^2}, \quad \chi'' = \chi_0\frac{\omega\gamma}{\omega^2+\gamma^2}, \tag{3.3}$$

where χ_0 is the zero field susceptibility, also called the initial susceptibility. The magnetic field is irrotational and may be derived from a potential ϕ_1 as $\boldsymbol{H}_1 = -\nabla\phi_1$. The potential satisfies Laplace's equation $\nabla^2\phi_1 = 0$ in the fluid. In complex notation we write $K(z,t) = K_0\mathrm{Re}\exp(ikz - i\omega t)$. We can then put $\phi_1(x,z,t) = \mathrm{Re}\, f(x)\exp(ikz - i\omega t)$ and find from the continuity of B_x and from the jump condition for H_z at the sheet

$$f(x) = \pm\frac{1}{2k}K_0e^{-k|x|}, \quad \text{for } x \gtrless 0. \tag{3.4}$$

Hence the first-order magnetic field components are given by

$$\begin{aligned} H_{1xc}(x) &= 0, \quad H_{1xs}(x) = \tfrac{1}{2}K_0e^{\mp k|x|},\\ H_{1zc}(x) &= \mp\tfrac{1}{2}K_0e^{-k|x|}, \quad H_{1zs}(x) = 0, \quad \text{for} \quad x \gtrless 0. \end{aligned} \tag{3.5}$$

From Eq. (3.2) we find for the components of the first-order magnetization:

$$\begin{aligned} M_{1xc}(x) &= \tfrac{1}{2}\chi''K_0e^{-k|x|}, \quad M_{1xs}(x) = \tfrac{1}{2}\chi'K_0e^{-k|x|},\\ M_{1zc}(x) &= \mp\tfrac{1}{2}\chi'K_0e^{-k|x|}, \quad M_{1zs}(x) = \pm\tfrac{1}{2}\chi''K_0e^{-k|x|},\\ &\text{for} \quad x \gtrless 0. \end{aligned} \tag{3.6}$$

This yields for the second-order magnetic torque density

$$\boldsymbol{N}_2 = \mu_0\boldsymbol{M}_1 \times \boldsymbol{H}_1 = (0,N_{2y},0), \tag{3.7}$$

with

$$N_{2y} = \pm Ce^{-2k|x|}, \quad C = \tfrac{1}{4}\mu_0\chi''K_0^2, \quad \text{for} \quad x \gtrless 0. \tag{3.8}$$

The torque density is independent of time. The magnetic force density can be expressed as the gradient of a pressure. The y component of Eq. (2.10) and the z component of Eq. (2.11) yield the pair of equations

$$\begin{aligned} &\eta'\frac{d^2\omega_{p2y}}{dx^2} - 2\zeta\left(\frac{dv_{2z}}{dx} + 2\omega_{p2y}\right) = \mp Ce^{-2k|x|},\\ &\text{for} \quad x \gtrless 0, \quad \eta\frac{d^2v_{2z}}{dx^2} + \frac{1}{2}\eta'\frac{d^3\omega_{p2y}}{dx^3} = 0. \end{aligned} \tag{3.9}$$

Note that in the second equation the Maxwell stress tensor does not contribute. The first term in Eq. (2.14) is balanced by the pressure gradient, and the second and third terms vanish because $\nabla \times \boldsymbol{M}_1 = 0$ and $\nabla \cdot \boldsymbol{M}_1 = 0$. The transverse component v_{2x} of the flow field vanishes. The transverse component of the force density is balanced by a transverse pressure gradient.

The Eqs. (3.9) have the solution

$$\begin{aligned} v_{2z}(x) &= U_2 + \frac{\eta'\kappa W}{2\eta}e^{-\kappa|x|} + \frac{\eta'kC}{\eta\xi}e^{-2k|x|},\\ \omega_{p2y}(x) &= \pm We^{-\kappa|x|} \pm \frac{C}{\xi}e^{-2k|x|}, \quad \text{for} \quad x \gtrless 0, \end{aligned} \tag{3.10}$$

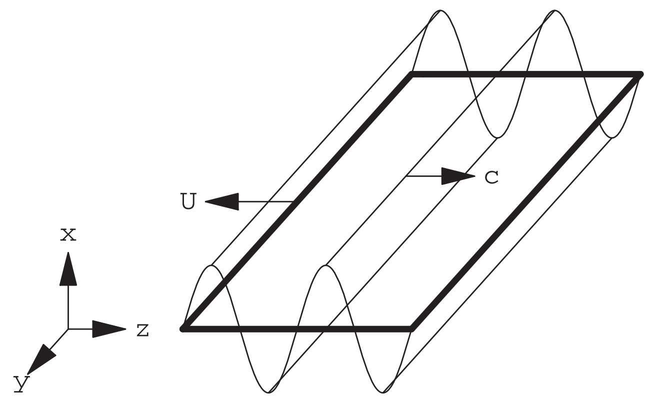

FIG. 1. Sketch of a planar sheet moving in the $-z$ direction due to a plane wave current density, polarized in the y direction, and running in the z direction with phase velocity $c = \omega/k$. The torque density in the ferrofluid, which generates the motion, is in the y direction.

where

$$\kappa = \sqrt{\frac{4\eta\zeta}{\eta'(\eta+\zeta)}}, \quad \xi = 4\zeta\left(1 - \frac{4k^2}{\kappa^2}\right). \tag{3.11}$$

The coefficients U and W can be determined from the boundary conditions. From the no-slip condition for $\boldsymbol{v}$ and from Eq. (2.14) for $\boldsymbol{\omega}_p$ one finds

$$\begin{aligned} U_2 &= \frac{C}{2(\eta+\zeta)(2k+\kappa)(1+\kappa\lambda_s)},\\ W &= \frac{C\kappa^2(1+2k\lambda_s)}{4\zeta(4k^2-\kappa^2)(1+\kappa\lambda_s)}. \end{aligned} \tag{3.12}$$

The flow velocity $v_{2z}(x)$ is even in x and the particle rotational velocity $\omega_{p2y}(x)$ is odd in x. At large distance $|x|$ the flow velocity tends to U_2. This implies that in the laboratory frame the sheet moves in the $-z$ direction with velocity $\boldsymbol{U} = -U_2\boldsymbol{e}_z$. In Fig. 1 we show a sketch of the geometry.

To second order in K_0 the dissipation in the system is purely magnetic. From the linear relaxation equation

$$\frac{\partial\boldsymbol{M}_1}{\partial t} = -\gamma(\boldsymbol{M}_1 - \chi_0\boldsymbol{H}_1), \tag{3.13}$$

one derives

$$\frac{\partial}{\partial t}\left(\frac{\mu_0}{2\chi_0}\boldsymbol{M}_1^2\right) = \mu_0\boldsymbol{H}_1\cdot\frac{\partial\boldsymbol{M}_1}{\partial t} - \frac{\mu_0\gamma}{\chi_0}(\boldsymbol{M}_1 - \chi_0\boldsymbol{H}_1)^2. \tag{3.14}$$

The left-hand side is the rate of change of the second-order magnetization energy density, the first term on the right represents the work done by the magnetic field as the magnetization varies, and the second term on the right represents the local rate of dissipation:

$$\Phi_{m2} = \frac{\mu_0\gamma}{\chi_0}(\boldsymbol{M}_1 - \chi_0\boldsymbol{H}_1)^2. \tag{3.15}$$

This is the heat produced locally by the relaxation process [6]. Substituting from Eqs. (3.1) and (3.2) we find

$$\Phi_{m2} = \frac{\mu_0}{4}\omega\chi''K_0^2e^{-2k|x|}, \tag{3.16}$$

independent of z and t. Integrating over the transverse coordinate we obtain

$$P_2 = \int_{-\infty}^{\infty} \Phi_{m2}\, dx = \frac{\mu_0}{4k}\omega\chi'' K_0^2. \tag{3.17}$$

This has the dimension power per unit area. The efficiency

$$\mathcal{E} = \eta\omega\frac{U_2}{P_2} = \frac{\eta}{2(\eta+\zeta)}\,\frac{k}{2k+\kappa}\,\frac{1}{1+\kappa\lambda_s} \tag{3.18}$$

is dimensionless and independent of frequency. The efficiency is of order unity provided k is of order κ or larger.

IV. PRIMARY SOLUTION

In the following we assume that the equilibrium equation of state appearing in the relaxation equation (2.8) can be approximated by the linear relation

$$\boldsymbol{M}_{eq}(\boldsymbol{H}) = \chi_0 \boldsymbol{H}. \tag{4.1}$$

The approximation is accurate provided the magnetization is small relative to the saturation magnetization. It is then clear that the solution of Eqs. (2.5), (2.8), (2.10), and (2.11) can be expressed as a sum of harmonics in the phase $kz-\omega t$. The same would be true if, for example, the Langevin equation of state were used, but we shall use Eq. (4.1) for simplicity. The primary solution of the problem is defined as the one obtained by truncation at first harmonics for field and magnetization and at zeroth harmonics for translational and rotational velocity. To second order in K_0 the primary solution reduces to the expressions found in the preceding section. In principle one can improve the solution by including higher harmonics up to a chosen order.

In the primary solution the flow field $\boldsymbol{v} = [0,0,v_z(x)]$ and the particle rotational velocity $\boldsymbol{\omega}_p = [0,\omega_{py}(x),0]$ depend only on the transverse coordinate x and are independent of time. In the present geometry the coupled partial differential equations (2.10) and (2.11) for flow velocity $\boldsymbol{v}$ and particle rotational velocity $\boldsymbol{\omega}_p$ reduce to ordinary differential equations which can be integrated. The integration of the equations leads to integral equations which relate the velocity components $v_z(x)$ and $\omega_{py}(x)$ to field and magnetization in self-consistent manner.

As in Eq. (3.1) it is convenient to decompose the x and z components of field and magnetization as

$$\begin{aligned} H_\alpha(x,z,t) &= H_{\alpha c}(x)\cos(kz-\omega t) + H_{\alpha s}(x)\sin(kz-\omega t),\\ M_\alpha(x,z,t) &= M_{\alpha c}(x)\cos(kz-\omega t) + M_{\alpha s}(x)\sin(kz-\omega t). \end{aligned} \tag{4.2}$$

By symmetry the components H_y and M_y vanish. Substituting the above expressions into the magnetic relaxation equation (2.8) with the approximation (4.1) and putting $\boldsymbol{v} = (0,0,v_z)$ and $\boldsymbol{\omega}_p = (0,\omega_{py},0)$, we find the relations

$$\begin{aligned} (\omega - kv_z)M_{xc} - \omega_{py}M_{zs} &= -\gamma(M_{xs} - \chi_0 H_{xs}),\\ -(\omega - kv_z)M_{xs} - \omega_{py}M_{zc} &= -\gamma(M_{xc} - \chi_0 H_{xc}),\\ (\omega - kv_z)M_{zc} + \omega_{py}M_{xs} &= -\gamma(M_{zs} - \chi_0 H_{zs}),\\ -(\omega - kv_z)M_{zs} + \omega_{py}M_{xc} &= -\gamma(M_{zc} - \chi_0 H_{zc}). \end{aligned} \tag{4.3}$$

We can write the relation between magnetization and field resulting from the solution of these equations in a form analogous to Eq. (3.2)

$$\boldsymbol{M}_c = \chi_f'\boldsymbol{H}_c + \chi_f''\boldsymbol{H}_s, \quad \boldsymbol{M}_s = \chi_f'\boldsymbol{H}_s - \chi_f''\boldsymbol{H}_c, \tag{4.4}$$

with χ_f' the real part and χ_f'' the imaginary part of the complex susceptibility in the presence of flow, $\chi_f = \chi_f' + i\chi_f''$,

$$\chi_f = i\gamma\chi_0\frac{\omega - kv_z + i\gamma}{(\omega - kv_z + i\gamma)^2 - \omega_{py}^2}, \tag{4.5}$$

in analogy to Eq. (3.3). The relations (4.4) are nonlinear, since the velocity components v_z and ω_{py} depend on field and magnetization.

The magnetic torque density,

$$\boldsymbol{N} = \mu_0\boldsymbol{M}\times\boldsymbol{H} = (0,N_y,0), \tag{4.6}$$

takes the form

$$N_y = \mu_0\chi_f''(H_{xc}H_{zs} - H_{zc}H_{xs}). \tag{4.7}$$

This is independent of z and t. The y component of Eq. (2.10) and the z component of Eq. (2.11) yield the pair of equations

$$\begin{aligned} \eta'\frac{d^2\omega_{py}}{dx^2} - 2\zeta\left(\frac{dv_z}{dx} + 2\omega_{py}\right) &= -N_y,\\ \eta\frac{d^2v_z}{dx^2} + \frac{1}{2}\eta'\frac{d^3\omega_{py}}{dx^3} &= -\overline{F_z}, \end{aligned} \tag{4.8}$$

in analogy to Eq. (3.9). Here $\overline{F_z}$ is the z component of the time-averaged magnetic force density

$$\overline{\boldsymbol{F}} = \frac{\mu_0}{T}\int_0^T \boldsymbol{M}\cdot\nabla\boldsymbol{H}\,dt + \frac{1}{2}\nabla\times\boldsymbol{N}, \tag{4.9}$$

where $T = 2\pi/\omega$. From Eqs. (4.2) and (4.4) we find

$$\begin{aligned} \overline{F_z} &= \frac{1}{2}\mu_0 k\chi_f''\left(H_{xc}^2 + H_{xs}^2 + H_{zc}^2 + H_{zs}^2\right) + \frac{1}{2}\frac{dN_y}{dx}\\ &= \mu_0 k\chi_f''\overline{H^2} + \frac{1}{2}\frac{dN_y}{dx}. \end{aligned} \tag{4.10}$$

Here we have used the relations

$$\frac{dH_{zc}}{dx} = kH_{xs}, \quad \frac{dH_{zs}}{dx} = -kH_{xc}, \tag{4.11}$$

which follow from $\nabla\times\boldsymbol{H} = 0$. To second order in K_0 one has $\overline{F_{z2}} = 0$. In a region of uniform susceptibility the two terms in Eq. (4.10) cancel. The pressure gradient does not contribute in Eq. (4.8) because the time-averaged pressure does not depend on z.

Differentiating the first equation (4.8) with respect to x, and eliminating ω_{py} one derives the fourth-order differential equation

$$\frac{d^2}{dx^2}\left(\frac{d^2v_z}{dx^2} - \kappa^2 v_z\right) = \frac{1}{2(\eta+\zeta)}\frac{d^2}{dx^2}\left(\frac{dN_y}{dx} - 2\overline{F_z}\right) + \frac{\kappa^2}{\eta}\overline{F_z}. \tag{4.12}$$

By symmetry the velocity component $v_z(x)$ is even in x, and the rotational velocity component $\omega_{py}(x)$ is odd in x. For simplicity we therefore consider in the following only the region $x > 0$. The solution of Eq. (4.12) which vanishes at $x = 0$ and tends to a constant U as $x \to \infty$ is given by

$$
\begin{aligned}
v_z(x) = {} & U(1 - e^{-\kappa x}) + V(x) - V(0)e^{-\kappa x} \\
& + \frac{1}{4(\eta+\zeta)}\biggl[e^{-\kappa x}\int_0^\infty e^{-\kappa x'}W_+(x')\,dx' - e^{\kappa x} \\
& \times \int_x^\infty e^{-\kappa x'}W_+(x')\,dx' + e^{-\kappa x}\int_0^x e^{\kappa x'}W_-(x')\,dx'\biggr],
\end{aligned}
\tag{4.13}
$$

where

$$V(x) = \frac{-1}{\eta}\int_x^\infty\int_{x'}^\infty \overline{F_z}(x'')\,dx''dx', \tag{4.14}$$

and

$$W_\pm(x) = N_y(x) \pm \frac{2\zeta}{\eta\kappa}\overline{F_z}(x). \tag{4.15}$$

Integrating the second equation in (4.8) twice we find

$$\frac{d\omega_{py}}{dx} = \frac{2\eta}{\eta'}[U + V(x) - v_z(x)]. \tag{4.16}$$

Substituting Eq. (4.13) and integrating we obtain for the particle rotational velocity

$$
\begin{aligned}
\omega_{py}(x) = {} & \frac{-2\eta}{\eta'\kappa}[U + V(0)]e^{-\kappa x} - \frac{1}{2\eta}\int_x^\infty \overline{F_z}(x')\,dx' \\
& + \frac{\eta}{2\eta'\kappa(\eta+\zeta)}\biggl[e^{-\kappa x}\int_0^\infty e^{-\kappa x'}W_+(x')\,dx' \\
& + e^{\kappa x}\int_x^\infty e^{-\kappa x'}W_+(x')\,dx' \\
& + e^{-\kappa x}\int_0^x e^{\kappa x'}W_-(x')\,dx'\biggr].
\end{aligned}
\tag{4.17}
$$

Hence, we find by use of the boundary condition (2.15) the relation

$$U = -V(0) + \frac{1}{2(1+\kappa\lambda_s)(\eta+\zeta)}\int_0^\infty e^{-\kappa x}N_y(x)\,dx. \tag{4.18}$$

The expression (4.17) may be cast in the alternative forms

$$
\begin{aligned}
\omega_{py}(x) &= \frac{1}{4\zeta}N_y(x) - \frac{\eta+\zeta}{2\zeta}\frac{dv_z}{dx} + \frac{\eta}{2\zeta}\frac{dV}{dx} \\
&= -\frac{\eta+\zeta}{2\zeta}\frac{dv_z}{dx} + \frac{\mu_0 k}{2\zeta}\int_x^\infty \chi_f''(x')\overline{H^2}(x')\,dx'.
\end{aligned}
\tag{4.19}
$$

The magnetic field must be determined from Maxwell's equations. From $\nabla \times \boldsymbol{H} = 0$ it follows that the field can be derived from a potential ϕ as $\boldsymbol{H} = -\nabla\phi$. In accordance with Eq. (4.2) we write

$$\phi(x,z,t) = \phi_c(x)\cos(kz - \omega t) + \phi_s(x)\sin(kz - \omega t). \tag{4.20}$$

From $\nabla \cdot \boldsymbol{B} = 0$ we find the pair of equations

$$
\begin{aligned}
\frac{d^2\phi_c}{dx^2} - k^2\phi_c &= \frac{dM_{xc}}{dx} + kM_{zs}, \\
\frac{d^2\phi_s}{dx^2} - k^2\phi_s &= \frac{dM_{xs}}{dx} - kM_{zc}.
\end{aligned}
\tag{4.21}
$$

The solution of these equations with proper behavior for $x \to \infty$ is

$$
\begin{aligned}
\phi_c(x) = {} & -\frac{1}{2}e^{kx}\int_x^\infty e^{-kx'}(\hat{M}_{xc}(x') + \hat{M}_{zs}(x'))\,dx' \\
& + \frac{1}{2}e^{-kx}\int_0^x e^{kx'}(\hat{M}_{xc}(x') - \hat{M}_{zs}(x'))\,dx', \\
\phi_s(x) = {} & \frac{K_0}{2k}e^{-kx} - \frac{1}{2}e^{kx}\int_x^\infty e^{-kx'}(\hat{M}_{xs}(x') - \hat{M}_{zc}(x'))\,dx' \\
& + \frac{1}{2}e^{-kx}\int_0^x e^{kx'}(\hat{M}_{xs}(x') + \hat{M}_{zc}(x'))\,dx',
\end{aligned}
\tag{4.22}
$$

where

$$\hat{M}_{\alpha\sigma}(x) = M_{\alpha\sigma}(x) - M_{1\alpha\sigma}(x), \quad \alpha = (x,z), \quad \sigma = (c,s) \tag{4.23}$$

are the contributions to the magnetization of order higher than first in K_0. The magnetic field components are given by

$$
\begin{aligned}
H_{xc}(x) &= -\frac{d\phi_c}{dx}, \quad H_{xs}(x) = -\frac{d\phi_s}{dx}, \\
H_{zc}(x) &= -k\phi_s(x), \quad H_{zs}(x) = k\phi_c(x).
\end{aligned}
\tag{4.24}
$$

The magnetization is found from Eq. (4.4).

The primary solution may be found to any desired numerical accuracy from the integral form of the equations by iteration, with use of the lowest order solution found in Sec. III as a starting point. The translational velocity U may be found from the flow velocity $v_z(x)$ at large x. In numerical examples the iteration scheme converges rapidly.

V. NUMERICAL RESULTS

In this section we show some numerical results for parameter values as in the calculation of Mao and Koser [1]. They consider $K_0 = 1000$ A/m, $\omega = \pi \times 10^5$ Hz, $k = 100$ m^{-1}, $\chi_0 = 1.7$, $\tau = 10$ μs, $\eta = 0.006$ kg/m s, $\zeta = 0.0008$ kg/m s, and $\eta' = 0$, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6} kg m/s (the units of η' are given incorrectly by Mao and Koser). We put $\eta' = 10^{-8}$ kg m/s. This is in the range of values found experimentally by Chaves *et al.* [12]. Then the length $1/\kappa$ equals 0.19 cm. Like Mao and Koser we put $\lambda_s = 0$. The

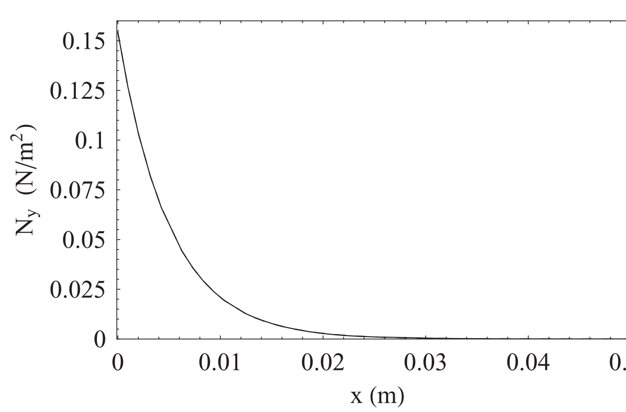

FIG. 2. Plot of the magnetic torque density $N_y(x)$, given by Eq. (4.14), in N/m^2 as a function of x for parameter values given in Sec. V.

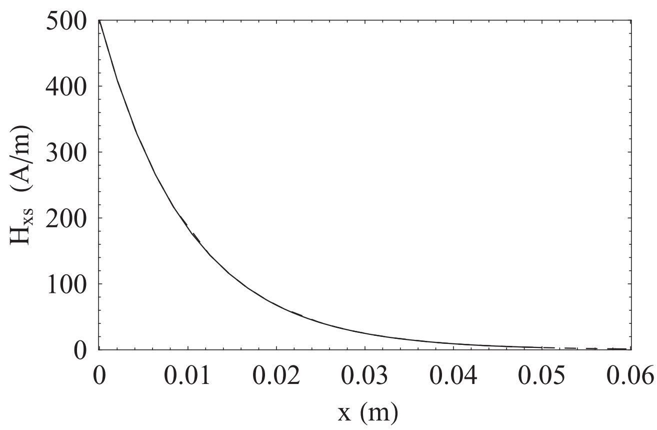

FIG. 3. Plot of the magnetic field component $H_{xs}(x)$ in A/m as a function of x for parameter values given in Sec. V.

validity of this boundary condition is supported by molecular dynamics calculations for water [13].

In Fig. 2 we show the torque density $N_y(x)$ as a function of x, as calculated from the perturbation calculation of Sec. III and from the calculation in first harmonic approximation of Sec. IV. On the scale of the figure the two curves cannot be distinguished. In Fig. 3 we show the magnetic field component $H_{xs}(x)$ as a function of x. The component $H_{xc}(x)$ is very small in comparison. In Fig. 4 we show the flow profile $v_z(x)$ as a function of x. In Fig. 5 we show the profile $\omega_{py}(x)$ as a function of x. The velocity of the sheet is $U = 0.016$ m/s. For all these quantities the second-order calculation and the primary solution yield nearly identical results. Presumably the corrections due to contributions from higher harmonics are quite small, but we have not investigated this in detail.

In the numerical example we find $\chi_f \approx \chi$; that is, the terms with kv_z and ω_{py} in Eq. (4.5) can be neglected to good approximation. In the approximation the susceptibility is spatially uniform and the magnetic force density has no effect on the flow. The latter statement follows from Eq. (2.14) and the fact that for uniform susceptibility $\nabla \cdot \boldsymbol{M} = 0$ and $\nabla \times \boldsymbol{M} = 0$, as follows from Maxwell's equations. The first term in Eq. (2.14) is canceled by a pressure gradient.

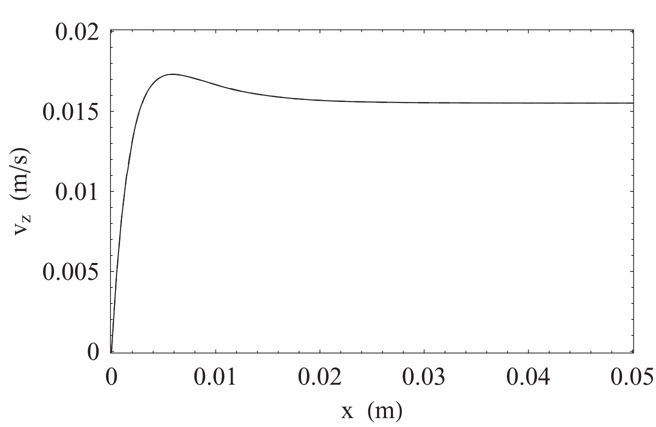

FIG. 4. Plot of the flow velocity $v_z(x)$ in m/s as a function of x for parameter values given in Sec. V.

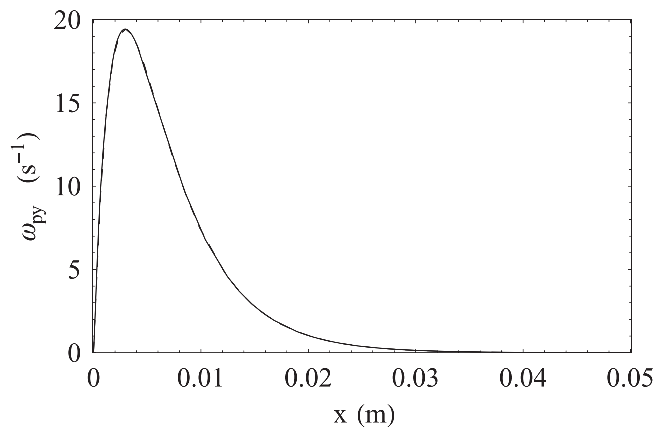

FIG. 5. Plot of the particle rotational velocity $\omega_{py}(x)$ in s^{-1} as a function of x for parameter values given in Sec. V.

VI. ELECTROHYDRODYNAMIC PROPULSION

The theory of electrohydrodynamic propulsion runs parallel to that for ferrohydrodynamic propulsion with slight changes due to the change in jump conditions. The magnetic field $\boldsymbol{H}$ is replaced by the electric field $\boldsymbol{E}$, the magnetization $\boldsymbol{M}$ is replaced by the reduced electric polarization $\boldsymbol{P}' = \boldsymbol{P}/\epsilon_0$, and the magnetic induction $\boldsymbol{B}$ is replaced by the electric displacement $\boldsymbol{D}$. The magnetic permeability of vacuum μ_0 is replaced by ϵ_0. The fields satisfy Maxwell's equations of electrostatics

$$\nabla \cdot \boldsymbol{D} = \rho_e, \quad \nabla \times \boldsymbol{E} = 0, \tag{6.1}$$

where ρ_e is the electric charge density. The charge density is taken to be

$$\rho_e(\boldsymbol{r},t) = \sigma(z,t)\delta(x), \tag{6.2}$$

where $\sigma(z,t)$ has the plane wave form

$$\sigma(z,t) = \sigma_0 \cos(kz - \omega t). \tag{6.3}$$

The relaxation of polarization is given by Eq. (2.8) with replacements as specified above and with electric susceptibility χ_0. For the first-order fields we get exactly the same expressions as in Sec. III with the replacement of K_0 by $-\sigma_0$. The second-order calculation is therefore exactly the same as in Sec. III, and we get the same expression (3.12) for the second-order speed. The calculation of Sec. IV holds with appropriate replacements.

For water the viscosity is [13] $\eta = 9.2 \times 10^{-4}$ kg/m s, the vortex viscosity is $\zeta = 1.7 \times 10^{-4}$ kg/m s, the spin viscosity is $\eta' = 3 \times 10^{-21}$ kg m/s, and the relaxation rate is $\gamma = 10^{11}$ Hz. Hence, the length $1/\kappa$ equals 2.3 nm, corresponding to the nanometer length scale. For typical values [3] $\omega = 10^8$ Hz, $k = 10^8$ m^{-1}, and $\sigma_0 = 10^5$ V/m the velocity U_2 equals 1.26 nm/s. The second-order calculation and the primary solution yield nearly identical results. The flow profile is qualitatively the same as that for the magnetic system studied in Sec. V.

VII. DISCUSSION

The perturbation calculation of Sec. III provides insight into the propulsion caused by a running plane wave current density in a planar sheet immersed in a magnetic ferrofluid, or by a running plane wave charge density in a sheet immersed in a polar liquid like water. We consider a slow time scale and consequently neglect inertial effects so that the equations of motion for flow velocity and particle rotational velocity reduce to the static equations (2.10) and (2.11).

The problem is nonlinear due to the bilinear expressions for force and torque density, as well as to the appearance of flow velocity and particle rotational velocity in the magnetic relaxation equation (2.8) or its equivalent in the electrical case. The perturbation calculation of Sec. III provides analytic insight into the dependence of the velocity of the sheet on the system parameters. The self-consistent integral equations, developed in Sec. IV and derived in the approximation of a linear magnetic equation of state and of neglect of higher harmonics, allow numerical calculation by means of an iterative scheme. We have found that for typical values of the parameters the perturbation scheme, evaluated to second order in the driving charge or current density, and the primary solution leads to nearly identical numerical results.

It would be of interest to study different geometries. For example, one could study the effect of confinement of the fluid to a planar duct on the propulsion of the sheet. For a cylindrical body confined to a cylindrical pipe the calculation will be quite similar. For both ferrofluids and polar liquids such as water it may be of interest to look for experimental realizations, and possibly, practical applications.

The method of self-consistent integral equations will have application in other situations, in particular in the problem of ferrohydrodynamic pumping of a ferrofluid [5]. The method provides an interesting resolution of the problem of coupling of translational fluid motion and rotational particle motion.

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