Dynamics and complexity of the Schelling segregation model

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In this paper we consider the Schelling social segregation model for two different populations. In Schelling's model, segregation appears as a consequence of discrimination, measured by the local difference between two populations. For that, the model defines a tolerance criterion on the neighborhood of an individual, indicating wether the individual is able to move to a new place or not. Next, the model chooses which of the available unhappy individuals really moves. In our work, we study the patterns generated by the dynamical evolution of the Schelling model in terms of various parameters or the initial condition, such as the size of the neighborhood of an inhabitant, the tolerance, and the initial number of individuals. As a general rule we observe that segregation patterns minimize the interface of zones of different people. In this context we introduce an energy functional associated with the configuration which is a strictly decreasing function for the tolerant people case. Moreover, as far as we know, we are the first to notice that in the case of a non-strictly-decreasing energy functional, the system may segregate very efficiently.

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I. INTRODUCTION

Forty years ago Thomas C. Schelling considered the problem of social segregation as an issue of social discrimination [1–3]. Schelling's seminal model of segregation considers two kinds of individuals of different characteristics living in a network, namely, a city. According to simple rules the individuals of different populations exchange their location. The majority rules model discriminates by measuring the mismatch of both populations in the neighborhood of a given individual; for example, if an individual is in numeric inferiority in its neighborhood, then he would feel "unhappy," and hence he would look for a possibility to change to a different place. Originally, Schelling considers this majority rule, but we can easily generalize it to perform a more realistic situation of more or less tolerant individuals.

Perhaps the most spectacular result of the Schelling model is the natural appearance of macroscopic segregation patterns, reminding us of the most classical motives of segregation in a physical system such as binary fluids. Naturally, the connection to physics is not evident; however, usually the motifs of nature do not depend on rules of the "micromotive behavior"—in Schelling's terms [3]—but only on the symmetries of the system; hence, it is not a surprise that social segregation patterns may be similar to physical motives. Pattern formation is almost independent of the social rules and the individuals; indeed, these are certainly different from physical interactions and atoms. Universal mechanisms of pattern formation in physics have been very well known since the second half of the 20th century [4,5]. Nature organizes itself in a variety of motives and patterns: skin motives of zebras, jaguars, fish, and insects; pigmentation of seashell patterns; convection in fluids; ripple formation in dunes; patterns in chemical reactions, laser physics, liquid crystals, etc. Despite many fundamental differences among these systems (some of them are actually living systems), these patterns have many similarities. Moreover, all those systems have something in common, which is the existence of an *instability*, expressing that the system is not stable in its actual configuration and tends to evolve; for example, a sand grain flies from one place to another because of air flow. Another common concept is the existence of a threshold for pattern formation which depends on a parameter, for example, the wind velocity for ripple formation on a dune: If the wind speed is slow, the air drag cannot overcome the weight of the grain, so the most preferable state of the system is repose; hence, a ripple would not develop.

The Schelling model is characterized by the existence of two populations of individuals, the $\bullet \equiv -1$ and the $\circ \equiv +1$, living in a lattice composed of habitations (in practice one can coarse-grain the residential information given at an individual scale and consider that a site contains the average information of many families belonging to a same group). If an individual, say $\bullet$, has a large number of $\circ$ in its neighborhood, it will be considered unhappy, thus preferring to move to a more *pleasant* place where the number of $\circ$ individuals is lower. Therefore, we can see that the $\bullet$ individual is *unstable*; this instability is what will end up forming a segregation pattern in the system. However, there are some constraints: An individual can move from one site to another only if the last site is available or empty; otherwise, the individual waits until he can exchange his habitation with an opposite individual $\circ$ in a similar unhappy state. Moreover, in the case of no empty habitations, the individual is only allowed to exchange if, and only if, another individual of the opposite population wants to exchange its habitation. Although, the happiness criterion concerns only the neighborhood, the swapping of two individuals of different populations could be a long-range process, as we exchange randomly two unhappy individuals of different populations, independently of the actual position of the individuals.

Usually, a pattern is characterized by a characteristic length which can be intrinsic, say a property of the system, which is the case of Turing patterns [6], or extrinsic, say imposed by the external world; for example, in fluid convection the

convective length is given by the boundaries. In Schelling's work this length is intrinsic because it is related to the size of the vicinity, for instance, in the explicit case considered by Schelling of a two-dimensional square lattice, if an individual vicinity is composed by his eight surrounding individuals (we omit the individual itself), the typical length is of order $\sqrt{8} \approx 3$, while in the case of a vicinity composed by the individuals inside a 5×5 region, a length of $\sqrt{24} \approx 5$ is obtained, and so forth. In general, a vicinity of size $|V|$ in a D-dimensional lattice implies a typical length of order of $|V|^{1/D}$. Similarly, the threshold corresponds to the *tolerance* parameter θ, which is the number of individuals of another population that an individual does actually tolerate in its neighborhood; for example, if an individual sees more than θ individuals of his opposite population, he is unhappy. This parameter is defined in a more general way in Sec. II B.

There has been a renewed interest in the Schelling segregation model over the last years. The close similarities of social segregation with the one observed in physical systems has driven attention to the existence of an *energy*, which is in some sense a measure of the perimeter of the interface between two individual populations. Using rules which are more or less similar to the ones used by Schelling, Pollicott, and Weiss [7], consider an exchange of individuals in a way that the *hapinness* always increases. In their model, they introduce an energy which is the familiar energy of the Ising model. This energy decreases strictly after a swap is performed, and because of the discrete characteristics of the segregation problem, the energy is bounded from below; therefore, a strictly decreasing energy dynamic does allow us to bound the duration of the process of segregation. Moreover, the Ising energy is a measure of the perimeter of the interfaces between the two populations of individuals. If the length of the total interface is reduced (thus reducing the energy), one would also diminish the contact between individuals of different populations. In this way, Vinković and Kirman [8] developed a macroscopical continuous model based on the geometry of an interface by following the rules of the surface energy of two phase systems. That is, to create segregation patterns one diminishes the energy of the system. The discrimination threshold parameter is replaced by a critical angle of occupancy by unlike neighbors. The model, naturally predicts the phenomenology of the Schelling segregation problem. Similarly, Stauffer and Solomon [9] put forward the similarities between segregation and phase separation. More recently Singh, Vainchtein, and Weiss [10], studied several scaling laws for segregation of two classes of individuals but with an important role of vacancies in the limit of very large cities, showing that segregation is independent of the size of the city. From a more economic point of view, Pancs and Vriend [11] consider a *utility* function implying that despite the preference for a *perfect* integration of the individuals, segregation is the robust behavior of the system.

In this article, we reexamine the original Schelling segregation problem in a regular lattice with two populations of individuals and we provide an exhaustive study of the system dynamics by varying the spatial dimensions of the lattice, the size of the neighborhood $|V|$, the degree of tolerance (the discrimination), and the initial populations. The case of very tolerant people $\theta \lesssim |V|$ is probably the simplest: Because almost all the individuals are happy, only a few exchanges are performed, making the segregation stop very quickly. The case of very intolerant people, $1 \lesssim \theta$, is, in some sense, very similar: Because almost no happy individuals are settled, they usually keep swapping in a randomlike fashion and no emergence of a segregation pattern may appear. In both cases of the extreme values of θ, the segregation is very inefficient. The cases near the majority rule, $\theta \equiv |V|/2$, are, without any doubt, the most interesting.

In Sec. II D we notice that the existence of a strictly decreasing energy after an exchange of individuals is performed does not apply for every value of the tolerance parameter θ; indeed, there are dramatic differences in the system behavior depending on if θ is greater or smaller than half of the vicinity size, $|V|$. If $\theta > |V|/2$ the energy is a strictly decreasing function and, as it has already been said, the dynamics stops after a finite time. Therefore, although the energy decreases the segregation is not really efficient because the dynamics eventually stop. This behavior does remind us of the nonequilibrium isolated thermodynamical system, which is driven irreversibly, by the existence of a thermodynamical potential (e.g., the entropy, a free energy, etc.), to an equilibrium, a minimum of the thermodynamical potential. In the absence of fluctuations, this minimum is not necessarily the ground state (the absolute minimum); it is simply the lowest energy that the segregation performs until it stops because of the absence of unhappy individuals of one of the populations. Although in various contexts, pattern formation arises as an energy argument: *It is energetically favorable to display a pattern*. This cannot be the general argument in favor of pattern forming structures; indeed, we do not see any energy involved in the underlying mechanism of skin motives in zebras, nor in other animals. The deep reason for pattern formation is not related to the existence of an energy, but it is tightly related to the existence of an instability; however, the existence of an energy simplifies the mathematics enormously.

In the case $\theta < |V|/2$ the energy is not a strictly decreasing quantity; thus, it cannot be interpreted as a thermodynamical potential. However, the story is not over, we first remarked that for $\theta < |V|/2$ the long-time dynamics show a very efficient segregation pattern, which is measured in terms of the aforementioned energy. We observe numerically that for $\theta \lesssim |V|/2$ the energy has a tendency to decrease in the long-time behavior, thus reaching the ground state, that is, the configuration with the minimum energy. The discovery of this critical behavior near $\theta = |V|/2$ leads one to explore the case of very large $|V|$, which is easily obtained in three-dimensional lattices or in two-dimensional lattices with a nonlocal neighborhood or indirected graphs.

The existence of an energy allows us to define with precision what we mean by *inefficient* or *efficient* segregation. If the energy approaches its minimum possible value (its ground state), we say that segregation is efficient; however, if the final energy is much larger than the ground-state energy, we say that the segregation is inefficient. From a qualitative point of view, the resulting patterns from an *efficient* segregation exhibit very well-defined domains with phase separation, while an *inefficient* segregation does not.

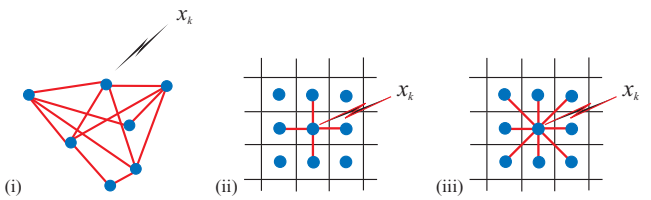

FIG. 1. (Color online) Examples of lattices and neighborhoods. We illustrate explicitly: an arbitrary undirected network (i), a regular square lattice in two dimensions, with a von Neumann neighborhood of 4 individuals (ii), and a Moore neighborhood of eight individuals (iii).

In Sec. II, we briefly discuss the general model, definitions, rules, and main properties, and we define the energy [7]. Next, in Sec. III, we study the behavior of segregation in various dimensions. First, in Sec. III A we pay attention to a one-dimensional periodical array, where each individual possesses two neighbors. Then in Sec. III B we examine in detail the case of a two-dimensional lattice with a von Neumann vicinity of four neighbors and a Moore vicinity of eight neighbors. Finally, in Sec. III C we study briefly the segregation in a three-dimensional lattice. Finally, we conclude and we discuss briefly further generalizations of the Schelling model and the computational complexity of the Schelling segregation problem.

II. MODEL AND DEFINITIONS

A. The lattice

We consider a Schelling model which consists of a general network with N nodes. Each node k possesses a neighbor V_k and the number of neighbors is named by the natural number $|V_k|$. Although nonregular lattices [see Fig. 1(i)] maybe considered, the special case of regular and periodic networks in D spatial dimensions are the most studied in our paper. Among them, the one-dimensional case with only two neighbors, the square lattice with a von Neumann vicinity $|V_k| = 4$ [see Fig. 1(ii)], and the Moore vicinity $|V_k| = 8$ [Fig. 1(iii)] are examples of regular lattices in two spatial dimensions. The cubic lattice with von Neumann vicinity (6 neighbors) and the cubic lattice with the Moore vicinity (26 neighbors) are examples of regular lattices in three spatial dimensions. For a sake of generality we describe the Schelling model and its properties in general; however, the detailed study of the dynamics are realized in regular lattice in two and three spatial dimensions.

Each node k possesses a discrete value x_k that may take values $+1$ and -1. At the initial state we consider N_+ habitants of one kind (state) $+1$ and N_- habitants of the other kind -1. Those numbers remain unchanged in the subsequently social evolution. Naturally, $N_+ + N_- = N$. Finally, the concentration of one population, say $+1$, is defined by $\phi = \frac{N_+}{N_+ + N_-} = \frac{N_+}{N}$ and is a parameter of the dynamical system.

B. The discrimination criterion

We say that an individual x_k at the node k (the house) is not happy at his site if there are more than θ_k neighbors at an opposite state. θ_k is a parameter that depends in principle on the node and it may take the following values: $\theta_k = \{0,1,2,\ldots,|V_k|\}$. Notice that in the case $\theta = 0$ all individuals are never happy, so the system is under a continuous swapping. There is no search of a coherent segregation. This case does not seem very interesting so we exclude it from our study.

Let $n_k(+1)$ be the number of neighbors of x_k that are at the state $+1$ and $n_k(-1)$ the number of neighbors of x_k at -1; naturally, $n_k(+1) + n_k(-1) = |V_k|$, then the satisfaction criterion reads as follows: If $x_k = +1$ and if $n_k(-1) > \theta_k$, then x_k is unhappy. This criterion may be rewritten as

$$\sum_{i\in V_k} x_i = n_k(+1) - n_k(-1) = |V_k| - 2n_k(-1) \leqslant |V_k| - 2\theta_k. \tag{1}$$

On the other hand, if $x_k = -1$ the satisfaction criterion reads

$$\sum_{i\in V_k} x_i = n_k(+1) - n_k(-1) = 2n_k(-1) - |V_k| \geqslant 2\theta_k - |V_k|. \tag{2}$$

Multiplying by x_k both sides of criteria (1) and (2) one gets an unifying criterion:

an individual x_k is unhappy at the node k if, and only if,

$$x_k \sum_{i\in V_k} x_i \leqslant |V_k| - 2\theta_k. \tag{3}$$

C. The exchange rule

At a given time t, the state is fully characterized by $\{x_k(t)\}$, and we build two distinct lists, $\ell_+(t)$ and $\ell_-(t)$, that contains all the unhappy nodes of the population $+1$ and -1, respectively. If both of these lists are not empty, then one takes randomly one element of each list and exchanges them; this is the *Schelling protocol*.

If k and l are these chosen elements, then the evolution dictates $x_k(t) \to x_k(t+1) = -x_k(t)$, $x_l(t) \to x_l(t+1) = -x_l(t)$, and all other nodes $i \neq k$ and $i \neq l$ remain unchanged, $x_i(t) \to x_i(t+1) = x_i(t)$. Then, at the next step $t+1$, one reactualizes both lists, getting $\ell_+(t+1)$ and $\ell_-(t+1)$, and proceeds to exchange randomly again two unhappy individuals of the actual lists, etc. The evolutions continue up to infinity or until one of the lists becomes empty, depending of the parameter θ_k.

Although other exchange protocols maybe implemented,[1] we think that the Schelling protocol we use is more natural because there is no clear mechanism of the sort where the happiness or unhappiness of the individuals causes them to move their habitation; therefore, we study the Schelling protocol hereafter.

D. The energy

This criterion provides us a general principle satisfied by the algorithm. Indeed the quantity, that we call the *energy*

[1]For example, both list $\ell_+(t)$ and $\ell_-(t)$ are ordered from most to less unhappy individuals, and then we take randomly among the most unhappy individuals one element of each list and exchange them.

by analogy with spin-glasses systems which posses the same energy expression (here $\{x\} = \{x_1, x_2, \ldots, x_N\}$ represents the full states of all nodes),

$$E[\{x\}] = -\frac{1}{2}\sum_{k=1}^{N} x_k \sum_{i\in V_k} x_i, \tag{4}$$

always decreases during the evolution if and only if $\theta_k \geqslant \frac{|V_k|}{2} + 1$. On the other hand, if $\theta_k \leqslant \frac{|V_k|}{2}$ the energy may increase or decrease during the evolution.

Proof. Let us consider two distinct unhappy individuals, $x_k = +1$ and $x_l = -1$, with opposite states that may exchange residence or node $x_k(t+1) = -x_k(t)$ and $x_l(t+1) = -x_l(t)$.

As an intermediate step, we rewrite the energy in a more symmetrical way, defining w_{ij} as the unity if $x_i \in V_j$ (and $x_j \in V_i$ since w_{ij} is symmetric) and zero elsewhere including the diagonal terms, $w_{ii} = 0$. One may split the terms into the sums that do involve or do not involve the indicies k and l:

$$\begin{aligned} E[\{x\}] &= -\frac{1}{2}\sum_{i,j=1}^{N} w_{ij}x_i x_j = -\frac{1}{2}\sum_{\{i,j\}\neq\{k,l\}}^{N} w_{ij}x_i x_j - w_{kl}x_k x_l \\ &\quad -x_l\sum_{j\neq k}^{N} w_{jl}x_j - x_k\sum_{j\neq l}^{N} w_{jk}x_j, \end{aligned} \tag{5}$$

where the notation $\{i,j\} \neq \{k,l\}$ means that $i \neq k$, $i \neq l$ and $j \neq k$, $j \neq l$.

The first term $-\frac{1}{2}\sum_{\{i,j\}\neq\{k,l\}}^{N} w_{ij}x_i x_j$ does not involve any k and l, and thus does not change after an exchange; similarly, the term $w_{kl}x_k x_l \equiv -w_{kl}$. The energy difference after the exchange of the individuals k and l is [below x_k' refers to $x_k(t+1) = -x_k(t) = -x_k$]

$$\begin{aligned} \Delta E &= E[\{x(t+1)\}] - E[\{x(t)\}] \\ &= (x_k - x_k')\sum_{j\neq l} w_{jk}x_j + (x_l - x_l')\sum_{j\neq k} w_{jl}x_j \\ &= 4w_{kl} + 2\left(x_k\sum_{j\in V_k} x_j + x_l\sum_{j\in V_l} x_j\right). \end{aligned} \tag{6}$$

Finally, because both k and l are unhappy individuals, applying the condition (3),

$$\Delta E \leqslant 2\,(2w_{kl} + |V_k| + |V_l| - 2\theta_k - 2\theta_l)\,. \tag{7}$$

In conclusion, if $\theta_k \geqslant \frac{|V_k|}{2} + 1$ any exchange $k \leftrightarrow l$ decreases the energy because

$$\begin{aligned} \Delta E &\leqslant 2\,(2w_{kl} + |V_k| + |V_l| - 2\theta_k - 2\theta_l) \\ &\leqslant 4\,(w_{kl} - 2) < -4 < 0, \end{aligned} \tag{8}$$

because w_{kl} is bounded by 1.

Remark 1. The energy functional (4) is bounded, indeed the lowest value of the energy is for a state composed by individuals of the same population. Therefore, for any configuration $\{x\}$ one has that $E[\{x\}] \geqslant E_0 = -\frac{1}{2}\sum_{k=1}^{N}|V_k|$, where E_0 is the lowest energy allowed by the functional $E[\{x\}]$.

Remark 2. For $\theta_k > \frac{|V_k|}{2}$, the evolution stops in finite time because the energy (4) is bounded and decreases by finite amounts.

Remark 3. For $\theta_k > \frac{|V_k|}{2}$, the energy decreases at least by an amount $\Delta E < -8$ for long-range exchange but $\Delta E < -4$ for near (inside) neighbor exchange (that is, whenever $w_{kl} = 1$).

Remark 4. For $\theta_k = \frac{|V_k|}{2}$ one has $\Delta E \leqslant 4w_{kl}$; thus, any long-range exchange does not increase the energy, but near neighbor exchange may increase the energy.

Remark 5. For $\theta_k < \frac{|V_k|}{2}$, the energy is not formally a decreasing functional and may increase or decrease after an exchange indistinctly.

Remark 6. If we have $\theta_k = \theta$, an integer, constant over the lattice, and we consider a regular lattice $|V_k| = |V|$, one has that the condition (7) simplifies to

$$\Delta E \leqslant 4\,(w_{kl} + |V| - 2\theta)\,.$$

Remark 7. The energy may be directly generalized for multiple state variable cases, in particular, when the lattice has empty places available for both kind of individuals.

Remark 8. The definition of the energy (4) is naturally valid for an arbitrary undirected graph. However, the existence of a strictly decreasing energy depends on the local values of the satisfaction parameter, θ_k, and the number of neighbors of the node, $|V_k|$. If all nodes satisfy $\theta_k > |V_k|/2$, then the energy is a strictly decreasing functional after any swap.

E. Energy in terms of the geometry of the interface

The energy (4) may be rewritten as

$$\begin{aligned} E[\{x\}] &= -\frac{1}{2}\sum_{k=1}^{N}\sum_{i\in V_k} 1 + \frac{1}{2}\sum_{k=1}^{N}\sum_{i\in V_k}(1 - x_k x_i) \\ &= E_0 + \frac{1}{2}\sum_{k=1}^{N}\sum_{i\in V_k}(1 - x_k x_i). \end{aligned}$$

The first term is the bulk energy defined previously, and the remainder term $(1 - x_k x_i)$ is different from zero (and equal to 2) only if the neighbors bond involves individuals of the opposite population, we may call this a "state-antistate bond." These kind of bonds are represented in Fig. 2 for the von Neumann [Fig. 2(i)] and Moore vicinities [Fig. 2(ii)].

In the case of the von Neumann vicinity [see Fig. 1(ii)] one has one state-antistate bond per interface or edge; more precisely, one has in Fig. 2(iii) one bond per one interface or edge. In conclusion, the energy is

$$E = E_0 + 2\sum \text{edges} = E_0 + 2 \times \text{perimeter}, \tag{9}$$

with $E_0 = -2N$.

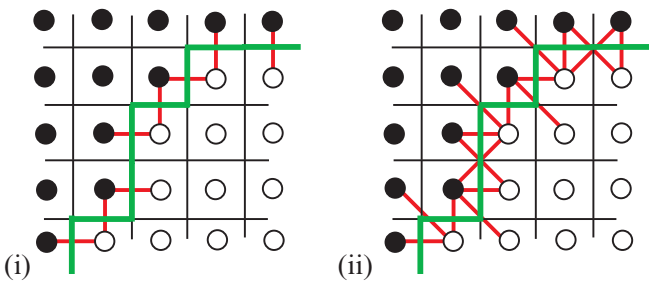

FIG. 2. (Color online) Scheme of an interface with explicit bond contribution depending on the neighborhood: (i) for von Neumann vicinity and (ii) for the Moore vicinity of eight neighbors.

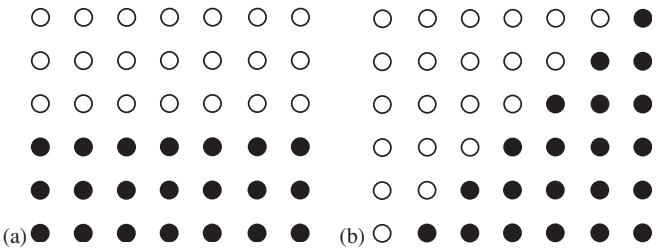

FIG. 3. (a) Staircase interface configuration. (b) Horizontal interface.

In the case of the Moore vicinity [see Fig. 1(ii)] one has in average three state-antistate bond per interface, but the corners modify slightly this equivalence. In a two-dimensional periodic lattice we have noticed that for closed interfaces the energy reads

$$E = E_0 + 2 \times \left(3 \sum \text{edges} - \sum \text{corners}\right) = E_0 + 2 \times (3 \times \text{perimeter} - \text{No. of corners}), \quad (10)$$

with $E_0 = -4N$. Notice that this result is not true in open interfaces, as we can see in the examples of Fig. 2.

In the same vein, in three spatial dimensions, one expects an energy expansion of the form

$$E = E_0 + c_2 \times \sum \text{faces} + c_1 \times \sum \text{edges} + c_0 \times \sum \text{corners}.$$

The constant $c_2 = 2$ in the case of the von Neumann vicinity and $c_2 = 2 \times 9$ for the Moore neighborhood. Other coefficients need a careful computation.

In conclusion, the energy of this system is under the class of universality of a Gibbs type of expansion in thermodynamics (L is the size of the system):

$$E = E_0 L^D + E_1 L^{D-1} + E_2 L^{D-2} + \cdots. \quad (11)$$

Remark. In two-dimensional lattices, the cases of von Neumann and Moore vicinities presents some differences. Basically, the Moore neighborhood is intrinsically more isotropic; indeed, the energy of a staircase interface [see in Fig. 3(a)] and of a horizontal [3(b)] or vertical interface have roughly—there is a minor difference because of the boundaries—the same energy; however, in the case of a von Neumann vicinity the energy of a horizontal or vertical interface is smaller than the energy of a diagonal (staircase) interface. More precisely, in the case drawn in Fig. 3(a) the energy is proportional to the interface length 17, while in Fig. 3(b) the perimeter is only 11. Though not clear in this small system, one can notice that the ratio between the staircase and a vertical or horizontal interface is $\sqrt{2}$. Indeed, a diagonal frontier in usual Euclidean geometry measures a length of $\sqrt{2}$ but because of the lattice a diagonal perimeter cost 2 in length in the von Neumann vicinity; therefore, the system prefers to enclose the individuals of one population inside a square domain oriented vertically and horizontally.

We see in next section that this fact emerges as a characteristic feature of the patterns displayed.

III. DYNAMICAL EVOLUTION OF THE SCHELLING MODEL

The Schelling model defined in Sec. II contains structural parameters of the lattice and two easily tunable parameters like the satisfaction parameter and the initial fraction of individuals of one population ϕ. From now on, we consider only regular lattices in one, two, and three spatial dimensions with a uniform value of the satisfaction parameter θ.

A. One-dimensional periodic lattice

The Schelling model in a one-dimensional lattice; that is, each site possesses two neighbors, divided into two different cases depending on the threshold θ. Because of the simplicity of this case, we treat, the one-dimensional lattice in some detail.

(i) The case $\theta = 2$. In this case an individual is unhappy if its two neighbors are in the opposite state. As an example, in the configuration ● ● ● ● ● ○ ● ● ● ● ● the site with a state in ○ is unhappy; in the configuration ○ ● ○ ● ○ ● ○ ● ○ ● all individuals are unhappy. Nevertheless, the configuration ○ ○ ○ ○ ○ ○ ○ ○ ○ ● ● ● ● ● ● ● is stable; moreover, any configuration where a population has a minimal length of two is always stable, for example, ○ ○ ● ● ○ ○ ● ● ● ○ ○ ○ ○ ● ● ● ○ ○ ● ●. Because of the large number of stable configurations, the Schelling dynamics quench very fast and the final state would consist of a scarcely segregated state.

(ii) The case $\theta = 1$. In this case an individual is unhappy if only one of its neighbors is of an opposite state. In this case the interfacial sites play an important role. As an example, in ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ● ● ● ● ● ● the underlined individuals at the interface are both in an unhappy state. The dynamics in this case is very rich because an interface is never stable. The previous configuration evolves via the Schelling dynamics through:

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ● ● ● ● ● ● → ○ ○ ○ ○ ○ ○ ○ ○ ○ ● ○ ● ● ● ● ●.

The only possible swap creates more unhappy individuals (underlined). For the next step one may go back to previous configuration or may depart further from the initial configuration, for example, ○ ○ ○ ○ ○ ○ ○ ● ● ○ ○ ● ● ● ● or ○ ○ ○ ○ ○ ○ ○ ● ○ ○ ● ● ● ● ●. Because of the natural instability of an interface the evolution never stops; moreover, although there is a visible evolution to segregation, the interface is never stable, as one may see in Fig. 4.

B. Two-dimensional periodic lattices

We analyze the dynamics in two-dimensional lattices. Among a great variety of regular lattices we consider the square lattice with a von Neumann vicinity $|V| = 4$ [see Fig. 1(ii)] and the Moore vicinity $|V| = 8$ [Fig. 1(iii)]. As in the one-dimensional lattice the dynamics depends on the parameters of intolerance θ, which varies between $1 \leqslant \theta \leqslant |V|$. The phenomenology also depends on the initial fraction of populations ϕ. For this purpose we consider an initially randomly distributed set of individuals of one population with a concentration ϕ in a large system size, usually 128×128. However, we emphasize that we ran simulations for various dimensions from 32×32 up to 512×512, noticing

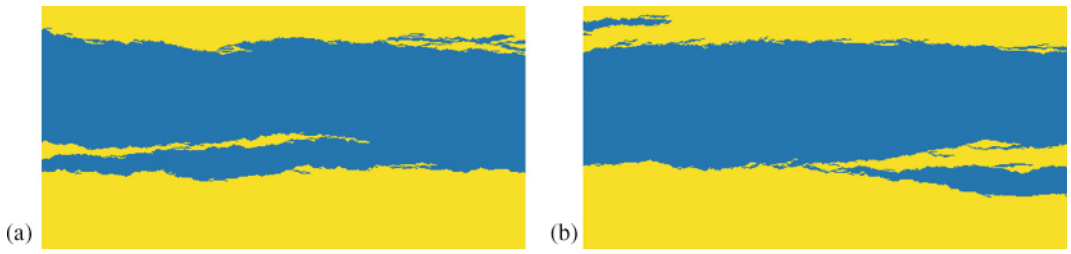

FIG. 4. (Color online) Spatiotemporal snapshot of the Schelling dynamics. The blue (dark gray) represents a state +1, while the yellow (light gray) represents a state −1. The horizontal axis represents the time (saved up to 512 steps) while the vertical axis represents the lattice (256 points) with periodic boundary conditions. The typical evolution for $\theta = 1$ is represented in these figures. In particular, the merging or coalescence of two domains is represented in (a) and a splitting or nucleation into two interfaces is captured in (b).

independence of the behavior in the system size. Despite the nonlocal character of the exchange rule (individuals are exchanged over all the system) the size of the system is not pertinent if the linear size of the system is much larger than the linear size of the neighbor.

1. The Schelling dynamics in a square lattice with four neighbors

As a first example we study the Schelling segregation problem with a von Neumann neighborhood composed of four nearest individuals. We recall that in this case the energy is just the perimeter of the interfaces between the regions of different individuals. The parameter θ thus runs from $\theta \in \{1,2,3,4\}$. One sees from the parameters phase space of cases $\theta = 1$ and $\theta = 4$ that the segregation is clearly inefficient. In the case $\theta = 1$ there is a constant swapping of unhappy individuals. The strongly unstable activity forbids any kind of self-organization. In the case $\theta = 4$ the number of unhappy individuals is very scarce; therefore, the dynamic stops very quickly because it reaches the end of one of the lists of unhappy individuals.

The case $\theta = 3$ is a situation in which the energy of the system diminishes after every swap. As already said, the evolution in this case ends in finite time because of the segregation quench in a semisegregated situation.

The case of $\theta = 2$, in which the energy does not necessarily decrease after a swap is, perhaps, the most interesting case: Because of the possibility of an increase of energy, the system evolves globally in a more efficient way the segregation so that it can exchange individuals increasing a bit the energy and allowing to decrease more and more the global energy. Indeed, after a long time the system is very efficient in segregation, as noticed in Fig 5. In this case the dynamics could be eventually in finite time. This happens if the domain containing a population closes its perimeter perfectly; thus, there is no more unhappy individual in the domain, and thus one list becomes empty (see Fig. 6). The Schelling dynamics stops this.

Perhaps the most relevant aspect of the case $\theta = 2$ is that segregation is not isotropic, indeed one observes that segregation leads to the emergence of square domains (see Fig. 5, case $\theta = 2$). From an energetic point of view, as it has already been said in Sec. II E, it is "cheaper" to make only vertical or horizontal frontiers. Finally, one notices that the energy decreases very fast for the cases $\theta = 4$ and $\theta = 3$ (see inset Fig. 7). In the case $\theta = 1$, a very unstable regime because of constant swapping is observed; hence, the energy fluctuates around a value which is relatively high with respect to the ground-state energy (despite an initial energy decreasing transient), which means that the segregation is not very efficient. The case of $\theta = 2$ is the most efficient because the energy diminishes until reaching the ground-state energy, which is a stripe (see Fig. 5, case $\theta = 2$ and $\phi = 50\%$).

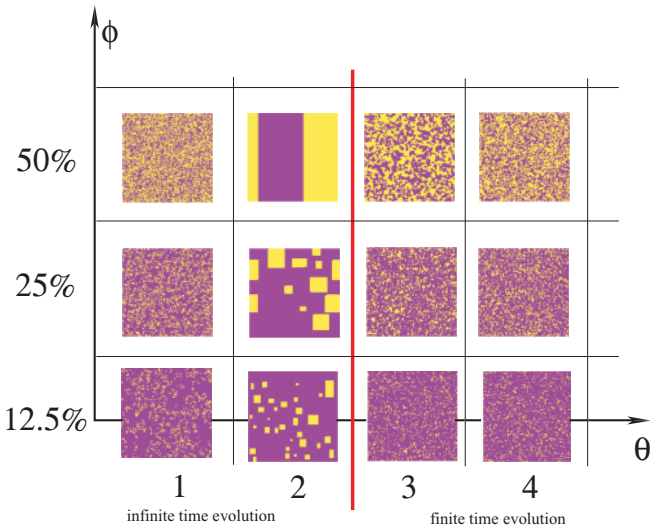

FIG. 5. (Color online) Phase diagram of the Schelling dynamics in a two dimensional square lattice with von Neumann neighbor. Each snapshot corresponds to a 128 × 128 numerical simulation, the violet (dark gray) represents a state +1, while the yellow (light gray) represents a state −1. This picture summarizes the phenomenology for different values of the initial concentration and the parameter θ.

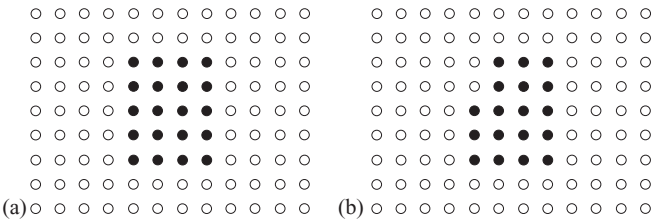

FIG. 6. (a) This configuration is stationary state of the Schelling segregation model with von Neumann neighborhood and with $\theta = 2$. Though there are four unhappy •'s, there is not a single ○ for an exchange. The configuration (b) is not a stationary state, because there are both unhappy •'s and ○'s. Schelling's rule runs forever in this case.

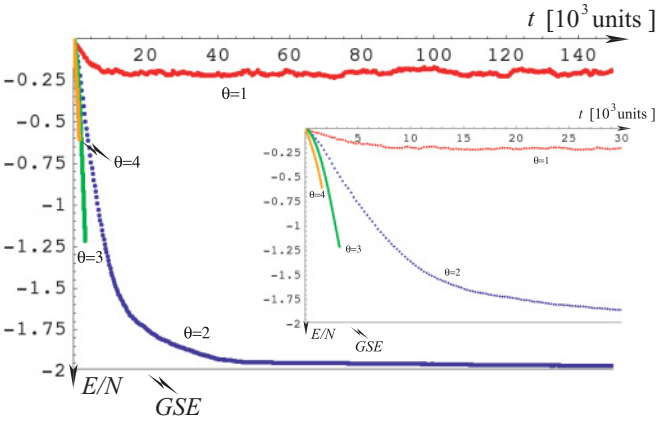

FIG. 7. (Color online) Plot of the Energy vs time for various values of θ and for a concentration of a 50% of individuals of both populations. The inset zooms of the early evolution of the energy. The GSE line represents the ground-state energy corresponding to the initial state of $\phi = 50\%$, given by the energy expression (9).

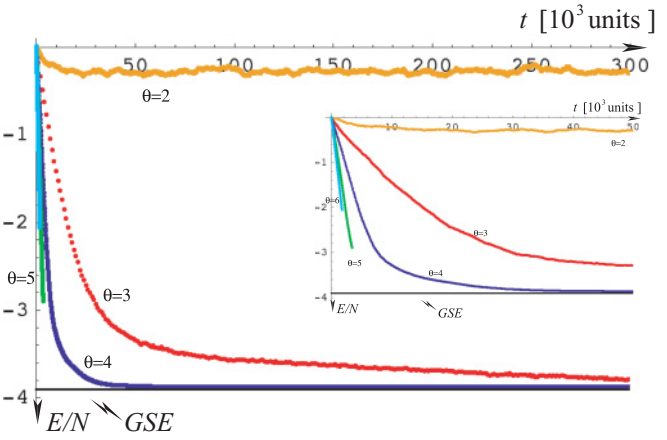

FIG. 8. (Color online) Plot of the energy vs time for various values of θ and for a concentration of 50% of individuals of both populations. The graphics plots the long-time evolution of the energy, while the inset plots a magnification of the early evolution of the energy. The GSE represents the ground-state energy (10) corresponding to a initial state of $\phi = 50\%$.

The square shape, which is observed for low concentrations, has a simple explanation. One needs to enclose a determined number of individuals ϕN inside a domain of a minimum possible perimeter. As has already been mentioned, vertical and horizontal lines are "cheaper"; therefore, the resulting geometrical figure would be a rectangle, indeed a square because it minimizes the rectangle's perimeter. The energy of a square of side a is $E = E_0 + 2 \times 4a$ with $a = \sqrt{N\phi}$; therefore, $E = E_0 + 8\sqrt{N\phi}$, while the energy of a stripe $E = E_0 + 2 \times 2\sqrt{N}$. Therefore, the square is advantageous for $\phi < 1/4$, while the stripe is the less energetic configuration for concentrations such a that $1/4 < \phi \leqslant 1/2$.

2. *The Schelling dynamics in a square lattice with the Moore neighborhood*

Let us consider now the case of the Moore vicinity [Fig. 1(iii)] where each individual is surrounded by eight neighbors; that is, θ rules between 1 and 8. Therefore, the range of parameter $\theta < 4$ and $\theta > 4$ is wider. A general result is that for $\theta \geqslant 5$ the energy only decreases (by a finite amount) after any exchange; consequently, the evolution is of finite time and ultimately stops. However, for $\theta \leqslant 4$ the dynamics may evolve forever.

Let us first consider the case of $\phi = 50\%$, that is, both populations are equally distributed. One sees that for $\theta = 5$ and $\theta = 6$, as in the case of the von Neumann vicinity, the system segregates until it reaches a well defined final state where the segregation is far from complete. The system quenches because one of the two list of unhappy individuals becomes empty.

In the same line, in these two cases one notices that the decreasing energy rate is larger for $\theta = 6$ than for $\theta = 5$ [in this case, $\Delta E \leqslant -4(2\theta - 9)$]; however, the final energy is less for $\theta = 5$ than for $\theta = 6$. Then in this sense $\theta = 5$ is more efficient than $\theta = 6$ which is more efficient than $\theta = 7$, etc. (see Figs. 8). The case $\theta = 4$, whenever the energy increases only in a limited number of cases, is the most efficient situation. The evolution finally approaches very near the ground-state energy (minimum energy possible allowed by the initial population conditions). As one approaches the ground sate the boundary fluctuates in a quite organized way. Sometimes the dynamics stop definitively and sometimes the dynamics continues indefinitely as in the case of Figs. 8 and 9.

In this case ($\theta = 4$) the dynamics is eventually restricted only to the swapping of individual on the interface. The interfaces play a major role, they are preferentially in angles 0°, 45°, and 90° with respect to the horizontal line. Therefore, the final state is composed of an assembly of facets; instead of an approximate circle, one has an octahedron. These facets are very characteristic in this case and they maybe seen in Fig. 10 (case $\theta = 4$). Because the dynamics is finally restricted only to the interface, this case may be mapped into a one-dimensional problem. Since in one dimension the interface fluctuates, in two spatial dimensions these fluctuations are also observed. The dynamics stops if the interfaces arrange their number of individuals to get at least one of the unhappy lists empty.

In the case $\theta = 3$ the energy reaches eventually the ground state; indeed, in Fig. 10 one sees that the state evolves to a lowest energy configuration. Although the dynamics allows an increment of energy, the energy changes are more frequent for a decreasing process than for an increasing process, so that, on average, the energy tends to the minimal allowed value, the ground-state energy. However, this process is less efficient than $\theta = 4$. The reason for that is because the existence of a larger fluctuating frontier, clearly observed in the sates of Fig. 10. Indeed, in the case $\theta = 4$ the frontier involves one layer of individuals; however, in the case $\theta = 3$ one sees a very rich dynamics in a thicker frontier, something similar to the one expected in two phase systems, for instance, a liquid droplet coexisting with vapor.

The minimal energy configuration is essentially realized via a minimal perimeter interface. The shape of the minimal perimeter interface depends on the initial fraction of individuals. If $\phi = 1/2$ (the two populations posses the same number of individuals) the minimum energy in the two-dimensional periodic square is a stripe (see Fig. 10, with $\theta = 3$ and

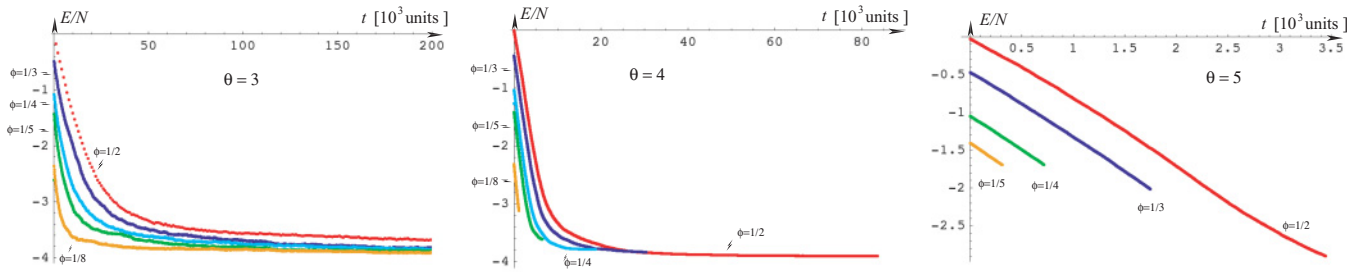

FIG. 9. (Color online) Plot of the energy per node E/N vs time for various values of θ and various values of the concentration of a ϕ of individuals. (a) Case $\theta = 3$. As said in the main text the energy decreases to reach efficiently almost the ground-sate energy. One notices, however, that the rate of energy decrease is quicker for smaller concentration of an initial population, for example, $\phi = 1/8$. (b) Case $\theta = 4$. Large concentrations tend to the ground-state energy, but the dynamics eventually stops in finite time for smaller concentrations ($\phi = 1/5$ and $\phi = 1/8$). (c) In the case of $\theta = 5$, the dynamics is of finite time. The decreasing energy rate is almost constant for all initial concentrations and takes values ranging from -14 to -15 units per step. This is roughly twice the maximum possible energy decreasing rate which is, after (8), $\Delta E < -8$ units for a not close neighbor exchange. Finally, one notices that for $\phi = 50\%$ the energy is the lowest at the end. This is clearly shown in Fig. 10, where, as one sees, that segregation effectively takes place and both communities separate and the perimeter decreases.

$\phi = 50\%$); however, as the initial fraction of individuals of one population is lowered the minimal perimeter curve becomes a circle (see Fig. 10, with $\theta = 3$ and $\phi = 25\%$ and $\phi = 12.5\%$).

These considerations follow from an energy argument. Let us consider the minimal energy of a state where the concentration of one population is ϕ. As already said, this is a problem of minimizing a perimeter under the constraint of having a total area of population, for example, N_+ fixed. In general this is a circle. The energy of a circular domain composed only by individuals of the population N_+ is $E = E_0 + c_1(2\pi r)$, with r the radius of the circle, which is given by the area of the region $N\phi = \pi r^2$. Thus, the total energy is $E = E_0 + c_1 2\sqrt{\pi N\phi}$, while the energy of a straight strip of individuals $+1$ is simply twice thst of the borderlines $E = E_0 + c_1 2\sqrt{N}$, so that the circular domain is the lowest energy if $\phi < \frac{1}{\pi}$ and the stripe domain is the lowest for $\frac{1}{\pi} < \phi < \frac{1}{2}$. The comparison of these energies is done in the Figs. 11(i) and 11(ii).

The highly intolerant case $\theta = 2$ is dramatically less efficient than the $\theta = 3$ case; the number of exchanges which increase the energy and the events that decreases the energy are roughly similar (excepting, perhaps, during the early evolution; see Fig. 8). For an initial concentration $\phi = 1/2$ the system essentially does not segregate, the high level of unhappiness makes exchanges so often that the system cannot self-organize in a coherent way (see Fig. 10 for $\theta = 2$ and $\phi = 50\%$). However, for lower concentration the system can self-organize and it does create a segregate domain in a very particular way (see Fig. 10 for $\theta = 2$ and $\phi = 25\%$ and $\phi = 12.5\%$).

Ending this section we discuss the energy long-time behavior. The long-time energy behaviors for the cases $\theta = 3$ and $\theta = 4$ are expounded in Fig. 8. As already said, the energy decreases faster in the case $\theta = 4$ than in the case $\theta = 3$; indeed, for $\theta = 4$ one has that the energy approaches the ground-state energy as [see Fig. 12(b)]

$$E/E_{GS} - 1 \sim t^{-3/2},$$

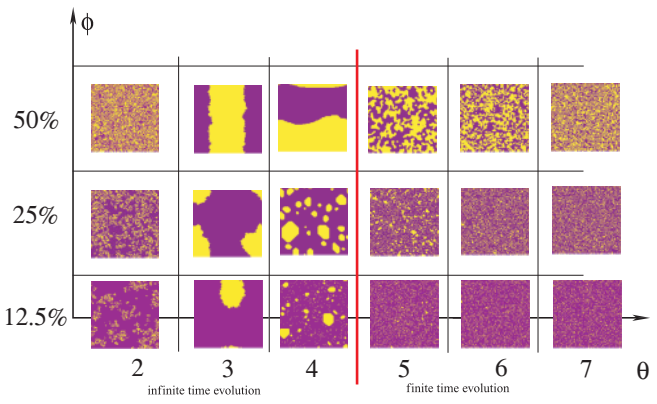

FIG. 10. (Color online) Phase diagram of the Schelling dynamics in a two-dimensional square lattice with the Moore neighbor. Each snapshot corresponds to a 128×128 numerical simulation, the violet (dark gray) represents a state $+1$, while the yellow (light gray) represents a state -1. This picture summarizes the phenomenology for different values of the initial concentration and for different values of the parameter θ.

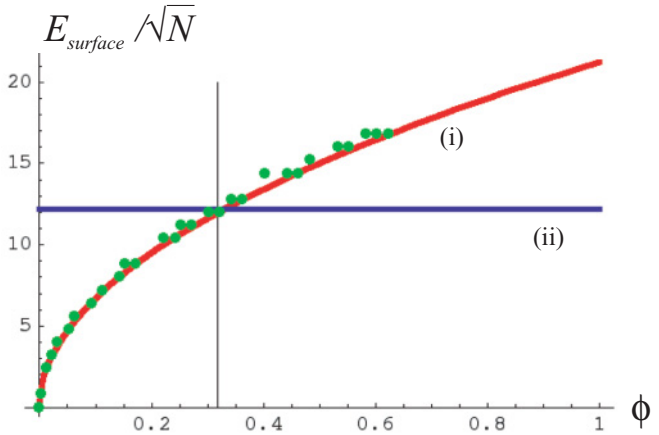

FIG. 11. (Color online) Plot of the rescaled perimeter energies ($E_{surface}/\sqrt{N}$) of (i) circular configuration and (ii) a planar domain. The dots correspond to the numerical values of the energies for these given states and for a concentration ϕ. The vertical line corresponds to $\phi = 1/\pi$; it intersects the two other energies $E_{surface}/\sqrt{N} \approx 6 \times 2 = 12$.

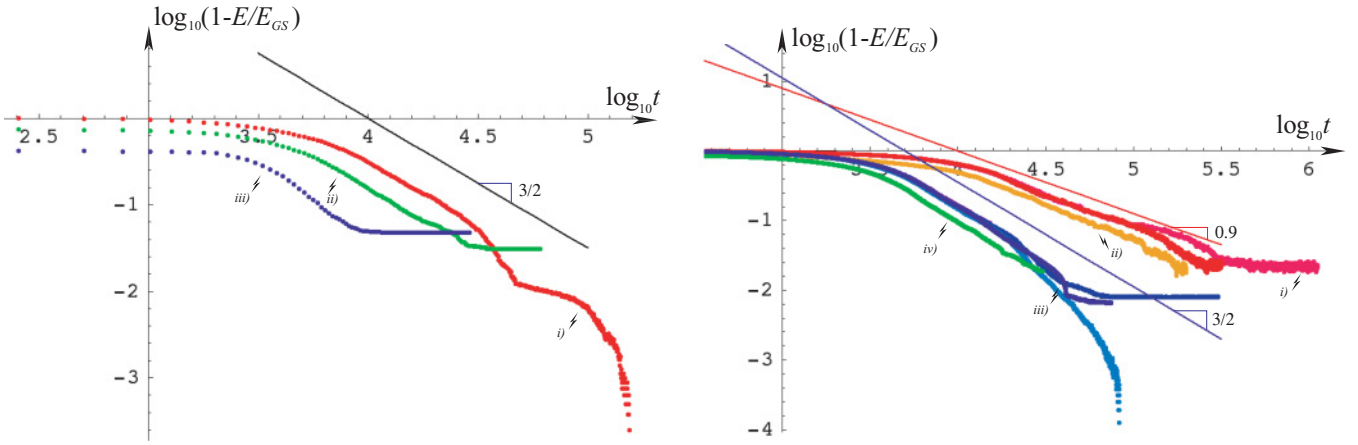

FIG. 12. (Color online) Plot of the intermediate evolution of the energy vs time in log-log scale for the case of the von Neumann (a) and Moore (b) vicinities. Precisely, we plot $\log_{10}(1 - E/E_{GS})$ vs $\log_{10} t$ with $E_{GS} = -32\,256$ in the case of the von Neumann neighborhood (a) and $E_{GS} = -128\,000$ for the case of the Moore vicinity (b). In plot (a), the parameter of tolerance is $\theta = 2$, and the curves are (i) $\phi = 1/2$, (ii) $\phi = 1/4$, and (iii) $\phi = 1/8$. For plot (b), one has that (i) $\theta = 3$ and $\phi = 1/2$ and (ii) $\theta = 3$ and $\phi = 1/3$. The evolution during the "coarsening" phase is consistent with a behavior $1 - E/E_{GS} \sim t^{-0.9}$. The cases (iii) $\theta = 4$ and $\phi = 1/2$ and (iv) $\theta = 4$ and $\phi = 1/3$ are also plotted. The evolution in this cases is faster during the "coarsening" phase, which is consistent with a behavior $1 - E/E_{GS} \sim t^{-3/2}$.

indicating that the perimeter decreases as $P \sim t^{-3/2}$. Similarly, in the case of von Neumann vicinity, with $\theta = 2$, one has also a $t^{-3/2}$ behavior in time. However, in the case $\theta = 3$ the evolution during the "coarsening" phase is consistent with a behavior $E/E_{GS} - 1 \sim t^{-0.9}$. We notice that these behaviors do not depend that much on the initial concentrations. One notices that the energy rates are much faster than a diffusive process. We do not have an explanation of any of these behaviors.

Discussion . In conclusion, segregation is more effective in the case of fairly intolerant cases $\theta = 4$ and $\theta = 3$. The mechanism of segregation may be explained simply from the phenomenology observed. In the cases $\theta = 5$ and $\theta = 6$ the decrease in energy rate is large but the system decreases so fast its energy and the dynamics end quenched in a frozen segregation structure because one of the lists of unhappy individuals becomes empty. Similarly, one cannot rearrange some exchanges because this implies that the energy must increase a bit, something forbidden in these cases. In the cases $\theta = 4$ and $\theta = 3$ the system is sometimes allowed to increase its energy and the system may reorganize the individuals to minimize energy. This happens naturally because of the fluctuations of the interface among individuals of different population. These interface fluctuations enhance the segregation effectivity.

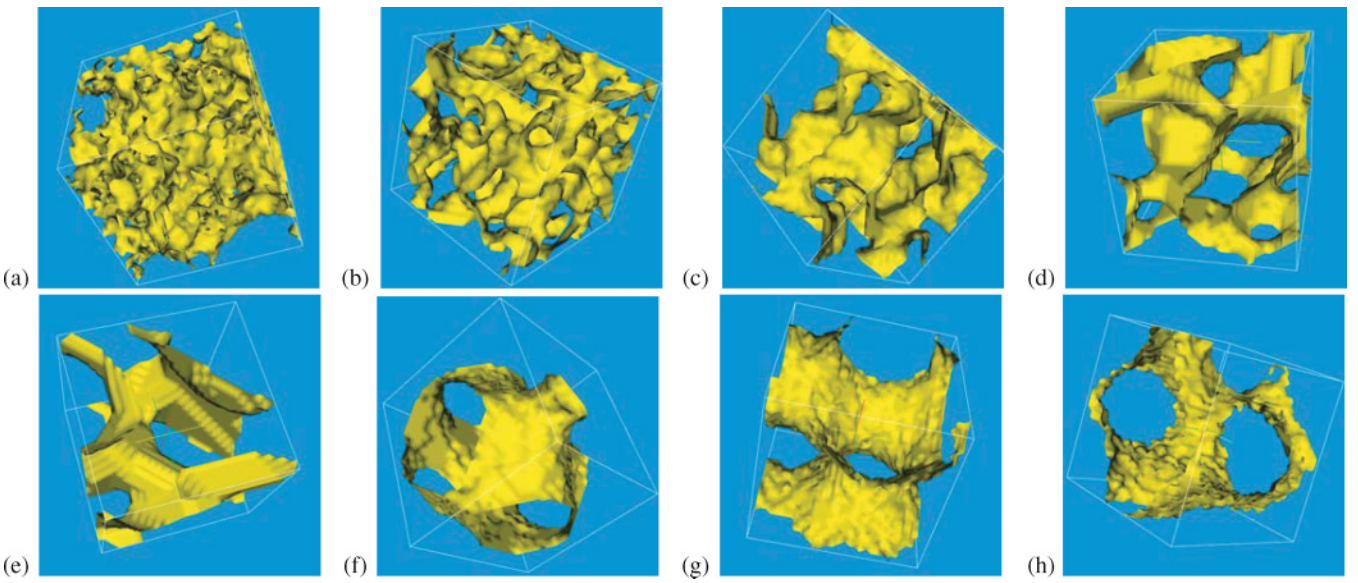

FIG. 13. (Color online) 3D snapshots of the interface (in yellow or light gray) between the two distinct populations after evolution of the Schelling segregation model in 3D with the Moore vicinity. The initial state is a random distribution of individuals with an initial concentration $\phi = 1/2$. Cases (a) $\theta = 18$, (b) $\theta = 16$, (c) $\theta = 15$, (d) $\theta = 14$, (e) $\theta = 13$, (f) $\theta = 12$, (g) $\theta = 11$, and (h) $\theta = 10$.

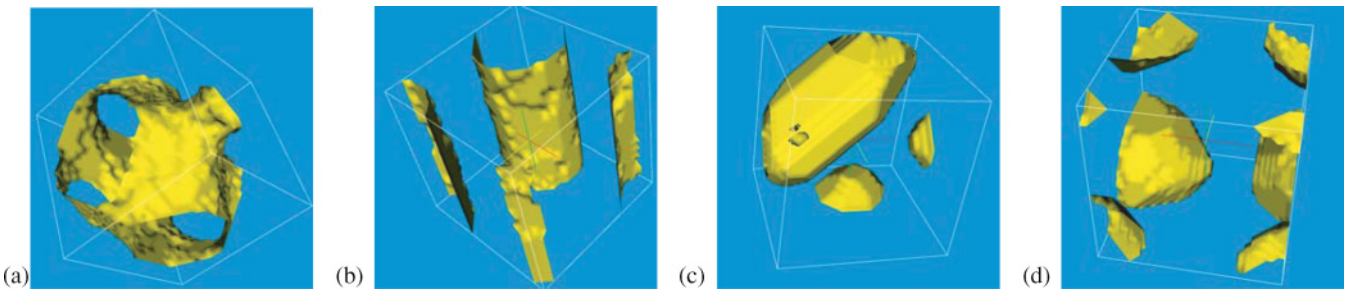

FIG. 14. (Color online) 3D Schelling segregation model with the Moore vicinity. Case $\theta = 12$ for initial concentrations (a) $\phi = 1/2$, (b) $\phi = 1/3$, (c) $\phi = 0.225$, and (d) $\phi = 1/5$.

C. Three-dimensional periodic lattices

Finally, we consider the Schelling dynamics in a three-dimensional lattice with a $3 \times 3 \times 3$ (excluding the center) neighborhood. This vicinity contains 26 sites. The values of the parameter θ ranges from $\theta = 1$ to $\theta = 26$. One expects to have a broader range to observe in more detail the regimes near the value $\theta = 13$ (in some sense the parameter θ becomes a continuous variable). The full phenomenology is represented in Fig. 13 for various values of the parameter θ. Although the segregation mechanism works perfectly and the images of segregation, in the phenomenology there is no big surprise; naturally, for $\theta \geqslant 14$ the dynamics decreases strictly energy at each swap, as in the one- and two-dimensional cases, and the system reaches in finite time to a quenched state, which is a kind of porous media. The porosity size depending on the value of θ. Indeed, as θ reaches the critical $\theta_c = 13$ the typical length of the porosity increases.

For $\theta = 13$ the system does not decrease strictly the energy and allows a search of an optimal configuration which eventually minimizes the energy. As in the case of a two-dimensional lattice with the Moore neighborhood, the system creates well defined facets, which minimizes the energy.

For θ less than, but close to, 12 the system evolves slowly to a global energy minimum, similar to the case $\theta = 3$ for the two-dimensional lattice with the Moore vicinity and $\theta = 3$. As in the two-dimensional case, the minimum energy is given by the minimum surface interface. The minimal surface, at constant volume, depends on the initial fraction ϕ. For $\phi = 1/2$ in the periodic three-dimensional cube the minimal surface is the so-called Schwarz surface [12,13]; as one decreases the initial fraction one has that the minimal surface becomes a cylinder, and for low initial concentrations it is a sphere (see Fig. 13 and 14).

As in the two-dimensional lattice the dynamics explore the configuration of a minimal surface. For a low concentration one has that the minimal surface problem is a spherical bubble. Similarly, one may observe the energy of a cylindrical concentration, which is essentially the circular interface in two dimensions. Finally, unlike the two-dimensional lattice, the lowest energy is not a straight strip for larger concentrations but a Schwarz minimal P surface [13], which is a periodic surface of genus 3.

The energy of a sphere is $E = E_0 + c_2(4\pi r^2)$, with $N\phi = \frac{4\pi}{3} r^3$, that is, $E = E_0 + c_2 6^{2/3} \pi^{1/3} \phi^{2/3} N^{2/3}$, while the energy of the cylindrical configuration is (the length of the cylinder is $N^{1/3}$) $E = E_0 + 2c_2\sqrt{\frac{\phi}{\pi}} N^{2/3}$.

So that the energy of a spherical distribution is the lowest if $\phi \leqslant \frac{4\pi}{81} \approx 0.155\,14$, while the cylindrical distribution of individuals is the lowest energy for $\frac{4\pi}{81} \leqslant \phi \leqslant \frac{1}{\pi}$, the Schwarz-P surface is the optimal for $\frac{1}{\pi} \leqslant \phi \leqslant \frac{1}{2}$. These behaviors maybe seen roughly in Fig. 14.

IV. CONCLUDING REMARKS, DISCUSSION

In this article we have discussed various properties of the Schelling segregation both from the points of view of physical science. Our study was focused on intensive numerical simulations of the Schelling segregation model for one-, two-, and three-dimensional lattices. More precisely, we varied the size of the neighborhood, $|V|$, the degree of tolerance parameter, θ, and the initial proportion of the populations, ϕ. We established a general satisfaction criterion which characterizes the fundamental instability of the segregation phenomena. Further, we characterize a macroscopical quantity, a kind of energy or "utility function," which decreases efficiently if $\theta \approx |V|/2$.

Perhaps the most interesting result of this article deals with the situation in which the tolerance parameter $\theta \lesssim |V|/2$ in which case the energy cannot be interpreted as a thermodynamical equilibrium. However, the numerical simulations show that the long-time evolution displays a very efficient segregation pattern with an energy close to the ground-state energy.

There are many variations on the Schelling segregation problem, for instance, to consider the existence of empty sites or vacancies. As we already said in Sec. II D (Remark 7) the energy introduced maybe easily generalized for multiple values of the variable x_k. On the other hand, one may undertake a mean field approach whenever the number of neighbors is much larger than unity and use the powerful techniques of the analysis. Similarly, no regular networks, as undirected graphs, may produce more realistic applications to segregation. In this case, from a phenomenological point of view, it seems that the relevant parameters are an average of discrimination parameter θ_k and the degree of the graph. Finally, another variant not considered in the present model is the following. Usually the satisfaction parameter θ_k depends on the individual, but here we considered that θ_k depends on the site k. We can state the following open problem: *What would happen if when one exchanges two different individuals, they keep their own θ?* These considerations will be treated elsewhere.

We conclude with some considerations of the computational complexity of a prediction problem associated to the Schelling segregation rule, in particular the connection of the physical system with the computational complexity.

One may define the complexity of a physical system as the number of computer steps needed to predict its outcome. In this context, we study the following prediction problem: *Given an initial condition, we want to know if a swap sequence exists such that one specific site will change in a finite number of steps*. The answer to this question is not always simple, and in some cases it can take a huge amount of time depending on the lattice dimensions and the happiness parameter. Hence, spending a large amount of time implies that predicting the system behavior is not always possible. In these cases the only feasible option to answer our prediction problem is to fully run the simulation.

We have proof, for the one-dimensional lattice case, that it is always possible to make a fast prediction. For the two-dimensional lattice with the von Neumann (four nearest neighbors in a square lattice) or the Moore neighborhood (the eight neighbors of the 3×3 neighbor in a square lattice), we prove that for very tolerant or intolerant individuals the system behavior is also easy to predict. However, when the tolerance paramenter is around $|V|/2$ to perform a prediction becomes a difficult problem, which means that we have to run the full simulation to obtain an answer. We sketch these considerations in the Appendix.

In summary, from a physical and computing complexity point of view, we notice that the case of very tolerant people $\theta \lesssim |V|$ and the case of very intolerant people $(1 \lesssim \theta)$ are very similar. However, although the cases $\theta \approx |V|/2$ are very complex from a computational point of view, from a physical point of view there are some differences: If $\theta > |V|/2$ the energy is a thermodynamical potential which drives the system to an equilibrium whence segregation stops.

ACKNOWLEDGMENTS

In conclusion, the authors thank Gonzalo Ruz for his comments and interest in this work. E.G. acknowledges the CNRS and the Laboratoire I3S de l'Université de Nice Sophia Antipolis, where part of this work was done. He also is grateful for Fondecyt Grant No. 1100003, Anillo ACT-88 (U. Adolfo Ibáñez), and Basal CMM-U.de Chile, and S.R. is grateful for Fondecyt Grant No. 1100289 (Chile) and the ANR SYSCOM COSTUME (France).

APPENDIX: COMPLEXITY OF THE SEGREGATION MODEL

The computational complexity of a model can be defined as the number of computer steps required to predict it. By "predict" we mean to know if a given site swaps after a given time. In this context, we associated to the segregation model the generic prediction problem.

Segregation prediction problem (SPP). Given an initial configuration of $+1$ and -1 in a D-dimensional lattice, consider a specific site, say 0, such that $x_0(t = 0) = -1$. Then, does a finite number of applications of the segregation model

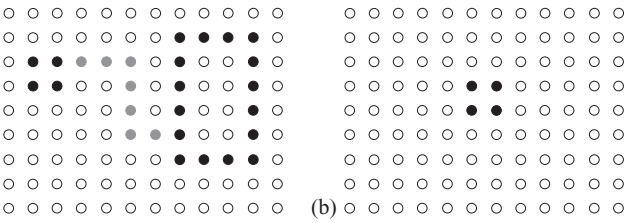

FIG. 15. (a) Fixed connected component. The symbol $\bullet \equiv -1$, $\circ \equiv +1$. The gray $\bullet \equiv -1$ states are stable because the are connected to two self-stable blocks (the $\bullet$). (b) Minimum fixed connected component.

exist $T \geqslant 1$, (a sequence of swaps) that change the value of this site to $+1$?

In some cases to find such a sequence could take a large amount of time, depending on the specific graph, the dimension of the lattice or the happiness parameter θ, as we sketch below. Essentially, we try to characterize such complexity in D-dimensional lattices and for any happiness parameter. To do that we consider the classic complexity classes P and NC. The class P takes into account the problems which can be solved, by a serial computer, in a polynomial time (N^{α}) on the problem size N. The class NC considers problems whose can be solved quickly, in polylogarithmic time of the order of $(\log N)^{\alpha}$ in a parallel computer with a polynomial number of processors. Clearly NC is contained in P. The deep question concerning both classes is the following: If they are really different, that is, if intrinsically sequential problems exist, so that it cannot be possible to translate them to a parallel computer with a logarithmic time performance. One of this candidates (to be in P but not in NC) is the *circuit value problem* (CVP): Given a Boolean circuit (a directed graph whose nodes are AND, OR, and NOT gates), and given the truth values of its inputs (say 0's and 1's), then, is the output true (1) or false (0)?

This intuition is associated with the observation that it is hard to imagine how to compute the output without sequentially going across each layer in the circuit, so the time is directly associated with the depth of the circuit. We say that a problem in P is P complete if it is in P and if any other problem in this class P can be reduced to it in log-space, that is, the amount of memory to codify the information scales as the logarithm of the size of the problem $(\log N)$. Further, the CVP and the particular case of the *monotone*

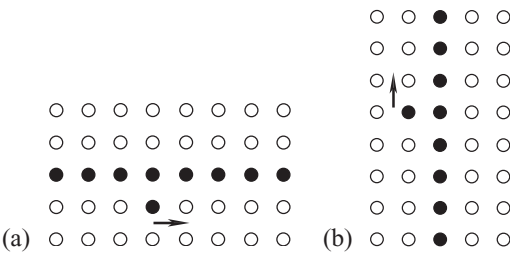

FIG. 16. Representation of wires which convey information, a $\bullet$ represents a -1 state, while a $\circ$ represents a $+1$ state. The isolated $\bullet$ represents a -1 traveling in the direction of the head of the arrow by swapping with its neighbor in the same line or column. (a) Horizontal wire and (b) vertical wire.

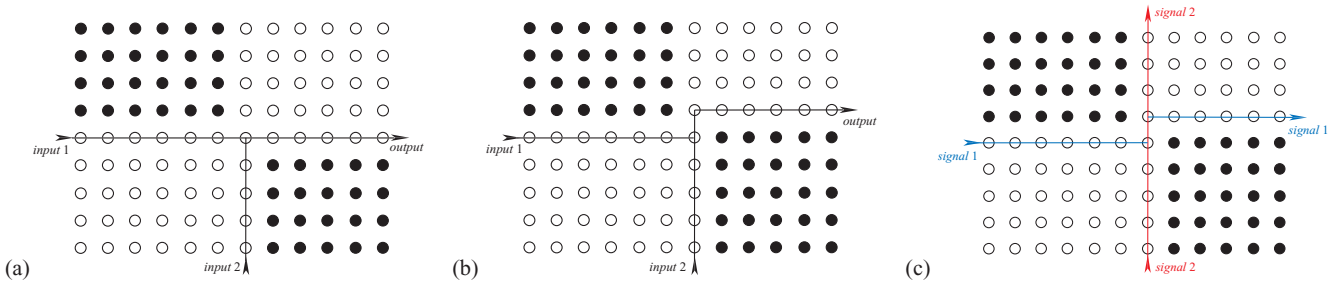

FIG. 17. (Color online) (a) AND gate; (b) OR gate; and (c) CROSS-OVER.

circuit value problem (MCV) (circuits without negations) are P complete [14,15]. From previous remarks we may say that P problems are inherently sequential unless P = NC. Thus, to prove or disprove the P completeness of a problem is a good way to taste its prediction possibilities. So in this context we study the complexity of the SPP in D-dimensional systems, with $D = 1,2,3$.

Let us consider the one-dimensional case, that is, a periodic lattice with N sites, such that each one is connected to its nearest neighbors and the happiness parameter may take the value $\theta = 1$ and $\theta = 2$. Because of simplicity we study in detail the case $\theta = 1$. We recall that in this situation a site is unhappy when at least one of its neighbors is in the opposite state. So a couple of consecutive different states are both unhappy and if we interchange them they remain unhappy. From that and the definition of SPP it is easy to see that to give the answer to the SPP one has to know the nearest site (left or right) to the site 0 such that it is at state $+1$. First, one computes the position of sites at state $+1$. By considering N processors we do that in constant time. After that, for each position k, such that it is in state $+1$, we compute $\min\{k, N-k\}$, and finally we compute the minimum of this value by using $\mathcal{O}(N)$ processors in $\log N$ steps. So the SPP is in the NC class.

For $\theta = 2$ the analysis is roughly similar, which implies that the SPP also belongs to the class NC.

As an illustration of our complexity study in two-dimensional lattices, let us consider the von Neumann neighborhood so $\theta \in \{1,2,3,4\}$. In this situation one may prove, by using a similar argument to the one-dimensional case, that for $\theta = 1$ and $\theta = 4$ the SPP belongs to the class NC. In the case $\theta = |V|/2 \equiv 3$ we have an energy principle so every initial configuration converges to a fixed point. The shape of a fixed point is such a that each site has at least two neighbors in its same value. Typically we have, say, for -1, fixed configuration, as in Fig. 15.

In Figs. 15 each site has at least two neighbors in its state. The square in Fig. 15(b) is the smallest fixed configuration. Then a connected component of -1 (the same argument holds for $+1$) will be fixed if and only if each site has two -1's. We say that a configuration of -1's ($+1$'s) is *self-stable* if, and only if, it is a rectangle or a circuit of -1's ($+1$'s). In this context a configuration of -1's ($+1$'s) is fixed if, and only if, (i) it is self-stable or (ii) each site is connected to two different self-stable blocks.

Therefore, an algorithm to know if the site, say (0,0) at a state -1 has some possibility to change in finite time consists essentially of computing the -1 connected components which contain (0,0) and verifying if this site belongs to a self-stable block or it is between two different self-stable blocks. Otherwise, if there are enough unhappy sites at value $+1$ the site (0,0) will swap to $+1$. Moreover, to compute a connected component in a parallel computer with a polynomial number of processor takes $\mathcal{O}(\log N)^2$ steps [16]; furthermore, computing if the site (0,0) belongs to a self-stable block, or it is connected to two self-stable blocks, is equivalent to computing if the site belongs to a biconnected component which can be done, in a parallel computer, also in $\mathcal{O}(\log N)^2$ steps [16]. Finally, one also may compute the number of unhappy 1'Õs in $\mathcal{O}(\log N)$ steps. In conclusion, the decision problem, to determine if a site may change its initial state, can be answered in a parallel computer in $\mathcal{O}(\log N)^2$, so SPP belongs to the class NC.

The most complex case happens for $\theta = 2$. In this situation we may prove the decision problem is P complete. Essentially, our proof consists to reduce the MCV (monotone circuit problem) to SPP. To achieve that we define as specific configurations of the segregation model wires (see Fig. 16), the AND and OR gates, as well as a CROSS-OVER (see Fig. 17). Clearly, for the CROSS-OVER we could create a misleading signal if, for instance, a vertical one (from the bottom to the top) in the intersection goes to the right. To avoid that we use a specific way to update the network. First, the CROSS-OVER, if it exists, will be only in sites (a,a) in the diagonal of the lattice, and the update policy we consider is the following: First, form the bottom to the top and from the left to the right, update by rows, contiguous horizontal sites $(x,y) \neq (a-1,a)$ for any $a \geqslant 1$, or else update the three couples of sites $\{(a-1,a),(a,a)\},\{(a,a),(a,a+1)\},\{(a+1,a+1),(a+2,a+1)\}$. When this procedure is finished, update every couple of neighbor sites from the bottom to the top and from left to right.

By doing so we evaluate any monotone circuit by using previous update rule as well as configurations defined below. Hence, the Schelling prediction problem is P complete.

One may study in a similar way the case of the Moore neighborhood in a two-dimensional lattice. Further, in the three-dimensional lattice with the nearest neighborhood and $\theta = 3$, the prediction problem is P complete. The P completeness is inherited from the two-dimensional case; the proof follows similarly as in the two-dimensional lattice with the von Neumann neighborhood and $\theta = 2$. Another possibility is to build logical gates directly in 3D, as in the majority automata studied in [14].

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