PHYSICAL REVIEW E **83**, 051134 (2011)

Heterogeneous k-core versus bootstrap percolation on complex networks

G. J. Baxter,[1,*] S. N. Dorogovtsev,[1,2] A. V. Goltsev,[1,2] and J. F. F. Mendes[1]

[1]*Departamento de Física, I3N, Universidade de Aveiro, Campus Universitário de Santiago, PT-3810-193 Aveiro, Portugal*

[2]*A. F. Ioffe Physico-Technical Institute, RU-194021 St. Petersburg, Russia*

(Received 20 December 2010; published 31 May 2011)

We introduce the heterogeneous k-core, which generalizes the k-core, and contrast it with bootstrap percolation. Vertices have a threshold r_i, that may be different at each vertex. If a vertex has fewer than r_i neighbors it is pruned from the network. The heterogeneous k-core is the subgraph remaining after no further vertices can be pruned. If the thresholds r_i are 1 with probability f, or $k \geqslant 3$ with probability $1 - f$, the process can be thought of as a pruning process counterpart to ordinary bootstrap percolation, which is an activation process. We show that there are two types of transitions in this heterogeneous k-core process: the giant heterogeneous k-core may appear with a continuous transition and there may be a second discontinuous hybrid transition. We compare critical phenomena, critical clusters, and avalanches at the heterogeneous k-core and bootstrap percolation transitions. We also show that the network structure has a crucial effect on these processes, with the giant heterogeneous k-core appearing immediately at a finite value for any $f > 0$ when the degree distribution tends to a power law $P(q) \sim q^{-\gamma}$ with $\gamma < 3$.

DOI: 10.1103/PhysRevE.83.051134 PACS number(s): 64.60.aq, 64.60.ah, 05.10.−a, 05.70.Fh

I. INTRODUCTION

Bootstrap percolation and the k-core are closely related concepts and, in fact, it is easy to confuse the two. Both belong to a new class of systems with hybrid phase transitions, yet it can be clearly shown that the two processes do not map onto each other. Here, we elucidate the relationship and differences between these two concepts by introducing a generalization of the k-core, the heterogeneous k-core.

The k-core is the maximal subgraph in which all vertices have an internal degree of at least k [1]. It has proved to be a useful tool, giving insight into the deep structure of complex networks [2–6], and has found applications in diverse areas, from rigidity [7] and jamming [8] transitions to social interactions [9], protein networks [10], real neural networks [11,12], and evolution [13]. The k-core has been extensively studied on treelike networks, starting with Bethe lattices [14,15] and random graphs [16–18], before finally being extended to arbitrary degree distributions [5,19–21]. Hyperbolic lattices have also been considered [22]. Other studies, mostly numerical, have considered the sizes of culling avalanches [23–25]. Results on nontreelike graphs have been largely numerical [26,27], although some analytic results that incorporate clustering have recently been obtained [28,29]. At the same time, bootstrap percolation has emerged as a useful model for a variety of applications such as neuronal activity [30–32], jamming and rigidity transitions and glassy dynamics [33,34], and magnetic systems [35]. In bootstrap percolation, a set of seed vertices is initially activated and other vertices become active if they have k active neighbors. This process has been investigated on two- and three-dimensional lattices (see Refs. [36–39] and references therein). Bootstrap percolation has been studied on the random regular graph [40,41], on infinite trees [42], and, most recently, on general complex networks [43]. Finite random graphs have also been studied [44]. An interesting alternative formulation is the Watts model of opinions, in which the threshold is defined as a certain fraction of the neighbors rather than an absolute number [45]. These processes may also be generalized so that the thresholds may be different at each vertex [43,46].

Here, we introduce a generalization of the k-core, the heterogeneous k-core. In the heterogeneous k-core, each vertex i in a network has a hidden variable, its threshold value r_i. The heterogeneous k-core is the largest subgraph whose members have at least as many neighbors within the subgraph as their threshold value, r_i. This may include finite clusters as well as any giant component. If r_i are all equal to each other we recover the standard k-core. We define a simple representative example of the heterogeneous k-core (HKC) in which vertices have a threshold of either one or $k \geqslant 3$, distributed randomly through the network with probabilities f and $(1 - f)$, respectively. This can be directly contrasted with bootstrap percolation, in which vertices can be of two types: with probability f they are 'seed' vertices which are always active, while with probability $(1 - f)$ vertices become active only if their number of active neighbors reaches a threshold k. The difference between these two processes arises because bootstrap percolation is an activation process beginning from a sparsely activated network while the heterogeneous k-core is a pruning process [16,17] beginning from a complete graph.

We observe two transitions in the size of the giant heterogeneous k-core (giant HKC): a continuous transition similar to that found in ordinary percolation and a discontinuous hybrid transition similar to that found for the ordinary k-core. We find a complex phase diagram for this giant HKC with respect to the proportion of each threshold and the amount of damage to the network, in which, depending on the parameter region, either transition may occur first. Two similar transitions are observed in the phase diagram of the giant component of active vertices in bootstrap percolation (which we will refer to as the giant BPC).

*gjbaxter@ua.pt

Finally, we show that network heterogeneity plays an important role. When the second moment of the degree distribution is finite but the third moment diverges, the giant HKC (or giant BPC) appears at a finite threshold not linearly but with a higher order transition. When the second moment of the degree distribution diverges—as in scale-free networks—the thresholds may disappear completely, so that the giant HKC (or giant BPC) appears discontinuously at a finite value for any $f > 0$ or $p > 0$.

II. THE HETEROGENEOUS *k*-CORE AND BOOTSTRAP PERCOLATION

Consider an arbitrary uncorrelated sparse complex network defined by its degree distribution $P(q)$. In the infinite size limit such networks are locally treelike, a property which enables the analysis we will use. The network may be damaged to some extent by the removal of vertices uniformly at random. The fraction of surviving vertices is p.

In the heterogeneous k-core, each vertex of a network is assigned a variable $r_i \in \{0,1,2,\ldots\}$. The r_i values are assumed to be uncorrelated, selected from a distribution $Q_k(r)$. The heterogeneous k-core is then the largest subgraph of the network for which each vertex i has at least r_i neighbors within the heterogeneous k-core. To find the heterogeneous k-core of a given network we start with the full network and prune any vertices whose degree is less than its value of r_i. As a result of this pruning, other vertices will lose neighbors and may thus drop below their threshold, so we repeat the pruning until a stationary state is reached. The remaining subgraph is the heterogeneous k-core. If it occupies a non-vanishing fraction of the original network in the limit that the size of the network goes to infinity, we say it is a giant heterogeneous k-core (giant HKC).

If all $r_i = 1$, then the HKC is simply the connected component of the network and the giant HKC is the giant connected component, exactly as in ordinary percolation. As is well known [6,47,48], this appears with a continuous transition at the critical point $p_c = \langle q\rangle/[\langle q^2\rangle - \langle q\rangle]$, where $\langle q^n\rangle = \sum_i q^n P(q)$. If $k = 2$, we again have a continuous transition similar to ordinary percolation, as the 2-core is obtained by simply pruning all dangling branches from the 1-core. If all r_i are equal to $k \geqslant 3$, then we have the ordinary k-core. In this case, the giant k-core appears with a discontinuous hybrid transition [5,8,19,49].

Let us briefly discuss the nature of hybrid phase transitions. These transitions form a specific new kind of phase transition that combines a discontinuity, like a first-order phase transition, with a critical singularity, like a continuous phase transition. Strong analogies between bootstrap percolation and metastable behavior in systems with first-order phase transitions were remarked upon in Ref. [50]. Here, we discuss this question from a physical point of view. In thermodynamics, where the changes of a control parameter (e.g., decreasing or increasing temperature) are assumed to be infinitely slow, a first-order transition has no hysteresis. In reality, the changes always occur with a small but finite rate and the "heating" and "cooling" branches of a first-order transition do not coincide, which indicates the presence of a metastable state and hysteresis. The width of the hysteresis increases with this rate until it reaches a limit that corresponds to the limiting metastable state. The resulting limiting curve for the order parameter has a square-root singularity at the breakdown point. For example, if we heat a ferromagnet with a first-order phase transition sufficiently fast, the curve "magnetization M versus temperature T" has a singularity $M(T) - M(T_b - 0) \propto \sqrt{T_b - T}$ at the breakdown point, T_b, and a discontinuity, that is, $M(T_b - 0) > M(T_b + 0) = 0$. One can see this, for example, using the Landau theory with a free energy $F(M) = -MH + a(T - T_0)M^2 - bM^4 + cM^6$, where the coefficients a, b, and c are positive and H is a magnetic field. A common simple analysis shows that metastable states with $M \neq 0$ emerge below the critical temperature $T_b = T_0 + b^2/(3ac)$ with a jump $M(T_b - 0) = \sqrt{b/3c}$. The first-order phase transition takes place at a lower temperature, T_c, below which only the state with $M \neq 0$ is stable (whereas the state with $M = 0$ is stable for $T > T_c$). After the breakdown, there is no singularity. The susceptibility $\chi(T) \equiv dM/dH$ also shows a singularity $\chi(T < T_b) \propto 1/\sqrt{T_b - T}$ but has no singularity above T_b. As we will show below, the hybrid (mixed) transition is analogous to a limiting metastable state for a first-order phase transition. The important property is that the transition is asymmetrical. This is evident from Eqs. (2) and (3) below, that are related to the divergence of the mean size of critical clusters appearing only on one side of the hybrid transition. These critical phenomena will be discussed further in Sec. III. There are critical fluctuations and a divergent correlation length on only one side of the critical point. A similar asymmetrical hybrid transition has been observed in the Kuramoto model of oscillator synchronization [51]. Compare this with the first-order phase transitions in which there are no critical fluctuations, and the correlation length is finite everywhere, including at the critical point T_c. On the other hand, continuous phase transitions demonstrate critical fluctuations and divergent correlation length on both sides of the transition.

In general, we might expect a combination of continuous and hybrid transitions. To this end, we first consider the simple case where r_i are distributed between two values, controlled by a parameter f. With probability f a vertex has a threshold equal to one, while with probability $(1 - f)$ the threshold is k for some integer $k \geqslant 3$. This condition allows for the presence of the hybrid transitions seen in the k-core when $k \geqslant 3$, but which is absent when $k = 1$ or 2. The threshold distribution is then

$$Q_k(r) = \delta_{r,1} f + \delta_{r,k}(1 - f). \tag{1}$$

This parametrized the HKC has as its two limits ordinary percolation ($f = 1$) and the original k-core ($f = 0$).

We now contrast this model with bootstrap percolation. In bootstrap percolation, with probability f a vertex is a "seed" that is initially active and remains active. The remaining vertices (a fraction $1 - f$) become active if their number of active neighbors reaches or exceeds a threshold value k. Once activated a vertex remains active. The activation of vertices may mean that new vertices now meet the threshold criterion and, hence, become active. This activation process continues iteratively until a stationary state is reached. The seed and activated vertices in bootstrap percolation are analogous to the threshold one and threshold k groups in the heterogeneous *k*-core. We might expect that the subgraph formed by the active vertices in the stationary state of bootstrap percolation might be related to the heterogeneous *k*-core. In fact, the two subgraphs are necessarily different, as we will describe in detail in Sec. III. Nevertheless, the two processes have some similar or analogous critical behaviors. Note that we are using a lower threshold of one for the generalized *k*-core, meaning isolated vertices are not counted as part of the HKC. We could also use a lower threshold of zero, which would include more vertices in the HKC, but it would yield an identical giant HKC. For this reason we can compare with bootstrap percolation, in which the seed vertices have effectively a threshold of zero.

In Fig. 1 we compare the activation in each case for the same parameters. Because bootstrap percolation is an activation process while the heterogeneous *k*-core is found by pruning, the curves are different for the two processes. Consider starting from a completely inactive network (that may be damaged so that some fraction p of vertices remains). As we gradually increase f from zero, under the bootstrap percolation process, more and more vertices become active (always reaching equilibrium before further increases of f) until at a certain threshold value, f_{c1-b}, a giant active component appears. As we increase f further, the size of the giant BPC traces the dashed curves shown in Fig. 1; see also Ref. [43]. The direction of this process is indicated by the arrows on these curves. Finally, at $f = 1$ all undamaged vertices are active. Now we reverse the process, beginning with a fully active network and gradually reducing f, deactivating (which is equivalent to pruning) vertices that fall below their threshold r_i under the heterogeneous *k*-core process. As f decreases, the solid curves in Fig. 1 will be followed in the direction indicated by the arrows. Notice that the size of the giant HKC for given values of f and p is always larger than the giant BPC. The explanation for this difference will be explored in the following section.

Let $\mathcal{S}_{\rm k}$ be the fraction of vertices that are in the heterogeneous *k*-core. That is, the total of all components, whether finite or infinite, that meet the threshold conditions. This is equal to the probability that an arbitrarily chosen vertex of the original network is in the heterogeneous *k*-core. Let $\mathcal{S}_{\rm gc\text{-}k}$ be the relative size of the giant heterogeneous *k*-core (that is, the subset of the heterogeneous *k*-core which forms a giant component)—also the probability that an arbitrarily chosen vertex is in the giant heterogeneous *k*-core. The fraction of vertices forming finite clusters is therefore $\mathcal{S}_{\rm k} - \mathcal{S}_{\rm gc\text{-}k}$. Note that in the standard *k*-core this is negligibly small. Similarly, let $\mathcal{S}_{\rm b}$ be the fraction of active vertices in the bootstrap percolation model, and $\mathcal{S}_{\rm gc\text{-}b}$ be the size of the giant component of active vertices. We construct self-consistency equations for $\mathcal{S}_{\rm k}$ and $\mathcal{S}_{\rm gc\text{-}k}$ in Appendix A. In networks without heavy-tailed degree distributions, i.e., whose second moments do not diverge in the infinite size limit, we find two different transitions for each process.

We first briefly describe the transitions observed in bootstrap percolation before comparing them with those found in the new heterogeneous *k*-core process. For bootstrap percolation above a certain value of p, the giant active component (giant BPC) may appear continuously from zero at a finite value of f, f_{c1-b}, and grow smoothly with f (see the dashed lines 1 and 2 in the top panel of Fig. 1). For larger p, after the giant active component appears, there may also be a second discontinuous hybrid phase transition at f_{c2-b}, as seen in Fig. 1 (see the dashed line 3). There is a jump in the size of the giant active component, $\mathcal{S}_{\rm gc\text{-}b}$, from the value at the critical point (marked by a circle on the dashed line 3). When approaching from below, the difference of $\mathcal{S}_{\rm gc\text{-}b}$ from the critical value goes as the square root of the distance from the critical point:

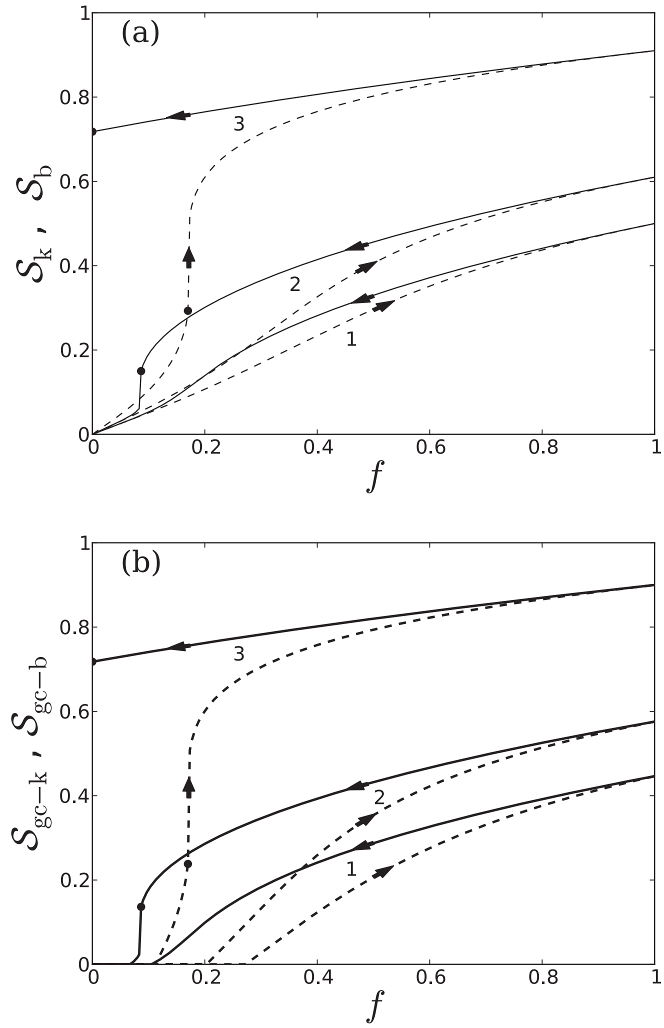


FIG. 1. (a) Relative size of the heterogeneous *k*-core, $\mathcal{S}_{\rm k}$ (solid curves), which is the subgraph that includes all vertices that meet the threshold requirements of Eq. (1), and the fraction of active vertices in bootstrap percolation, $\mathcal{S}_{\rm b}$ (dashed curves) as a function of f for the same network—an Erdős-Rényi graph of mean degree five—with the same $k = 3$ at three different values of p corresponding to different regions of the phase diagrams (see Fig. 2): (1) $p = 0.5$, which is between $p_{\rm c}$ and $p_{\rm s}$ for both models, (2) $p = 0.61$, which is above $p_{\rm s\text{-}k}$ but still below $p_{\rm s\text{-}b}$, and (3) $p = 0.91$, which is above $p_{\rm f\text{-}k}$ and $p_{\rm s\text{-}b}$. Each numbered pair indicates the same choice of parameters p and k with the activation process of bootstrap percolation indicated by arrows to the right and the pruning process of the heterogeneous *k*-core by arrows to the left. (b) Size $\mathcal{S}_{\rm gc\text{-}k}$ of the giant heterogeneous *k*-core (solid) and the size $\mathcal{S}_{\rm gc\text{-}b}$ of the giant BPC (dashed) as a function of f for the same network and the same values of p.

$$\mathcal{S}_{\rm b}(f) = \mathcal{S}_{\rm b}(f_{\rm c2\text{-}b}) - a(f_{\rm c2\text{-}b} - f)^{1/2}. \qquad (2)$$

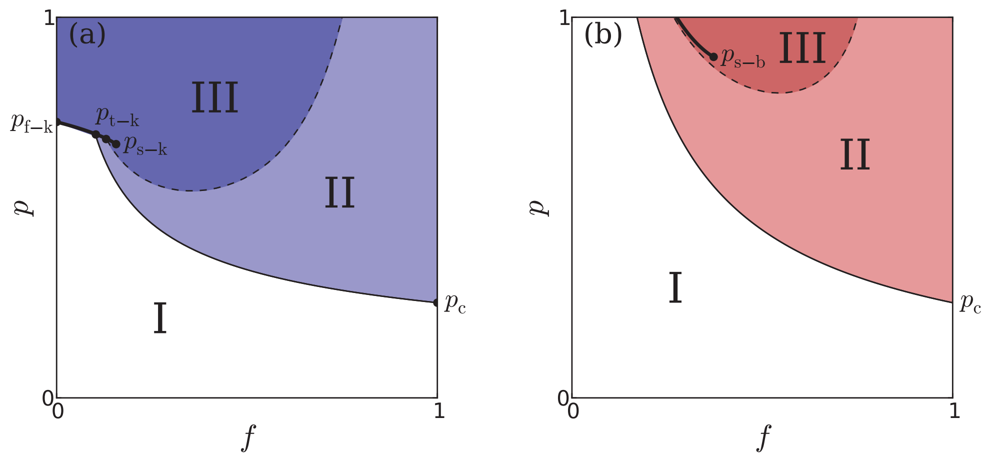


FIG. 2. (Color online) (a) Phase diagram for the heterogeneous k-core in the f-p plane. The giant HKC is absent in region I. In region II the giant HKC is present but vertices with threshold k do not form a giant component within it. In region III the vertices with threshold k in the HKC form a giant connected component. The giant HKC appears continuously at the threshold marked by the thin black curve. The hybrid discontinuous transition occurs at the points marked by the heavy black line beginning from p_{s-k}. The giant component of threshold vertices appears with a continuous transition marked by a thin dashed curve. Above p_{t-k} the first appearance of the giant HKC is with a discontinuous transition. Above p_{f-k} the giant HKC appears discontinuously for any $f > 0$. (b) Phase diagram for boostrap percolation. The giant BPC is absent in region I. The giant BPC is present in region II. In region III the giant BPC is present and the active vertices with threshold k form a giant connected component. The continuous appearance of the giant BPC is marked by the thin solid curve and the hybrid transition (beginning at p_{s-b}) by a heavy solid curve. These diagrams are for a Bethe lattice with degree five and for $k = 3$, but the diagram for any network with finite second moment of the degree distribution will be qualitatively the same. Note that the locations of the continuous transitions from region I to II and from II to III for BPC with $k \to k - 1$ coincide (up to the point where the discontinuous transition is encountered) with those for the heterogeneous k-core. The special critical point, p_{s-b}, with k reduced by 1 also coincides with p_{s-k} (see also Appendix A).

See Ref. [43] for a complete description of the bootstrap percolation results.

For the heterogeneous k-core we see a different but analogous pair of transitions. Again, for a given p the giant HKC appears continuously from zero above some critical value of f, f_{c1-k}. There may also be a second transition at f_{c2-k}, where, again, we see a discontinuity in the size of the giant HKC. Now, however, the square-root scaling occurs as the critical point is approached from above:

$$\mathcal{S}_{\rm k}(f) = \mathcal{S}_{\rm k}(f_{\rm c2\text{-}k}) + a(f - f_{\rm c2\text{-}k})^{1/2}, \quad (3)$$

this occurs in the solid line 2 in Fig. 1. Another important difference is that for bootstrap percolation the discontinuous transition is always above the first appearance of the giant BPC, i.e., $f_{c2-b} > f_{c1-b}$, but for the heterogeneous k-core f_{c2-k} may be greater than f_{c1-k}—so that the first appearance of the giant HKC is similar to that found in ordinary percolation—or less than f_{c1-k}, with the giant HKC appearing discontinuously from zero, as in the ordinary k-core [8,19,49] (see Fig. 1).

The overall behavior with respect to the parameters p and f of each model is summarized by the phase diagrams in Fig. 2. The diagrams are qualitatively the same for any degree distribution with a finite second moment. Considering first the heterogeneous k-core, a giant HKC is absent in region I. It is present in region II but the vertices with threshold k do not form a giant component. A giant HKC is present also in region III and now the threshold-k vertices also form a giant connected component within the giant HKC. For p below the percolation threshold p_c the giant HKC never appears for any f. Above p_c the giant HKC appears with a continuous transition growing linearly with f (or p for that matter) close to the critical point. The threshold is indicated by the thin black line in the Figure, which divides regions I and II. From p_{s-k} a second discontinuous hybrid transition appears. This is marked by the heavy black curve in Fig. 2. As already noted in Eq. (3), the size of the giant HKC grows as the square root of the distance above this second critical point. At the special point p_{s-k}, however, the size of the discontinuity reduces to zero and the scaling near the critical point is cube root:

$$\mathcal{S}_{\rm k}(f) = \mathcal{S}_{\rm k}(f_{\rm c2\text{-}k}) + a(f - f_{\rm c2\text{-}k})^{1/3}. \quad (4)$$

At first, the hybrid transition occurs after the continuous appearance of the giant HKC (see line 2 in Fig. 1), but at p_{t-k} the two transitions cross and the giant HKC begins to appear immediately with a jump. Finally, at p_{f-k} the giant HKC begins to appear discontinuously immediately from $f = 0$ (see line 3 in Fig. 1). Thus, the first appearance of the giant HKC has a classical percolation-like transition below p_{t-k}, while above p_{t-k} the appearance is similar to that found in the ordinary k-core. Within the heterogeneous k-core, the vertices with threshold k may form a giant component themselves. This occurs in the region labeled III. This giant component appears with a continuous transition.

For bootstrap percolation the giant active component again appears only above the percolation threshold, p_c. It also first appears with a linear continuous transition (when the degree distribution has finite third moment) but at a larger value of f. The point of appearance is marked by the thin curve in Fig. 2 that divides regions I and II. Again, a second transition appears for larger p beginning at the special critical point, p_{s-b}. This transition is marked by the heavy solid curve in the Figure (see also line 3 of Fig. 1). The scaling near the hybrid transition is again square root but this time only when approaching the transition from below. At the special critical point, p_{s-b}, the scaling becomes cube root:

$$\mathcal{S}_{\rm b}(f) = \mathcal{S}_{\rm b}(f_{\text{c2-b}}) - a(f_{\text{c2-b}} - f)^{1/3}. \tag{5}$$

Note the difference between Eqs. (2) and (3) and between (5) and (4). In bootstrap percolation the hybrid transition always occurs above the continuous one and neither reaches $f = 0$. Again, the active threshold k vertices form a giant component in region III. Note also that at the special critical point, $p_{s-b} > p_{f-k}$, so that for a given p we may have a hybrid transition for the HKC or BPC but not for both. It turns out that the special critical point, p_{s-b}, for bootstrap percolation, whose value of k is one less, coincides with the special critical point for the heterogeneous k-core. Furthermore, the location of the continuous appearance of the giant heterogeneous k-core for a given k also coincides with the appearance of the giant component of bootstrap percolation for $k-1$. The same is also true for the appearance of the giant components of vertices with threshold k. This is clear from the equations given in Appendix A. If the value of k is increased, the locations of the hybrid transitions move toward larger values of p and there is a limiting value of k after which these transitions disappear altogether. For both the heterogeneous k-core and bootstrap percolation this limit is proportional to the mean degree. Note also that the continuous transition also moves slightly with increasing k and in the limit $k \to \infty$ tends to the line $pf = p_c$ for both processes.

III. SUBCRITICAL CLUSTERS, CORONA CLUSTERS, AND AVALANCHES

It is clear from Figs. 1 and 2 that even though bootstrap percolation and the heterogeneous k-core described above have the same thresholds and proportions of each kind of vertex, the equilibrium size of the respective giant components is very different. The difference results from the top-down versus bottom-up ways in which they are constructed. To find the heterogeneous k-core we begin with the full network and prune vertices which do not meet the criteria until we reach equilibrium. In contrast, bootstrap percolation begins with a largely inactive network and successively activatable vertices until equilibrium is reached. To see the effect of this difference we now describe an important concept: the subcritical clusters of bootstrap percolation.

A subcritical cluster in bootstrap percolation is a cluster of activatable vertices (i.e., not seed vertices) that have exactly $k-1$ active neighbors external to the cluster. Under the rules of bootstrap percolation such clusters cannot become activated. The vertices within the cluster block each other from becoming active (see Fig. 3). Now compare the situation for the heterogeneous k-core. Any cluster of threshold-k vertices with $k-1$ neighbors in the core external to the cluster is always included in the heterogeneous k-core. The HKC vertices in clusters like those in Fig. 4 assist one another. Thus the exclusion of subcritical clusters from activation in bootstrap percolation accounts for the difference in sizes of the BPC and the HKC.

Subcritical clusters have another important property, which helps us to understand the discontinuity at the second transition. A vertex in a subcritical cluster becomes active if it gains an extra active neighbor (e.g., through an infinitesimal change in p or f). This in turn allows each of its neighbors in the cluster to activate. A domino-like effect ensues leading to an avalanche of activations—one extra active neighbor of any vertex in the subcritical cluster leads to the whole cluster becoming active. Thus the rate of change of $\mathcal{S}_{\rm b}$ is related to the sizes of the subcritical clusters. Almost everywhere, if we choose a subcritical vertex at random, the mean size of the subcritical cluster to which it belongs is finite. However, exactly at the second threshold, the mean size of the subcritical cluster to which a randomly chosen vertex belongs diverges as we approach from below. This was shown in Ref. [43]. Thus, approaching this point, an infinitesimal increase in f (or p) leads to a finite fraction of the network becoming activated, hence, a discontinuity in $\mathcal{S}_{\rm b}$ (and also in $\mathcal{S}_{\text{gc-b}}$) appears. The distribution of avalanche sizes near the transition is determined by the size distribution $G(s)$ of subcritical clusters, which at the critical point follows $G(s) \sim s^{-3/2}$. This can be shown using a generating function approach, as demonstrated in Ref. [43]. A similar method can be found in Refs. [19,49,52,53].

We can understand the hybrid transition in the heterogeneous k-core by considering the relevant clusters with similar properties, the so-called corona clusters. Corona clusters are clusters of vertices with threshold k that have exactly k neighbors in the HKC. These clusters are part of the heterogeneous k-core, but if any member of a cluster loses a neighbor a domino-like effect leads to an avalanche as the entire cluster is removed from the HKC. The corona clusters are finite everywhere except at the discontinuous transition, where the mean size of corona cluster to which a randomly chosen vertex belongs diverges as we approach from above

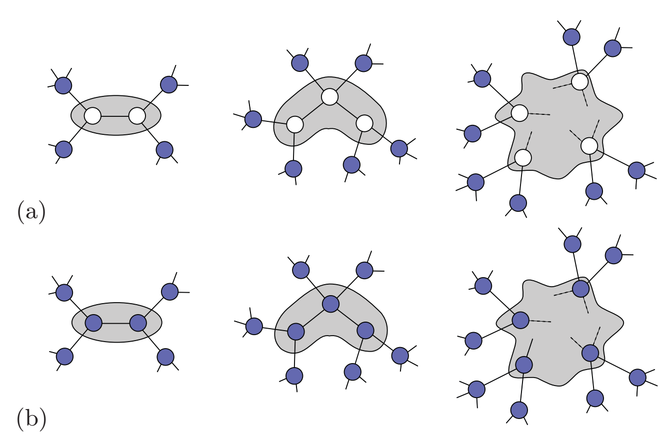


FIG. 3. (Color online) (a) Subcritical clusters of different sizes in bootstrap percolation. Left: Because they start in an inactive state two connected vertices (shaded area) cannot become active if each has $k-1$ active neighbors (in this example $k = 3$). The same follows for clusters of three (center) or more vertices (right). If any member of a subcritical cluster gains another active neighbor, an avalanche of activations encompasses the whole cluster. (b) Similar clusters would be included in the heterogeneous k-core.

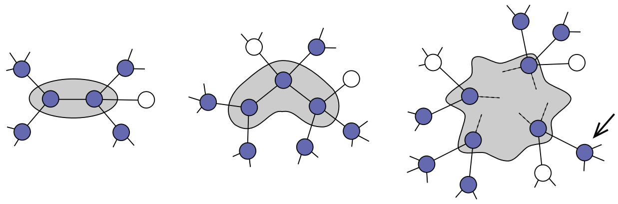


FIG. 4. (Color online) Corona clusters of different sizes in heterogeneous k-core. Left: Because they are included unless pruned, two connected vertices (whose threshold is k) form part of the heterogeneous k-core if each has $k-1$ other neighbors in the core, as each is "assisted" by the other. Right: In general, a corona cluster consists of vertices (with threshold k) that each have exactly k active neighbors, either inside or outside the cluster. If one neighbor of any of the cluster vertices is removed from the core (for example, the one indicated by the arrow), an avalanche is caused as the entire cluster is pruned.

[6,19,49]. Thus, an infinitesimal change in f (or p) leads to a finite fraction of the network being removed from the HKC, hence, a discontinuity in $\mathcal{S}_{\rm k}$ (and also in $\mathcal{S}_{\rm gc\text{-}k}$) arises. The size distribution of corona clusters and, hence, avalanches at this transition also goes as a power law with exponent $-3/2$.

IV. SCALE-FREE NETWORKS

The results presented above, in Sec. II and in Figs. 1 and 2, are qualitatively the same for networks with any degree distribution that has finite second and third moments. When only the second moment is finite, the phase diagram remains qualitatively the same but the critical behavior is changed. Instead of a second-order continuous transition, we have a transition of a higher order. When the second moment diverges, we have quite different behavior.

To examine the behavior when the second and third moments diverge we consider scale-free networks with degree distributions of the form

$$P(q) \approx q^{-\gamma} \tag{6}$$

for large q. At present we consider only values of $\gamma > 2$.

To find the behavior near the critical points we begin with a self-consistency equation for X, which is the probability that an arbitrarily chosen edge leads to an infinite $(r_i - 1)$ tree [see Eq. (A4)]. The probability $\mathcal{S}_{\rm gc\text{-}k}$ can be written in terms of X and, in fact, both X and $\mathcal{S}_{\rm gc\text{-}k}$ grow with the same exponent near the appearance of the giant active component. We expand the right-hand side of Eq. (A4) near the appearance of the giant HKC (i.e., near $X = 0$). When $\gamma < 4$, the third and possibly second moments of the degree distribution diverge. This means that coefficients of integral powers of X diverge and we must instead find leading nonintegral powers of X.

When $\gamma > 4$, the second and third moments of the distribution are finite, so the behavior is the same as already described in Sec. II. Equation (A4) leads to

$$X = c_1 X + c_2 X^2 + \text{higher order terms}, \tag{7}$$

which gives the critical behavior $X \propto (f - f_{\rm c2\text{-}k})^\beta$ with $\beta = 1$.

When $3 < \gamma \leqslant 4$, the linear term in the expansion of Eq. (A4) survives but the second leading power is $\gamma - 2$:

$$X = c_1 X + c_2 X^{\gamma-2} + \text{higher terms}, \tag{8}$$

where the coefficients c_1 and c_2 depend on the degree distribution, the parameters p and f, and the (nonzero) value of Z. The presence of the linear term means that the giant HKC appears at a finite threshold, but because the second leading power is not 2, the giant HKC grows not linearly but with an exponent $\beta = 1/(\gamma - 3)$. This means that the phase diagram remains qualitatively the same as in Fig. 2; however, the size of the giant HKC grows as $(f - f_{\rm c1})^\beta$ with $\beta = 1/(\gamma - 3)$. This is the same scaling as was found for ordinary percolation [54].

For values of γ below 3, the change in behavior is more dramatic. When $2 < \gamma \leqslant 3$, the second moment of $P(q)$ also diverges, meaning the leading order in the equation for X is no longer linear but $\gamma - 2$:

$$X = d_1 X^{\gamma-2} + \text{higher terms}. \tag{9}$$

From this equation it follows that there is no threshold for the appearance of the giant HKC (or giant BPC). The giant HKC appears immediately and discontinuously for any $f > 0$ (or $p > 0$) and there is also no upper limit to the threshold k. This behavior is the same for bootstrap percolation, so the (featureless) phase diagram is the same for both processes, even though the sizes of the giant HKC and giant BPC are different.

The critical thresholds are affected by finite-size effects when $\gamma \leqslant 3$. In finite networks, the degree distribution has an upper cutoff that depends on the system size N. For $\gamma > 3$ this does not have a significant effect. However, for $2 < \gamma \leqslant 3$, the leading term in Eq. (9) results from a singularity that occurs only in the limit $N \to \infty$. For finite N the leading terms are of order X and X^2, so the giant HKC appears at finite values of f and p. These thresholds tend to zero as $N \to \infty$.

V. DISCUSSION

We have introduced a new concept, the heterogeneous k-core (HKC), an extension of the well-known k-core of complex networks in which the thresholds may be different at each vertex. The HKC has potential applications in models of neuronal networks, as real neurons may have different activation thresholds from one another.

A simple representative example of the HKC has vertices with randomly assigned thresholds of either one (with probability f) or $k \geqslant 3$ (with probability $1 - f$). This HKC has a complex phase diagram including two types of phase transition: a continuous transition at the appearance of the giant HKC, similar to the ordinary percolation transition, and a second discontinuous hybrid phase transition. This second transition is similar to the ordinary k-core but it may occur after the first continuous appearance of the giant HKC or before. The first transition occurs when the vertices of both kinds reaching their threshold form a giant percolating cluster. The second transition occurs when the mean size of avalanches of pruned vertices diverges. This can be understood by considering corona clusters (clusters of vertices that exactly meet the upper threshold). The mean size of the corona cluster to which an arbitrarily chosen vertex belongs diverges as we approach the second transition from above. The size of pruning avalanches are determined by these corona clusters and so the transition is discontinuous.

We have contrasted the heterogeneous k-core with bootstrap percolation, in which there are also two kinds of vertices, but the core is defined by an activation process, rather than a pruning process. The phase diagram for the BPC therefore does not coincide with that of the HKC. The difference between the giant HKC and the giant BPC results from the subcritical clusters of bootstrap percolation. These clusters cannot be activated because all members have $k-1$ active neighbors outside the cluster. However, the equivalent clusters would be included in the giant HKC.

All of these results are strongly dependent on the network structure. If the third moment of the degree distribution is finite, we obtain the described results with the giant HKC growing linearly above the continuous threshold. This is the case if the degree distribution decays faster than a power law $q^{-\gamma}$ with $\gamma > 4$ for large degree q. If instead the degree distribution tends to a power law with $3 < \gamma \leqslant 4$, so that the second moment is finite while the third moment diverges, the phase diagram is qualitatively the same but the continuous transition is of higher order. If $2 < \gamma \leqslant 3$, the second moment of the degree distribution diverges and the situation is more extreme. The giant HKC appears immediately at a finite size for any $f > 0$ or $p > 0$, showing that, in common with similar behavior found in other systems, such scale-free networks are extremely resilient to damage.

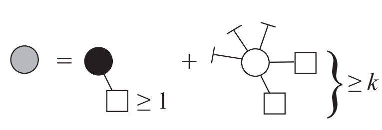
ACKNOWLEDGMENTS

This work was partially supported by the following projects PTDC: FIS/71551/2006, FIS/108476/2008, SAU-NEU/103904/2008, and MAT/114515/2009, and also by the SOCIALNETS EU project.

APPENDIX A: SELF-CONSISTENCY EQUATIONS

Here, we construct the self-consistency equations that the probabilities $\mathcal{S}_{\text{k}}$, $\mathcal{S}_{\text{gc-k}}$, $\mathcal{S}_{\text{b}}$, and $\mathcal{S}_{\text{gc-b}}$ must obey. The (usually numerical) solution of these equations leads to the phase diagrams and other results presented in Sec. II. We have already given them for the case of bootstrap percolation [43].

The probability $\mathcal{S}_{\text{k}}$ that an arbitrarily chosen vertex belongs to the heterogeneous k-core (HKC) is the sum of the probabilities that it has $r_i = 1$ and at least one neighbor in the core, or $r_i = k$ and has at least k neighbors in the core. We can represent this diagrammatically as



We define Z in terms of an $(r_i - 1)$-ary tree, a generalization of the $(k-1)$-ary tree. An $(r_i - 1)$-ary tree is a subtree in which, as we traverse the tree, each encountered vertex has at least $r_i - 1$ child edges (i.e., edges leading from the vertex not including the one by which we entered). The variable r_i can be different at each vertex. In our example, r_i is either 1, in which case the vertex does not need to have any children (though it may have them) and is only required to be connected to the tree, or $r_i = k$, in which case it must have at least $k-1$ children.

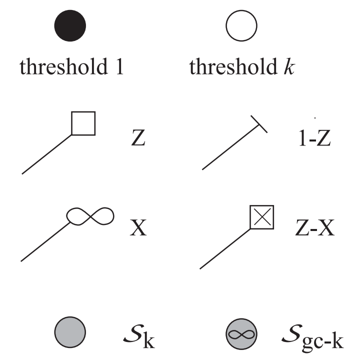


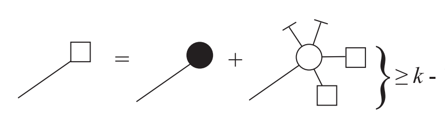
FIG. 5. Symbols used in graphical representations of self-consistency equations for the heterogeneous k-core.

The probability Z can then be very simply stated as the probability that, on following an arbitrarily chosen edge in the network, we reach a vertex that is a root of a $(r_i - 1)$-ary tree. For the specific case considered in this paper, the encountered vertex either has $r_i = 1$ or it has $k-1$ children leading to the roots of $(r_i - 1)$-ary trees. The probability Z is represented by a square in the diagram. A bar represents the probability $(1-Z)$, a black circle represents a vertex with $r_i = 1$, and a white circle is for a vertex with $r_i = k$ (see Fig. 5). These conditions can be written as binomial terms and sums over all possible values of the degree of i, so this diagram can be written in a mathematical form as

$$\mathcal{S}_{\text{k}} = pf \sum_{q=1}^{\infty} P(q) \sum_{l=1}^{q} \binom{q}{l} Z^l (1-Z)^{q-l}$$
$$+ p(1-f) \sum_{q=k}^{\infty} P(q) \sum_{l=k}^{q} \binom{q}{l} Z^l (1-Z)^{q-l}, \quad \text{(A1)}$$

where the factor p accounts for the probability that the vertex has not been damaged.

To calculate Z we construct a recursive (self-consistent) expression in a similar way based on the definition given above. This is represented by the diagram

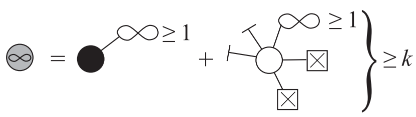


which in equation form is

$$Z = pf + \\ \times p(1-f) \sum_{q \geqslant k} \frac{q P(q)}{\langle q \rangle} \sum_{l=k-1}^{q-1} \binom{q-1}{l} Z^l (1-Z)^{q-1-l} \\ \equiv \Psi(Z,p,f). \quad \text{(A2)}$$

We have used the fact that $q P(q)/\langle q \rangle$ is the probability that the vertex reached along an arbitrary edge has degree q. Solving Eq. (A2), usually numerically, for Z and then substituting into Eq. (A1) allows the calculation of $\mathcal{S}_{\text{k}}$.

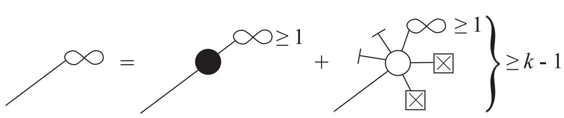
We follow a similar procedure to calculate the size of the giant HKC that is equal to the probability $\mathcal{S}_{\text{gc-k}}$ that an arbitrarily chosen vertex is a member of a the HKC component of infinite size. We denote by X the probability that an arbitrarily chosen edge leads to a vertex that is the root of an infinite $(r_i - 1)$-ary tree. This definition is similar to Z but with the extra condition that the subtree must extend indefinitely. We represented X by an infinity symbol (see Fig. 5). The diagram for $\mathcal{S}_{\text{gc-k}}$ is



which is equivalent to the equation

$$\mathcal{S}_{\text{gc-k}} = pf \sum_{q=0}^{\infty} P(q) \sum_{m=1}^{q} \binom{q}{m} X^m (1-X)^{q-m} + p(1-f) \sum_{q=k}^{\infty} P(q) \sum_{l=k}^{q} \binom{q}{l} (1-Z)^{q-l} \times \sum_{m=1}^{l} \binom{l}{m} X^m (Z-X)^{l-m}. \tag{A3}$$

To find X, we construct a self-consistency equation from the diagram



leading to

$$X = pf \sum_{q=0}^{\infty} \frac{qP(q)}{\langle q \rangle} \sum_{m=1}^{q-1} \binom{q-1}{m} X^m (1-X)^{q-1-m} + p(1-f) \sum_{q=k}^{\infty} \frac{qP(q)}{\langle q \rangle} \sum_{l=k-1}^{q-1} \binom{q-1}{l} (1-Z)^{q-1-l} \times \sum_{m=1}^{l} \binom{l}{m} X^m (Z-X)^{l-m}. \tag{A4}$$

The solution of Eqs. (A2) and (A4) allows the calculation of $\mathcal{S}_{\text{gc-k}}$ through Eq. (A3). Note that when there are multiple solutions of Z, we choose the largest solution as the "physical" one.

To find the appearance of the giant component for a given p, we find leading terms for $X \ll 1$, and solve for f, as described in Sec. IV. To calculate the location of the hybrid transition we note that at this critical point a second solution to Eq. (A2) appears. This occurs when the function $\Psi(Z)$ just touches the line Z, which must be at a local extremum of Ψ/Z,

$$\frac{d}{dZ}\left(\frac{\Psi}{Z}\right) = 0. \tag{A5}$$

The scaling near the critical point, $f_{\text{c2-k}}$, is found by expanding Eq. (A2) to leading orders about Z_c. Making use of the fact that Eq. (A5) is satisfied at Z_c, we find

$$Z \approx Z_c + c(f - f_{\text{c2-k}})^{1/2}. \tag{A6}$$

Substituting this into Eq. (A1) gives Eq. (3).

Furthermore, at the special point, p_{s-k}, where the second transition disappears, by a similar argument a further condition must also be satisfied:

$$\frac{d^2}{dZ^2}\left(\frac{\Psi}{Z}\right) = 0. \tag{A7}$$

Thus, p_{s-k} is determined by a simultaneous solution of Eqs. (A2), (A5), and (A7). This in turn leads to cube-root scaling above the threshold, hence, Eq. (4).

For bootstrap percolation we can construct similar self-consistency equations in order to calculate $\mathcal{S}_{\text{b}}$ and $\mathcal{S}_{\text{gc-b}}$. Let Y be the probability (counterpart of Z) that while following an arbitrary edge we encounter a vertex that is either a seed or has k active children. As discussed in Sec. III, an activation in bootstrap percolation must spread through the network, meaning that the vertex needs k active downstream neighbors in order to become active (and thus provide an active neighbor to its upstream "parent"). Repeating the diagrammatic method described above, we arrive at the equation

$$Y = pf + p(1-f) \sum_{q \geqslant k+1} \frac{qP(q)}{\langle q \rangle} \times \sum_{l=k}^{q-1} \binom{q-1}{l} Y^l (1-Y)^{q-1-l} \equiv \Phi(Y,p,f). \tag{A8}$$

Note that Eq. (A8) differs from Eq. (A2) because the number of children required is k not $k-1$. This is equivalent to excluding the subcritical clusters represented in Fig. 3. As an aside, consider the probability $\mathcal{P}_{\text{sub}}$ that an arbitrary edge leads to a vertex in a subcritical cluster. This is simply the probability that the vertex has exactly $k-1$ active neighbors, and thus

$$\mathcal{P}_{\text{sub}} = p(1-f) \sum_{q \geqslant k} \frac{qP(q)}{\langle q \rangle} \binom{q-1}{k-1} Y^{k-1} (1-Y)^{q-k}. \tag{A9}$$

Comparing Eqs. (A2) and (A8), we see that the right-hand side of Eq. (A9) contains precisely the terms that are counted in Eq. (A2) but absent from Eq. (A8). Thus clusters of vertices all having exactly $k-1$ active neighbors are excluded from the active component in bootstrap percolation, while vertices having $k-1$ neighbors in the HKC are always included in the HKC. Of course, because of the self-recursion the value of Z is necessarily different from that of Y by more than just the amount of these terms.

Drawing diagrams similar to those given for the HKC allows the construction of further self-consistency equations for the remaining quantities of interest. The probability that an arbitrarily chosen vertex is active, $\mathcal{S}_{\mathrm{b}}$, is then identical to Eq. (A1) but with Y replacing Z,

$$\mathcal{S}_{\mathrm{b}} = pf\sum_{q=1}^{\infty}P(q)\sum_{l=1}^{q}\binom{q}{l}Y^{l}(1-Y)^{q-l} + p(1-f)\sum_{q=k}^{\infty}P(q)\sum_{l=k}^{q}\binom{q}{l}Y^{l}(1-Y)^{q-l}. \tag{A10}$$

The construction of the equation for $\mathcal{S}_{\text{gc-b}}$ is similar. We introduce the probability W that, upon following an arbitrary edge, we reach a vertex that is active and also has at least one edge leading to an infinite active subtree. Then W obeys

$$W = pf\sum_{q=0}^{\infty}\frac{qP(q)}{\langle q\rangle}\sum_{m=1}^{q-1}\binom{q-1}{m}W^{m}(1-W)^{q-1-m} + p(1-f)\sum_{q=k+1}^{\infty}\frac{qP(q)}{\langle q\rangle}\sum_{l=k}^{q-1}\binom{q-1}{l}(1-Y)^{q-1-l} \times\sum_{m=1}^{l}\binom{l}{m}W^{m}(Y-W)^{l-m}, \tag{A11}$$

which again differs from Eq. (A4) since the limit is $k-1$ not k. Then $\mathcal{S}_{\text{gc-b}}$ obeys

$$\mathcal{S}_{\text{gc-b}} = pf\sum_{q=0}^{\infty}P(q)\sum_{m=1}^{q}\binom{q}{m}W^{m}(1-W)^{q-m} + p(1-f)\sum_{q=k}^{\infty}P(q)\sum_{l=k}^{q}\binom{q}{l}(1-Y)^{q-l} \times\sum_{m=1}^{l}\binom{l}{m}W^{m}(Y-W)^{l-m}, \tag{A12}$$

which is identical in form to Eq. (A3), though the values of Y and W (for bootstrap percolation) will be different from those of Z and X (for the HKC). Note also that, for bootstrap percolation the physical solution for Y is always the smallest of Eq. (A8). The discontinuous transition occurs at the point where

$$\frac{d}{dY}\left(\frac{\Phi}{Y}\right) = 0 \tag{A13}$$

and at the special point, p_{s-b}, one more condition,

$$\frac{d^{2}}{dY^{2}}\left(\frac{\Phi}{Y}\right) = 0, \tag{A14}$$

has to be satisfied, where the function $\Phi(Y)$ is defined by Eq. (A8).

Finally, we note that the appearance of the giant component of threshold-k vertices in the HKC process can be found in a similar way as the appearance of the giant HKC. We define R to be the probability that an arbitrarily chosen edge leads to the root of an infinite subtree that is an (r_i-1)-ary tree with all of the $r_i=1$ vertices removed. This probability obeys a self-consistency equation similar to that for X:

$$R = p(1-f)\sum_{q=k}^{\infty}\frac{qP(q)}{\langle q\rangle}\sum_{l=k-1}^{q-1}\binom{q-1}{l}(1-Z)^{q-1-l} \times\sum_{m=1}^{l}\binom{l}{m}R^{m}(Z-R)^{l-m}. \tag{A15}$$

The appearance of the giant component of threshold-k vertices is then found by expanding this equation to leading order with respect to R and solving for f. This leads to an equation that gives the dashed line in Fig. 2. Recall that $Z(f,p)$ is determined by Eq. (A2). A similar procedure yields the corresponding transition in bootstrap percolation.

APPENDIX B: GENERAL FORM OF EQUATIONS

In this paper, we have examined only a special case of the HKC, in which the vertices have a threshold value equal to either one or $k\geqslant 3$. For completeness, we now give the self-consistency equations for an arbitrary threshold distribution $Q(r)$. The size $\mathcal{S}_{\mathrm{k}}$ of the HKC is

$$\mathcal{S}_{\mathrm{k}} = p\sum_{r\geqslant 1}Q(r)\sum_{q=r}^{\infty}P(q)\left[\sum_{l=r}^{q}\binom{q}{l}Z^{l}(1-Z)^{q-l}\right], \tag{B1}$$

where, as above, Z is the probability of encountering a vertex i, which is the root of an (r_i-1)-ary tree,

$$Z = p\sum_{r\geqslant 1}Q(r)\sum_{q=r}^{\infty}\frac{(q)P(q)}{\langle q\rangle}\sum_{l=r-1}^{q-1}\binom{q-1}{l}Z^{l}(1-Z)^{q-1-l}. \tag{B2}$$

Similarly, the equation for the size of the giant active component is

$$\mathcal{S}_{\text{gc-k}} = p\sum_{r\geqslant 1}Q(r)\sum_{q=r}^{\infty}P(q)\sum_{l=r}^{q}\binom{q}{l}(1-Z)^{q-l} \times\sum_{m=1}^{l}\binom{l}{m}X^{m}(Z-X)^{l-m},$$

where X obeys

$$X = p\sum_{r\geqslant 1}Q(r)\sum_{q=r}^{\infty}\frac{qP(q)}{\langle q\rangle}\sum_{l=r-1}^{q-1}\binom{q-1}{l}(1-Z)^{q-1-l} \times\sum_{m=1}^{l}\binom{l}{m}X^{m}(Z-X)^{l-m}.$$

We do not derive any results for this general case but we can speculate that a more complicated phase diagram would appear. If any vertices have threshold less than three, that is, $Q(1)+Q(2)>0$, we would find a continuous appearance of the giant HKC. Thresholds of three or more, on the other hand, contribute to discontinuous transitions, and it may be that there are multiple such transitions.

[1] B. Bollobas, in *Graph Theory and Combinatorics: Proceedings of the Cambridge Combinatorial Conference in Honour of Paul Erdos*, edited by B. Bollobas (Academic Press, New York, 1984), pp. 35–37.
[2] S. Carmi, S. Havlin, S. Kirkpatrick, Y. Shavitt, and E. Shir, Proc. Natl. Acad. Sci. USA **104**, 11150 (2007).
[3] J. I. Alvarez-Hamelin, L. Dall'Asta, A. Barrat, and A. Vespignani, in *Advances in Neural Information Processing Systems 18*, edited by Y. Weiss, B. Schölkopf, and J. Platt (MIT Press, Cambridge, MA, 2006), pp. 41–50.
[4] J. I. Alvarez-Hamelin, L. Dall'Asta, A. Barrat, and A. Vespignani, Networks Heterogeneous Media **3**, 371 (2008).
[5] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **96**, 040601 (2006).
[6] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[7] C. F. Moukarzel, Phys. Rev. E **68**, 056104 (2003).
[8] J. M. Schwarz, A. J. Liu, and L. Q. Chayes, Europhys. Lett. **73**, 560 (2006).
[9] M. Kitsak, L. K. Gallos, S. Havlin, F. Liljeros, L. Muchnik, H. E. Stanley, and H. A. Makse, e-print arXiv:1001.5285.
[10] S. Wuchty and E. Almaas, Proteomics **5**, 444 (2005).
[11] N. Chatterjee and S. Sinha, Progr. Brain Res. **168**, 145 (2008).
[12] D. J. Schwab, R. F. Bruinsma, J. L. Feldman, and A. J. Levine, Phys. Rev. E **82**, 051911 (2010).
[13] P. Klimek, S. Thurner, and R. Hanel, J. Theoret. Biol. **256**, 142 (2009).
[14] G. R. Reich and P. L. Leath, J. Stat. Phys. **19**, 611 (1978).
[15] J. Chalupa, P. L. Leath, and G. R. Reich, J. Phys. C **12**, L31 (1979).
[16] B. Pittel, J. Spencer, and N. Wormald, J. Comb. Theory B **67**, 111 (1996).
[17] D. Fernholz and V. Ramachandran, University of Texas Computer Science Technical Report TR04-13, 2004 (unpublished).
[18] M. Molloy, Random Struct. Algorithms **27**, 124 (2005).
[19] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **73**, 056101 (2006).
[20] O. Riordan, Combin. Probab. Comput. **17**, 111 (2008).
[21] B. Corominas-Murtra, J. F. F. Mendes, and R. V. Solé, J. Phys. A **41**, 385003 (2008).
[22] F. Sausset, C. Toninelli, G. Biroli, and G. Tarjus, J. Stat. Phys. **138**, 411 (2009).
[23] C. L. Farrow, P. Shukla, and P. M. Duxbury, J. Phys. A **40**, F581 (2007).
[24] P. Shukla, Pramana J. Phys. **71**, 319 (2008).
[25] M. Iwata and S. Sasa, J. Phys. A **42**, 075005 (2009).
[26] P. M. Kogut and P. L. Leath, J. Phys. C **14**, 3187 (1981).
[27] G. Parisi and T. Rizzo, Phys. Rev. E **78**, 022101 (2008).
[28] J. P. Gleeson and S. Melnik, Phys. Rev. E **80**, 046121 (2009).
[29] J. P. Gleeson, S. Melnik, and A. Hackett, Phys. Rev. E **81**, 066114 (2010).
[30] J.-P. Eckmann, O. Feinerman, L. Gruendlinger, E. Moses, J. Soriano, and T. Tlusty, Phys. Rep. **449**, 54 (2007).
[31] J. Soriano, M. R. Martínez, T. Tlusty, and E. Moses, Proc. Natl. Acad. Sci. USA **105**, 13758 (2008).
[32] A. V. Goltsev, F. V. de Abreu, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **81**, 061921 (2010).
[33] M. Sellitto, G. Biroli, and C. Toninell, Europhys. Lett. **69**, 496 (2005).
[34] C. Toninelli, G. Biroli, and D. S. Fisher, Phys. Rev. Lett. **96**, 035702 (2006).
[35] S. Sabhapandit, D. Dhar, and P. Shukla, Phys. Rev. Lett. **88**, 197202 (2002).
[36] A. E. Holroyd, Probab. Theory Relat. Fields **125**, 195 (2003).
[37] A. E. Holroyd, Electron. J. Probab. **11**, 418 (2006).
[38] J. Balogh and B. Bollobas, Probab. Theory Relat. Fields **134**, 624 (2006).
[39] R. Cerf and E. N. Cirillo, Ann. Probab. **27**, 1837 (1999).
[40] J. Balogh and B. G. Pittel, Random Struct. Algorithms **30**, 257 (2007).
[41] L. R. G. Fontes and R. H. Schonmann, J. Stat. Phys. **132**, 839 (2008).
[42] J. Balogh, Y. Peres, and G. Pete, Combin. Probab. Comput. **15**, 715 (2006).
[43] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E **82**, 011103 (2010).
[44] D. E. Whitney, e-print arXiv:0911.4499.
[45] D. J. Watts, Proc. Natl. Acad. Sci. USA **99**, 5766 (2002).
[46] J. P. Gleeson, Phys. Rev. E **77**, 046117 (2008).
[47] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[48] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[49] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Physica D **224**, 7 (2006).
[50] M. Aizenman and J. L. Lebowitz, J. Phys. A **21**, 3801 (1988).
[51] D. Pazó, Phys. Rev. E **72**, 046211 (2005).
[52] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[53] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[54] R. Cohen, D. ben-Avraham, and S. Havlin, Phys. Rev. E **66**, 036113 (2002).