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Transmission of information between complex systems: $1/f$ resonance

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We study the transport of information between two complex systems with similar properties. Both systems generate non-Poisson renewal fluctuations with a power-law spectrum $1/f^{3-\mu}$, the case $\mu = 2$ corresponding to ideal $1/f$ noise. We denote by μ_S and μ_P the power-law indexes of the system of interest S and the perturbing system P, respectively. By adopting a generalized fluctuation-dissipation theorem (FDT) we show that the ideal condition of $1/f$ noise for both systems corresponds to maximal information transport. We prove that to make the system S respond when $\mu_S < 2$ we have to set the condition $\mu_P < 2$. In the latter case, if $\mu_P < \mu_S$, the system S inherits the relaxation properties of the perturbing system. In the case where $\mu_P > 2$, no response and no information transmission occurs in the long-time limit. We consider two possible generalizations of the fluctuation dissipation theorem and show that both lead to maximal information transport in the condition of $1/f$ noise.

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I. INTRODUCTION

Linear response theory (LRT) [1] is one of the basic ways of obtaining information from fluctuations in nonequilibrium statistical physics [2,3] that is currently adopted to address new phenomena such as glassy systems [2] and granular matter [3]. An even more challenging issue is the application of LRT to complex processes such as physiological processes and especially the understanding of brain dynamics.

It is becoming more widely accepted that the brain operates at criticality [4,5] and that the critical condition of a phase transition has manifestations that extend beyond the conventional condition of temperature driven systems [6]. Frantsuzov *et al.* [7] adopted a model of cooperatively interacting units to propose a solution to the long-standing mystery of the origin of the power-law distribution of the blinking times in colloidal quantum dot fluorescence. On the other hand, this form of intermittence is characterized by the condition of renewal aging [8] and by the consequent ergodicity breakdown [9] that makes it impossible to use conventional LRT. We refer to these systems as *complex systems*. It is now very well understood [10] that the breakdown of the ergodic condition is caused by the occurrence of *crucial events*. It is interesting to remark that the same perspective applies to complex networks, when the cooperation-induced phase transition produces temporal complexity and crucial events as illustrated in in the work of Ref. [11]. These events are renewal, namely, the time interval between two consecutive events does not have any relation whatsoever with the earlier or later time intervals between two consecutive events. Yet, if a Gibbs ensemble is suitably *prepared* initially, namely, in all the systems of the ensemble an event occurs at the time origin, then the rate of event production turns out to be time dependent rather than constant as in the ordinary Poisson case. The authors of Refs. [12,13] applied this theoretical perspective to the liquid crystal dynamics and experimentally realized a true cascade of renewal events. The time interval between two consecutive crucial events is given by a waiting times probability density function (PDF) $\psi(\tau)$ with the following asymptotic form

$$\psi(\tau) = (\mu - 1)\frac{T^{\mu-1}}{(\tau + T)^{\mu-1}}, \tag{1}$$

and the power-law index μ fulfilling the inequality

$$1 < \mu < 3. \tag{2}$$

It is important to explain the origin of the special form of Eq. (1). First of all we want to stress that according to a point of view in the field of complexity, only the asymptotic time behavior matters, namely, $\psi(\tau) \propto 1/\tau^{\mu}$. The adoption of this widely shared point of view, as we shall see hereby, would prevent us from establishing a correct accordance with the experiments on the response of complex systems to external perturbations. Thus, the choice of Eq. (1) is dictated by the need for defining a border between the asymptotic time regime ($\tau \gg T$) and the microscopic time regime ($\tau \leqslant T$). Given the neurophysiology interest of this paper and especially the focus on brain dynamics, we refer the interested reader to the work of Ref. [14], where the waiting times PDF of Eq. (1) is obtained by means of a Fechner transformation [14] from the conventional Poisson distribution.

The rate of cascade of renewal events tends to a vanishing value as $1/t^{2-\mu}$ when $\mu < 2$ and to a constant value as $1/t^{\mu-2}$ when $\mu > 2$. It is evident that in both cases the time duration of the out-of-equilibrium condition is infinite, thereby raising the challenging task of going beyond conventional LRT to describe the dynamics.

Conventional LRT is given by the following expression:

$$\sigma(t) = \langle \xi_S(t) \rangle = \epsilon \int_0^t ds\, \chi(t,s) \xi_P(s), \tag{3}$$

where $\xi_S(t)$ is the fluctuation produced by the system of interest S. The symbol $\langle \xi_S(t) \rangle$ denotes the Gibbs average

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over the fluctuations. In the absence of perturbation this average is assumed to vanish. The variable $\xi_P(t)$ denotes the time-dependent perturbation and ϵ its intensity. LRT predicts the response of S on the basis of the unperturbed correlation function of ξ_S. In fact, the function $\chi(t,s)$, called the linear response function, is related to the correlation function of the fluctuation ξ_S, whose quadratic mean value is assumed for simplicity to be normalized to unity

$$\Psi_S(t,s) \equiv \langle \xi_S(t)\xi_S(s)\rangle, \tag{4}$$

by the following expression

$$\chi(t,s) = \frac{d}{ds}\Psi_S(t,s). \tag{5}$$

Note that the traditional LRT refers to the stationary case

$$\Psi_S(t,s) = \Psi_S(t-s), \tag{6}$$

and as a consequence

$$\frac{d}{ds}\Psi_S(t,s) = -\frac{d}{dt}\Psi_S(t,s). \tag{7}$$

This condition is not fulfilled by complex systems. For these latter systems the choices of linear response functions

$$\chi(t,s) = \frac{d}{ds}\Psi_S(t,s), \tag{8}$$

and

$$\chi(t,s) = -\frac{d}{dt}\Psi_S(t,s), \tag{9}$$

are not equivalent.

The authors of Refs. [15–18] have discussed the foundation of both choices and have established that the new form of LRT is determined by the physical way through which perturbation determines a bias. For the sake of simplicity these authors have made the assumption that $\xi_S(t)$ is a dichotomous signal. Using the jargon of turbulence theory they called the time intervals between two consecutive crucial events *laminar regions*. At the moment of a crucial event occurrence, unperturbed dynamics are realized by the random selection of either the positive, $\xi_S = 1$, or the negative value, $\xi_S = -1$. In other words, they assume that the occurrence of a crucial event generates the tossing of a fair coin which determines the sign of the next laminar region. Consequently, the external perturbation can generate a bias in two different ways. The first way rests on affecting the fairness of the coin tossing process. If $\xi_P(t) > 0$ ($\xi_P(t) < 0$) the choice of the positive (negative) sign is more probable than the choice of the negative (positive) sign. This prescription leads to the choice of Eq. (8) and is denoted as *phenomenological* LRT.

The experiments done by the authors of Refs. [12,13] show that nature prefers Eq. (9), the dynamic LRT. What is the theoretical argument in favor of the dynamical theory? To afford a convincing answer to this important question, let us go back to the special form of Eq. (1). We note that we do not know the Hamiltonian of our complex system, and we do not even know if a satisfactory discussion of the complex dynamics can be made using a Hamiltonian formalism. Let us assume that Eq. (1) is a reliable representation for the distribution length of the laminar region. In this case, a reasonable conjecture is that the external perturbation affects either μ or T, or both parameters defining the form of Eq. (1). We know that μ defines the system's complexity and emerges from the cooperative interaction among interacting units. A weak external perturbation is not expected to change the system's complexity. It is therefore reasonable to assume that the external perturbation affects T, by enlarging (reducing) its value if ξ_P and ξ_S have the same (opposite) sign. This assumption leads to the choice of Eq. (9), as shown earlier [15–18].

It is important to notice that the response of a complex system of the same nature as the one discussed in this paper has been studied by many authors [19–28]. These authors did not establish a connection between their results and the LRT of Refs. [15–18] and in some cases they made the misleading conjecture that their results establish the “death of LRT.” Actually, these theoretical treatments are asymptotic in time and the only possible connection with LRT is through the adoption of the phenomenological theory of Eq. (8), as the readers can establish by a careful reading of Ref. [28].

We are now in a position to define the main purpose of this paper. We draw the attention of the readers to the recent results of the authors Ref. [29]. This paper addresses the important issue of studying the response of a complex system to a complex external perturbation with the surprising result that a complex system does not respond to stimuli that are not complex (i.e., that have a stationary Fourier spectrum). It is important to stress that the authors of Ref. [29] focused on the correlation between $\xi_S(t)$ and $\xi_P(t)$ in the long-time limit. This is an ideal condition that has the effect of restraining the definition of complexity to the systems with $\mu \leqslant 2$. In fact, in the long-time limit a system with $\mu > 2$ reaches the normal condition of a constant rate of event production, thereby recovering the ordinary Poisson condition. The condition $\mu = 2$ is of fundamental importance for brain function. In fact, recent work [30,31] established that the brain works with $\mu = 2$, which, in turn, is known [32] to correspond to making the brain action become the source of ideal $1/f$ noise. The results of the authors of Ref. [29] may therefore have important applications to design the most convenient stimuli to drive complex networks, and especially brain dynamics, via what was defined as “complexity management” [29]. However, an apparent weakness found in Ref. [29] is that these results are derived from the adoption of the phenomenological LRT, thereby raising doubt that those complex systems, which have been proven to obey the dynamical LRT [12,13], may not obey the principle of *complexity management* (CM) established in Ref. [29]. Herein we prove that the more realistic dynamical LRT generates CM. In addition to this main purpose, the present paper affords technical details on the theory developed in Ref. [29] that, due to space limitations, were not conveniently illustrated.

II. FDT FOR THE NONERGODIC RENEWAL REGIME: PHENOMENOLOGICAL AND DYNAMICAL APPROACH

The authors of Refs. [15,16] discovered a form of FDT that applies to systems or networks whose dynamics are dominated by non-Poisson renewal events. In the stationary case this

FDT becomes indistinguishable from the ordinary theoretical prediction [1].

Herein we investigate the consequences of the adoption of either the "phenomenological" choice Eq. (8) or the "dynamical" choice (9), in the special case where both $\xi_P(t)$ and $\xi_S(t)$ are event-dominated processes and show that the transmission of information from P to S is determined by the dialogue between the critical events of $\xi_S(t)$ and the critical events of $\xi_P(t)$. Specifically, this discussion is devoted to studying the transport of information from P to S, using both forms of generalized FDT (gFDT). Note that there is no limitation on the form of ξ_P provided that the coupling is weak enough as to be compatible with the emergence of the linear response form of Eq. (3).

To discuss the problem of the transmission of information from P to S, it is convenient to imagine the ideal case of a Gibbs ensemble of composite systems $S + P$. Thus, in this paper we imagine that P generates a fluctuating signal $\xi_P(t)$ and that for any signal $\xi_P(t)$ there exists a response $\xi_S(t)$. For each perturbing signal we have to make infinitely many experiments and average over all possible responses. For simplicity we take both signals $\xi_S(t)$ and $\xi_P(t)$ to be dichotomous and fluctuating between values ± 1. It is important to remark that Eq. (3), for the response of the system S to the perturbation P, is valid when the system is prepared at time $t = 0$ and placed at the beginning of a laminar phase, and the interaction with the perturbation P is turned on at the same time.

In the general case of a dicothomous renewal process $\xi(t)$, generated with a waiting-times PDF $\psi(t)$, the probability density that fixed a time t', the first next event is observed at time $t > t'$ is given [33] by

$$\psi(t,t') = \psi(t) + \sum_{n=1}^{\infty} \int_0^{t'} R(t'')\psi(t - t'')dt'', \qquad (10)$$

with

$$R(t) = \sum_{n=1}^{\infty} \psi_n(t), \qquad (11)$$

where $\psi_n(t)$ denotes the n-times convolution of $\psi(t)$. $R(t)$ is therefore the probability density of having an event occurring exactly at time t.

It can be shown as well, see Ref. [33], that the autocorrelation function of the process is related to $\psi(t,t')$ by

$$\langle \xi(t)\xi(t')\rangle = \int_t^{\infty} dx\,\psi(x,t') = \Psi(t,t'), \qquad (12)$$

and therefore coincides with the survival probability $\Psi(t,t')$ for the first event (i.e., the probability that, for fixed t', no event is observed until time $t > t'$). We assume that the fluctuation $\xi_S(t)$ generated by S is a dicothomous renewal process defined by the probability density

$$\psi_S(t) = (\mu_S - 1)\frac{{T_S}^{\mu_S - 1}}{(t + T_S)^{\mu_s}}. \qquad (13)$$

We therefore name, respectively, $\psi_S(t,t')$, $R_S(t)$, and $\Psi_S(t,t')$ the functions obtained by replacing in Eqs. (10), (11), and (12) $\psi(t)$ with $\psi_S(t)$.

Let us consider now the Gibbs ensemble of composite systems $S + P$, and evaluate the average $\langle \xi_S(t)\rangle_{SP}$. Note that the average is over the separate statistics of the two systems S and P

$$\langle \xi_S(t)\rangle_{SP} = \langle\langle \xi_S(t)\rangle_S\rangle_P. \qquad (14)$$

We select all the responses to the same perturbation, characterized by a given $\xi_P(t)$, we evaluate their average, denoted by $\langle \xi_S(t)\rangle_S$, and finally we construct the average over all possible perturbations denoted by $\langle ...\rangle_P$ so as to obtain the final result denoted by $\langle ...\rangle_{SP}$. In conclusion with this procedure we obtain

$$\langle \sigma(t)\rangle = \langle\langle \xi_S(t)\rangle\rangle = \epsilon \int_0^t dt' \chi(t,t')\langle \xi_P(t')\rangle, \qquad (15)$$

where, for notational convenience, we drop the subscripts, but we understand the averages in the sense described above.

If necessary, the signal $\xi_P(t)$ must share the same properties as $\xi_S(t)$ and for simplicity they are both assumed to be dichotomous signals with random renewal fluctuations between the values $+1$ and -1. $\xi_P(t)$ is therefore a non-Poissonian dichotomic fluctuation with the following waiting-time PDF:

$$\psi_P(t) = (\mu_P - 1)\frac{T^{\mu_P - 1}}{(t + T_P)^{\mu_P}}. \qquad (16)$$

It is therefore convenient to define the additional functions $\psi_P(t,t')$, $R_P(t)$, and $\Psi_P(t,t')$ obtained, analogously as for S, from Eqs. (10), (11), and (12) after replacing $\psi(t)$ with the waiting-time distribution $\psi_P(t)$.

The spectrum of this type of fluctuating signal, in the absence of perturbation, as calculated in Refs. [32,34], is

$$S(f) \propto L^{\mu - 2} f^{\mu - 3}, \qquad (17)$$

valid for $\mu < 2$, remarkably, even though a stationary autocorrelation function cannot be defined in this case. In the case $\mu > 2$, $S(f) = A/f^{3-\mu}$, with A independent of L, the length of the sequence under study.

At this point it should be clear to the reader that to get the important results of this paper on the transmission of the statistical properties of P to S we must use Eq. (15). This leads us to give a prescription to define $\langle \xi_P(t)\rangle$. We assume that the perturbation P is prepared at $t = 0$ at the beginning of a laminar phase (i.e., with the analogous prescription adopted for the system S). This assumption allows us to replace $\langle \xi_P(t')\rangle$ with $\Psi_P(t')$ in Eq. (15).

III. PHENOMENOLOGICAL APPROACH

In this section we study the response of a complex system producing non-Poissonian renewal fluctuations to a perturbing signal with similar properties within the phenomenological approach. We analyze both the average response and the input-output correlation [i.e., the correlation between the perturbing fluctuating signal (input) and the signal produced by the system of interest S (output)]. In the phenomenological approach, the waiting times between the events generating the dichotomic fluctuations remain unchanged by the perturbation. The external perturbation introduces a bias so that when an event occurs the probability that the dichotomic variable changes or keeps its value is slightly different. The function $\chi(t,s)$ in this approach is given by [15,16]

$$\chi(t,t') = \frac{d\Psi_S(t,t')}{dt'} = R_S(t')\Psi_S(t - t'). \qquad (18)$$

The function $\Psi_S(t,t')$ is the autocorrelation function of $\xi_S(t)$, namely, the survival probability of age t', and $R_S(t)$ for the case of the discrete signals considered here, is the rate at which events are produced by S prepared at $t = 0$ [i.e., the bits per second encoded in $\xi_S(t)$]. This rate is time independent only in the Poisson case. In the non-Poisson case it depends on time, thereby making $\Psi_S(t,t')$ nonstationary. The brand new survival probability $\Psi_S(t) = \Psi_S(t,t'{=}0)$ is given by [15,16,18]

$$\Psi_S(t) = (1 + t/T_S)^{1-\mu_S}, \tag{19}$$

from which the corresponding waiting-times PDF $\psi_S(t) = -d\Psi_S(t)/dt$ is derived. In the range of parameters $1 < \mu_S < 3$ considered here, it is known [10] that

$$R_S(t) \approx -\frac{\sin \pi\mu_S}{T_S}(T_S/t)^{2-\mu_S}, \quad \text{for } 1 < \mu_S < 2, \tag{20}$$

$$R_S(t) \approx \frac{1}{\tau_S}[1 + (T_S/t)^{\mu_S-2}], \quad \text{for } 2 < \mu_S < 3, \tag{21}$$

with $\tau_S = T_S/(\mu_S - 2)$ the mean value of $\psi_S(t)$.

When $\mu_S < 2$ the experimental preparation of S induces a sequence of events whose rate R_S tends to vanish for $t \to \infty$, yielding a perennial out-of-equilibrium condition, and an explanation of the death of linear response [18–24,28] as well. In fact, the response to a harmonic perturbation of frequency f is proportional to $1/(ft)^{2-\mu_S}$ [18]. In the case $2 < \mu_S < 3$, on the contrary, the preparation-induced cascade of events, in the limit $t \to \infty$, becomes stationary and virtually identical to that of a Poisson process. The theoretical analysis of this paper is done in the asymptotic time regime. Thus, we refer to the case $2 < \mu < 3$ as *stationary*, in contrast to the *nonstationary* case $\mu \leqslant 2$ of perennial transition. Similarly to the rate of events $R_S(t)$ the spectral intensity per unit time tends to vanish for $\mu < 2$ as an effect of increasing L [see Eq. (17)]. The ideal $1/f$-noise condition, corresponding to $\mu = 2$, generates instead a logarithmic decrease of the spectral intensity with time, and consequently a spectrum virtually independent of L.

A. Average response to perturbation

As previously mentioned, the nonstationary LRT (NSLRT) of Eq. (3) rests on the preparation of S at time $t = 0$. We apply the same preparation condition to P, thereby generating the cascades $R_S(t)$ and $R_P(t)$ described by Eqs. (20) and (21), with the appropriate indexing. Under this condition the relaxation of $\langle\xi_P(t)\rangle$ becomes identical to the survival probability $\Psi_P(t)$. Assuming the condition of Eq. (18) we have the following expression for the average response

$$\langle\sigma(t)\rangle = \varepsilon \int_0^t R_S(t')\Psi_S(t-t')\Psi_P(t')dt'. \tag{22}$$

The preparation of both S and P makes the average over many realizations of the response $\sigma(t)$ to a given stimulus P vanish for $t \to \infty$.

Stationary case: $2 < \mu_S < 3$.

In this regime a finite time scale for the fluctuation ξ_S exists and $R_S(t)$ reaches the constant value $1/\tau_S$. The inverse power-law relaxation of $\Psi_S(t)$ allows us to approximate Eq. (22) by replacing $R_S(t')$ with its value for $t' \simeq t$, that is,

$$\langle\sigma(t)\rangle \simeq \epsilon \int_0^t dt'\Psi_S(t-t')\Psi_P(t')/\tau_S, \tag{23}$$

which becomes exact for $t \to \infty$. The asymptotic behavior of Eq. (23) is easily obtained in the Laplace domain

$$\langle\hat{\sigma}(s)\rangle \simeq \epsilon\frac{1}{\tau_S}\frac{1-\hat{\psi}_S(s)}{s}\frac{1-\hat{\psi}_P(s)}{s}, \tag{24}$$

which can be studied in the limit of small s. In fact, since [35]

$$\hat{\psi}(s) \simeq 1 - \tau s + \Gamma(\mu-2)s^{\mu-1}, \quad \mu > 2, \tag{25}$$

and

$$\hat{\psi}(s) \simeq 1 + \Gamma(\mu-2)s^{\mu-1}, \quad \mu < 2, \tag{26}$$

it follows that for $1 < \mu_P < 2$ and $2 < \mu_P < \mu_S$, the time-asymptotic behavior is $\langle\sigma(t)\rangle \sim t^{1-\mu_P}$, which is proportional to $\langle\xi_P(t)\rangle$ for large t, meaning that the system S "inherits" the relaxation properties of the perturbation P.

For $2 < \mu_S < \mu_P$, instead, the asymptotic dominant term is $\langle\sigma(t)\rangle \sim t^{1-\mu_S}$, which is proportional to the ordinary unperturbed relaxation to equilibrium $\langle\xi_S(t)\rangle$, when an initial bias for $\xi_S(t)$ is introduced. We see therefore that for $\mu_P < \mu_S$, when $\xi_P(t)$ is slower than $\xi_S(t)$, the perturbation P imposes on the system S its own relaxation properties thereby allowing one to "manage" the complexity of a system or network by using an appropriate stimulus.

Nonstationary case: $1 < \mu_S \leqslant 2$.

In this regime, the system S violates the finite-time scale condition necessary for stationary dynamics and, in fact, $R_S(t) \propto t^{\mu_S-2}$, see Eq. (20). With such replacement in Eq. (22), a convolution form appears which can easily studied via a Laplace transformation. In the Laplace domain (see Appendix A for details on coefficients)

$$\langle\hat{\sigma}(s \simeq 0)\rangle \simeq a_S\, s^{\mu_S-2} + a_P\, s^{\mu_P-2}, \tag{27}$$

which implies that if $\mu_P > 2$ or if $1 < \mu_S < \mu_P$ then $\langle\sigma(t)\rangle \sim t^{1-\mu_S}$. When $1 < \mu_P < \mu_S$, we have $\langle\sigma(t)\rangle \sim t^{1-\mu_P}$. Also in this case the perturbation P forces onto S its own relaxation properties to equilibrium.

B. Input-output correlation function

We study the cross-correlation (or input-output correlation) function between the system S and the stimulus $P : C(t) \equiv \langle\langle\xi_S(t)\xi_P(t)\rangle\rangle$, which is also used as an indicator of aperiodic stochastic resonance [36]. Multiplying both sides of Eq. (3) by $\xi_P(t)$ and averaging over the fluctuations of the perturbation P we obtain

$$\Phi(t) \equiv C(t)/\varepsilon = \int_0^t dt' R_S(t')\Psi_S(t-t')\Psi_P(t,t'). \tag{28}$$

Note that both Eqs. (22) and (28) depend on the survival probability of P, but in the former such survival probability depends on the single time t' whereas in the latter it depends on both t' and t. We limit ourselves to report the results for the asymptotic value Φ_∞ of $\Phi(t)$. When $\xi_S(t)$ and $\xi_P(t)$ are not stationary, that is, when $1 < \mu_S \leqslant 2$ and $1 < \mu_P \leqslant 2$, Eq. (28), in the limit $t \to \infty$, gives

$$\Phi_\infty = \zeta(\mu_S,\mu_P) \equiv \Gamma(\mu_S + \mu_P - 2) \times \frac{{}_3F_2\,[\{\mu_P - 1,\mu_P - 1,\mu_P + \mu_S - 2\},\{\mu_P,\mu_P\},1]}{\Gamma(2-\mu_P)\Gamma(\mu_P)^2\Gamma(\mu_S - 1)}, \tag{29}$$

where ${}_3F_2$ is the generalized hypergeometric function. For more details see Appendix A. In the case $2 < \mu_P < 3$, Φ_∞ is simply zero.

In the case $2 < \mu_S < 3$, inserting into Eq. (28) expression (21) for $R_S(t)$, leads to

$$\begin{aligned}\Phi(t) = \int_0^t dt' \frac{\Psi_S(t-t')}{\tau_S}\Psi_P(t,t') = \tilde{\Psi}_S(0)\Psi_P(t,t)\\ - \tilde{\Psi}_S(t)\Psi_P(t,0) - \int_0^t dt' \tilde{\Psi}_S(t-t')\frac{d\Psi_P(t,t')}{dt'},\end{aligned} \tag{30}$$

where $\tilde{\Psi}_S(t)$ is given by Eq. (19) after replacing μ_S with $\mu_S - 1$. Equation (30) is exact for $t \gg \tau_S$ and for $1 < \mu_P \leqslant 2$ it leads to $\Phi_\infty = 1$ since both the second and third terms disappear for $t \to \infty$ and the first is trivially 1 in the same limit. For Eq. (30) $2 < \mu_P < 3$ it yields

$$\begin{aligned}\Phi_\infty &= 1 - (\mu_P - 2)T_P^{\mu_P-2}T_S^{\mu_S-2}\Delta T^{4-\mu_S-\mu_P}\\ &\quad\times B\left[\begin{matrix}\Delta T/T_P\\ \Delta T/T_S\end{matrix},\mu_S + \mu_P - 4,\begin{matrix}2-\mu_P\\ 3-\mu_S\end{matrix}\right] \overset{\Delta T\to 0}{=}\\ &\quad\times \frac{\mu_S - 2}{\mu_P + \mu_S - 4},\end{aligned} \tag{31}$$

where $\Delta T = |T_S - T_P|$, $B[x,a,b]$ is the incomplete Beta function and the upper (lower) choice of the parameters refers to the case $T_S > (<)T_P$. The final expression in Eq. (31) corresponds to the case $T_S = T_P$. The results are summarized in Table I. For illustrative purposes, we supplement Table I with Fig. 1, showing the three-dimensional (3D) plot of the cross-correlation function Φ_∞ in the same parameter range: Square II and square III correspond to the condition of minimal and maximal correlation, respectively. Intuitively it is so because of the difference of time scales between S and P in such regions. In III fluctuations $\xi_S(t)$ and $\xi_P(t)$ have a finite and an infinite time scale, respectively, thereby allowing $\xi_S(t)$ to adapt to the stimulus-induced bias so as to yield maximal correlation. In II the role of the time scales is inverted, the bias induced by P on the longer (diverging) time scale of the process $\xi_S(t)$ is asymptotically averaged out due to the many intervening switching events of $\xi_P(t)$, producing no correlation. The vertex $\mu_S = \mu_P = 2$, representing a $1/f$-noise generating system under the stimulus of a $1/f$-noise perturbation, marks the abrupt transition from vanishing (square II) to maximal correlation (III).

TABLE I. Summary of the asymptotic values of the cross-correlation function $\Phi(t)$ in the phenomenological case.

$\mu_{S\downarrow}\ \mu_{P\to}$	$1 < \mu_P \leqslant 2$		$2 < \mu_P < 3$	
$1<\mu_S\leqslant 2$	$\Phi_\infty = \zeta(\mu_S,\mu_P)$*	I	$\Phi_\infty = 0$	II
$2<\mu_S<3$	$\Phi_\infty = 1$	III	$\Phi_\infty = \frac{\mu_S-2}{\mu_S+\mu_P-4}$	IV

*See Eq. (29).

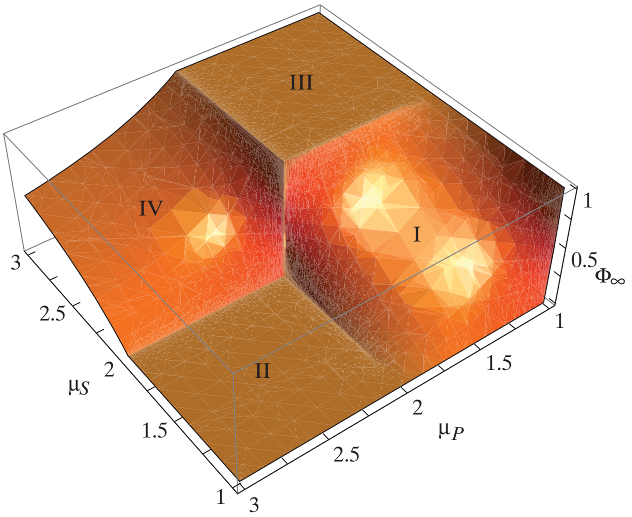

FIG. 1. (Color online) The asymptotic limit of $\Phi(t)$ is displayed for $\mu_S,\mu_P \in]1,3[$. The vertex $\mu_S = \mu_P = 2$ marks the transition to a condition of maximal input-output cross correlation.

IV. DYNAMICAL APPROACH

In this section we extend the analysis of the previous section to the dynamical approach. Within such approach we derive both the average response and the input-output correlation function.

A. Average response to perturbation

Starting from the property

$$\langle\sigma(t)\rangle = \langle\langle\xi_S(t)\rangle\rangle, \tag{32}$$

obtained by averaging over the fluctuations of both S and P and using the dynamical condition of Eq. (9), one obtains

$$\chi(t,t') = -\frac{d}{dt}\Psi_S(t,t') = \psi_S(t,t'). \tag{33}$$

Equation (3) then becomes

$$\langle\sigma(t)\rangle = \epsilon\int_0^t dt'\psi_S(t,t')\langle\xi_P(t')\rangle. \tag{34}$$

For simplicity we prepare the perturbation $\xi_P(t)$ at $t = 0$. To observe the influence of P on S we select all the systems of the Gibbs ensemble so that $\xi_P(0) = 1$. In this case $\langle\xi_P(t')\rangle$ is given by the survival probability $\Psi_P(t)$ [33]. Thus, Eq. (34) yields

$$\langle\sigma(t)\rangle = \epsilon\int_0^t dt'\psi_S(t,t')\Psi_P(t'), \tag{35}$$

where $\Psi_P(t')$ is the survival probability for the process $\xi_P(t)$

$$\Psi_P(t) = \int_t^\infty dx\,\psi_P(x) = \left(\frac{T_P}{t+T_P}\right)^{\mu_P-1}. \tag{36}$$

This slow decay corresponds to the probability that no perturbation event occurs up to time t. The system S evolves in time so as to reach a steady value that corresponds to a constant perturbation abruptly applied at $t = 0$. However, during this process a perturbation event occurs that has the effect of suddenly changing the external field. Thus, $\sigma(t)$ does not reach a steady value, but after reaching a maximum value will decay. Under the specific conditions discussed in this section, the time asymptotic decay of $\langle\sigma(t)\rangle$ in Eq. (35) has the same power-law index as that of survival probability $\Psi_P(t)$. We interpret this phenomenon as the transmission into S of the statistics of P.

We explore again the whole range of parameters $1 < \mu_S < 3$ and $1 < \mu_P < 3$, respectively, for the system S and the perturbation P, depending on the values of the power-law indexes μ_S, μ_P characterizing their waiting-times PDFs. The value $\mu = 2$ marks the transition from a finite to an infinite mean time (i.e., the transition to a nonergodic, nonstationary condition. In fact, while for $\mu > 2$ a mean time exists, a finite time scale can be defined and a stationary condition is reached, for $\mu < 2$ such condition is never achieved, not even in the infinite mean time.

Stationary case: $2 < \mu_S < 3$.

In this regime the waiting-times PDF $\psi_S(t)$ has a finite mean value τ_S. Therefore a finite time scale t_C exists such that for $t > t' > t_C$ the following approximation corresponding to reaching a stationary condition, is valid [33]

$$\psi_S(t,t') \simeq \frac{1}{\tau_S}\int_t^\infty dx\,\psi_S(x-t') = \frac{\Psi_S(t-t')}{\tau_S}, \tag{37}$$

where $\tau_S = T_S/(\mu_S - 2)$ is the mean value of $\psi_S(t)$. Equation (37) is exact for $tc \to \infty$. We can also use Eq. (3) for the response in the case when the interaction is turned on at a later time t_C that we assume to be so large as to satisfy the approximation Eq. (37) using the following procedure. We introduce into Eq. (3) an effective perturbation which is turned on at time $t = 0$ but is zero until t_C [i.e., $\xi_P^{\text{eff}}(t) = \theta(t - t_C)\xi_P(t - t_C)$]. In this scheme we can assume that the "real" process $\xi_P(t)$ is prepared at time $t = t_C$ in a brand new condition. Time t_C therefore corresponds to the age of the system S. The average response $\langle\sigma(t)\rangle$ of S in this case reads

$$\begin{aligned}\langle\sigma(t)\rangle &= \varepsilon\int_0^t dt'\psi_S(t,t')\langle\theta(t'-t_C)\xi_P(t'-t_C)\rangle\\ &= \varepsilon\int_{t_C}^t dt'\psi_S(t,t')\Psi_P(t'-t_C),\end{aligned} \tag{38}$$

and the approximation (37) can be used to replace $\psi_S(t,t')$. With the substitution $\tau = t' - t_C$ and after renaming $t - t_C$ as t back again, the average response of the system S of age t_C reads

$$\begin{aligned}\frac{\langle\sigma(t)\rangle}{\varepsilon}\Big|_{t_C} &= \int_0^t d\tau\frac{\Psi_S(t-\tau)}{\tau_S}\Psi_P(\tau) = \tilde{\Psi}_S(0)\Psi_P(t)\\ &\quad - \tilde{\Psi}_S(t)\Psi_P(0) - \int_0^t dt'\tilde{\Psi}_S(t-t')\frac{d}{dt'}\Psi_P(t'),\end{aligned} \tag{39}$$

where $\tilde{\Psi}_S(t)$ is defined as

$$\tilde{\Psi}_S(t) = (1 + t/T_S)^{\mu_S - 2}. \tag{40}$$

Equation (39) is exact for $t_C \to \infty$ and coincides with the expression (23) obtained in the phenomenological case in the same regime. Therefore, in the limit $t \gg T_S, T_P$, the same considerations apply, that is, if $1 < \mu_P < 2$ and $2 < \mu_S < \mu_P$, the time-asymptotic behavior is $\langle\sigma(t)\rangle \sim t^{1-\mu_P}$, which is proportional to $\langle\xi_P(t)\rangle$ for large t, therefore the system S always "inherits" the relaxation properties of the perturbation. If $2 < \mu_S < \mu_P$, the asymptotic dominant term is $\langle\sigma(t)\rangle \sim t^{1-\mu_S}$, which is proportional to the ordinary unperturbed relaxation to equilibrium $\langle\xi_S(t)\rangle$.

Nonstationary case: $1 < \mu_S < 2$.

Let us make the assumption that, although at time $t = 0$ half of the S elements of the Gibbs ensemble are in the state $\xi_S = +1$ and half in the state $\xi_S = -1$, all of them are at the beginning of their sojourn in the corresponding states. This is an out-of-equilibrium condition, corresponding to preparing the system at $t = 0$. The calculations are detailed in Appendix B. Using the dynamic theory we obtain

$$\frac{\langle\sigma(t)\rangle}{\epsilon} \approx \frac{k_1(\mu_S,\mu_P)}{t^{\mu_P - 1}} + \frac{k_2(\mu_S,\mu_P)}{t^{\mu_P + 1 - \mu_S}}, \tag{41}$$

where the first coefficient is given by

$$k_1(\mu_S,\mu_P) = \frac{T_P{}^{\mu_P - 1}\sin(\pi\mu_S)\Gamma(2-\mu_S)\Gamma(1-\mu_P+\mu_S)}{\pi\Gamma(3-\mu_P)}, \tag{42}$$

and the second coefficient is determined to be

$$k_2(\mu_S,\mu_P) = \frac{[\sin(\pi\mu_S)\Gamma(3-\mu_S)\Gamma(1-\mu_P+\mu_S)\Gamma(2\mu_S-3) - (2-\mu_S)\Gamma(2\mu_S-\mu_P-1)]}{T_P{}^{1-\mu_P}T_S{}^{\mu_S-2}\pi(\mu_P-2)(2-\mu_S)\Gamma(3-\mu_S)\Gamma(\mu_S-\mu_P)\Gamma(2\mu_S-3)}. \tag{43}$$

In this range of parameters the dominant term is always the first term in Eq. (41) which, if $\mu_P < \mu_S$, is also slower than the unperturbed relaxation to equilibrium of ξ_S. In the latter range, therefore, the system S relaxes to equilibrium inheriting the same properties of the perturbation P.

The result in Eq. (41) is of special interest since it discriminates between the two approaches in the nonstationary regime. It is this difference that allowed to determine that liquid crystals [13] follow the prediction of the dynamical approach. In fact, the phenomenological approach disregards the influence of the perturbation on the occurrence time of the S events [19,28], while the dynamical theory does not, thereby affording a criterion for information transport that we judge to be a more appropriate representation of the communication among complex systems with $\mu < 2$. However, the equivalence between the phenomenological and the dynamic theories in the case when the system S is infinitely aged (i.e., for $\mu_S > 2$) indicates that the generalization of FDT given by Eq. (9), namely, the dynamical theory, becomes active only when the system S is in a far from equilibrium condition and begins drifting toward equilibrium. Although equilibrium is never reached when $\mu < 2$, the correlation function $\Psi_S(t,t')$ tends to recover the property $\Psi_S(t,t') = \tilde{\Psi}_S(t-t')$ that makes the phenomenological theory formally equivalent to the dynamical theory.

B. Input-output correlation function

Herein we study the asymptotic limit of the input-output correlation function

$$\Phi_\infty \equiv \lim_{t\to\infty} C(t)/\epsilon, \tag{44}$$

within the dynamical approach. The input-output correlation function is again defined by the average over the fluctuations in both the system S and perturbation P

$$C(t) \equiv \langle \xi_S(t)\xi_P(t)\rangle. \tag{45}$$

The asymptotic limit of $C(t)$ is independent of the way the system and the perturbation are prepared, so we can use the prescription leading to Eq. (3) obtained assuming that both S and P are prepared at time $t = 0$. We therefore use the same arguments as those yielding Eq. (28) and, adapting them to the dynamic theory, we obtain

$$\Phi(t) = C(t)/\epsilon = \int_0^t dt' \psi_S(t,t')\Psi_P(t,t'). \tag{46}$$

Nonstationary case I: $\mu_S < 2$, $\mu_P < 2$.

We use Eq. (46) with the general expressions for $\psi_S(t,t')$ and $\Psi_P(t,t')$ as obtained through Eqs. (10) and (12), respectively. In this case, taking the limit $t \to \infty$ yields (see Appendix B for details)

$$\begin{aligned}\zeta_D = \lim_{t\to\infty} \Phi(t) = -\frac{\sin \pi\mu_P}{\pi}\frac{\Gamma(\mu_P+\mu_S-1)}{(\mu_P-1)\Gamma(\mu_P+1)\Gamma(\mu_S-1)}\\ \times F\left[\{\mu_P-1,\mu_P-1,\mu_P+\mu_S-1\},\{\mu_P,\mu_P+1\},1\right].\end{aligned} \tag{47}$$

Nonstationary case II: $\mu_S < 2$, $\mu_P > 2$.

In this case we can assume that, when the interaction is turned on, the perturbation P has already reached a stationary condition, so that in Eq. (46) $\psi_S(t,t')$ is given again by Eq. (10) but $\Psi_P(t,t') = \tilde{\Psi}_P(t-t')$, with

$$\tilde{\Psi}_P(t) = (1 + t/T_P)^{\mu_P-2}, \tag{48}$$

and this expression has to be directly inserted into Eq. (46).

The power-law index $\mu_P - 2$ reflect the stationary condition realized with the preparation of the perturbation P at a time $t_P = -\infty$. In this case, we obtain

$$\lim_{t\to\infty} \Phi(t) = 0. \tag{49}$$

In the time asymptotic limit, the system S turns out to be independent of P in spite of the fact that at $t = 0$ we switch on the S-P interaction.

Stationary case I: $\mu_S > 2$, $\mu_P > 2$.

We again use Eq. (46) and assume that, when the interaction is turned on, the perturbation P has already reached a stationary condition so that in Eq. (46) $\Psi_P(t,t') = \tilde{\Psi}_P(t-t')$ with $\tilde{\Psi}_P(t)$ given by Eq. (48). For $\mu_S > 2$ a finite mean time τ_S of $\psi(t)$ exists, therefore a finite time scale $t_C \propto \tau_S$ exists such that for $t > t_C$ Eq. (37) can be used again to approximate $\psi_S(t,t')$. With such substitutions, the expression for the correlation function becomes asymptotically equal to the correlation obtained in the phenomenological approach in the same regime [i.e., Eq. (30), leading to the final result Eq. (31)].

Stationary case II: $\mu_S > 2$, $\mu_P < 2$.

In this case again a finite time scale t_C can be found such that the approximation in Eq. (37) is valid. Thus, again Eq. (46) can be rewritten

$$\Phi(t) \simeq \int_0^{t_C} dt' \psi_S(t,t')\Psi_P(t,t') + \int_{t_C}^t dt' \left[\frac{d}{dt'}\tilde{\Psi}_S(t-t')\right]\Psi_P(t,t'). \tag{50}$$

This case is therefore equivalent to what is obtained for the phenomenological case in the same range of parameters. Again, in the asymptotic limit $t \to \infty$, the first term in Eq. (50) vanishes and after integrating the second term by parts, one obtains

$$\begin{aligned}\Phi(t) \simeq \tilde{\Psi}_S(0)\Psi_P(t,t) - \tilde{\Psi}_S(t-t_C)\Psi_P(t,t_C)\\ - \int_{t_C}^t dt' \tilde{\Psi}_S(t-t')\frac{d\Psi_P(t,t')}{dt'}.\end{aligned} \tag{51}$$

In the asymptotic limit the second term on the right-hand side trivially vanishes and also the third term side can be shown to vanish (see Appendix B). The only remaining contribution in Eq. (51) is given by the first term, which is exactly one. It follows

$$\Phi_\infty = \lim_{t\to\infty} C(t)/\epsilon = 1. \tag{52}$$

Let us make here a few remarks on these results. Of course in the absence of coupling, $\epsilon = 0$, which means that $C(t)$ always vanishes. We switch the interaction on at $t = 0$. Thus, we always have the zero cross-correlation initial condition $C(0) = 0$.

As an effect of switching on the interaction at $t = 0$ we realize the condition $C(t) \neq 0$, for $t > 0$. However, we find that there exist special conditions (region II in Fig. 2) for which the cross correlation again goes to zero, asymptotically

$$\Phi_\infty = \lim_{t\to\infty} C(t)/\epsilon = 0. \tag{53}$$

This indicates that only in such conditions, the system S, after a transient, recovers the condition of statistical independence of the perturbation $\xi_P(t)$. The fluctuations ξ_S with $\mu_S < 2$ turn out to be statistically independent of $\xi_P(t)$ only when $\mu_P > 2$. The environmental perturbation with $\mu_P < 2$, on the other hand, is characterized by the remarkable property of forcing the system S to respond, regardless of the value of μ_S.

Table II and the plot in Fig. 2 summarize such results and show the same qualitative properties observed for the correlation function in the phenomenological case, with the correlation functions for the two approaches being identical in squares I, II, and IV. The condition of ideal $1/f$ noise (i.e., $\mu_S = \mu_P = 2$) marks the transition from a condition of zero to maximal correlation.

TABLE II. Summary of the asymptotic values of the cross-correlation function $\Phi(t)$ in the dynamical case.

$\mu_{S\downarrow}\ \mu_{P\rightarrow}$	$1 < \mu_P \leqslant 2$		$2 < \mu_P < 3$	
$1<\mu_S\leqslant 2$	$\Phi_\infty = \zeta_D(\mu_S,\mu_P)$*	I	$\Phi_\infty = 0$	II
$2<\mu_S<3$	$\Phi_\infty = 1$	III	$\Phi_\infty = \frac{\mu_S-2}{\mu_S+\mu_P-4}$	IV

*See Eq. (47).

V. FINITE RESPONSE AND 1/*f* RESONANCE

Herein we proceed to demonstrate that the intensity of the response $\sigma(t)$ to a single realization of the stimulus does not decay if $\Phi_\infty \neq 0$. This demonstration, therefore, is valid for both the dynamical and the phenomenological approach, leading to a general result. Let us define with $p_t(\xi_S^i)$ the probability that at time t the variable ξ_S takes the value $i = \pm 1$ and with $p_t(\xi_S^i|\xi_P^j)$ the conditional probability for the occurrence, at time t, of a value $\xi_S = i = \pm 1$, given the occurrence of a value $\xi_P = j = \pm 1$. By definition, the nonvanishing Φ_∞ yields

$$C(t) \equiv \sum_{i,j=\pm 1} ij\ p_t\left(\xi_S^i|\xi_P^j\right)\ p_t\left(\xi_P^j\right) \rightarrow \varepsilon\Phi_\infty. \tag{54}$$

We note that for $t \rightarrow \infty$, on a time scale such that $\langle \xi_P(t)\rangle$, which decays as $\langle \xi_P(0)\rangle t^{1-\mu_P}$, is a second-order quantity $O(\varepsilon^2)$ we have that

$$p_t\left(\xi_P^j\right) = 1/2 + O(\varepsilon^2), \tag{55}$$

and

$$\Phi(t) = \Phi_\infty + O(\varepsilon^2). \tag{56}$$

Thus is due to Eq. (54) and to the symmetry of the considered dichotomous processes

$$p_t\left(\xi_S^i|\xi_P^j\right) \rightarrow \frac{1}{2} + i\ j\varepsilon\frac{\Phi_\infty}{2}. \tag{57}$$

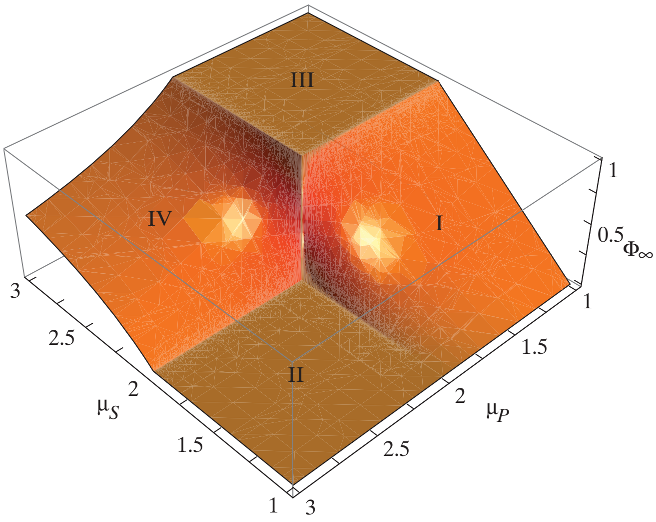

FIG. 2. (Color online) The asymptotic limit of $\Phi(t)$ is displayed for the different regimes of parameters for $1 < \mu_S < 3$ and $1 < \mu_P < 3$, in the dynamical approach.

In the same long-time scale, Eq. (57) yields

$$\langle \sigma(t)\rangle_\pm \equiv \sum_i p_t\left(\xi_S^i|\xi_P^{\pm 1}\right)\ i \simeq \pm\varepsilon\Phi_\infty, \tag{58}$$

where the subscript $\pm$ indicates the value of ξ_P at time t. Summing Eq. (58) over the two values of ξ_P gives a total average null response, as expected. But if the magnitude $|\sigma(t)|$ of the response to a single instance of the input $\xi_P(t)$ is considered instead, its total average is

$$\langle |\sigma(t)|\rangle = \frac{1}{2}\sum_\pm \langle |\sigma(t)|\rangle_\pm \gtrsim \frac{1}{2}\sum_\pm |\langle \sigma(t)\rangle_\pm| \simeq \varepsilon\Phi_\infty, \tag{59}$$

where an equality holds if terms of order $O(\varepsilon^2)$ are neglected. Thus when $\Phi_\infty > 0$, the response $\sigma(t)$ to a single instance of the input $\xi_P(t)$ *does not die out and remains proportional to the stimulus intensity*, no matter how large t becomes. Square III in both Figs. 1 and 2 is the plateau region of maximal cross correlation and response. Claims regarding the *death of linear response*, as in Refs. [19–28], are therefore appropriate only in relation to the vanishing correlation of square II. The total average response $\langle \sigma(t)\rangle$ always tends to vanish for $t \rightarrow \infty$ for reasons that do not imply a lack of response except in the case of square II.

The reason for the striking difference between the response to a harmonic perturbation and the response to a nonergodic stimulus is intimately related to the emergence of $1/f$ noise and to its spectrum described by Eq. (17), which assigns the weight $S(f)/L = 1/(fL)^{3-\mu_P}$ to the spectral component of frequency f of a nonergodic stimulus. As a consequence, the stimulus generates, in time, lower and lower frequencies f so as to keep $1/(fL)^{2-\mu_S}$ (i.e., the response intensity to frequency f [18]) finite, thereby yielding Eq. (59). The death of linear response [19–28] is caused by the fact that stimuli with fixed frequencies cannot cope with the decreasing frequency of the cascade of events of Eq. (20).

We have afforded a compelling proof that the intensity of the single realizations of $\sigma(t)$, with $\mu_S < 2$, does not decay if the perturbation $\xi_P(t)$ falls in the same complexity basin ($\mu_P < 2$). This is the phenomenon of *complexity management* which allows to define the right stimulus to obtain a response from a system or network producing nonergodic fluctuations. Now we argue that $1/f$ stimuli generate the maximum information transport by looking at the mutual information

$$I(t) = \sum_{i,j} p_t\left(\xi_P^j\right) p_t\left(\xi_S^i|\xi_P^j\right) \log\left[p_t\left(\xi_S^i|\xi_P^j\right)/p_t\left(\xi_S^i\right)\right]. \tag{60}$$

Using Eqs. (55), (56), and (57) it follows that

$$I(t \rightarrow \infty) \simeq \varepsilon^2\Phi_\infty^2, \tag{61}$$

and the information transmission rate is obtained by multiplying $I(t)$ by the input rate [37], given by $R_P(t)$. If $\mu_P > 2$, Fig. 1 shows that $\Phi_\infty < 1$. Although square III in Figs. 1 and 2, indicates that all stimuli with $\mu_P \leqslant 2$ induce maximal correlation, $\mu_P < 2$ corresponds to a stimulus with decaying events rate (input bits/s) $R_P(t)$. So even if a response is produced in this regime, the rate of information vanishes in time.

Only at the crucial condition $\mu_P = 2$, of ideal $1/f$ noise, does this algebraic decay become logarithmic, and, consequently, a steady and maximal information transmission rate is achieved. This is the phenomenon that we call $1/f$ resonance. The above considerations are valid for both the dynamical and phenomenological approaches, therefore we consider the condition of maximal information transmission achieved in the ideal $1/f$-noise condition, a fundamental property of $1/f$-noise renewal processes.

VI. CONCLUDING REMARKS

The growing interest for the dynamics of complex networks is shifting the attention of the researchers from the synchronization of two stochastic units [38] to the synchronization of a large number of units [39], an interesting phenomenon that is closely related to the very popular model of Kuramoto [40]. The single units of the processes of chaos synchronization are chaotic and do surprisingly synchronize while maintaining the erratic dynamics that they have in isolation. Although the single units of the Kuramoto model are regular, it is becoming increasingly evident that the emergence of a global synchronization is a condition independent of whether the single units are regular or stochastic. The single units of the work of Refs. [11,41] are Poisson processes and if one of them drove the other, they would obey the principle of aperiodic stochastic resonance [42]. If the two units are bidirectionally coupled they are expected to undergo a condition of perfect synchronization if the coupling is sufficiently intense. When the number of interacting units is very large a phase transition occurs from the noncooperative to the cooperative behavior [11,41]. It is important to stress that at criticality no permanent consensus is reached, and the mean value of the global field vanishes. Yet, this condition is strikingly different from the noncooperative condition. The whole network remains in the "yes" ("*no*") state for an extended time before making a transition to the "*no*" ("*yes*") state.

It is surprising that the phase-transition literature seems to have overlooked, with only a few exceptions [11,41,43], that the transitions from the "*yes*" ("*no*") to the "*no*" ("*yes*") state occurring at criticality are the "crucial" events defined in Sec. I. In other words, the time interval between two consecutive transitions is derived from a PDF that has the asymptotic time structure of Eq. (1) with a power index μ fitting the inequality condition of Eq. (2). Some authors [11,41] argued that $\mu = 1.5$ and others, [7], releasing the condition that all the units share the same Poisson rate, generate a global condition with crucial events characterized by $\mu < 2$, but significantly departing from the value $\mu = 1.5$. Note that the theoretical arguments of Ref. [44], yield the misleading impression that the crucial value of μ is a consequence of ordinary statistical physics.

According to some authors [45,46] the Kuramoto phenomenon can be defined as *inner synchronization*. A network of cooperating units located on the nodes of a complex network may reach inner synchronization with different values of the control parameter, depending on the network topology [47]. This is a subject of increasing interest with attractive applications to the dynamics of the human brain [45]. If we adopt this perspective, we can address the problem of information transmission from one to another complex network as a process of *outer synchronization*. This is an interesting issue, but the conditions to fulfill to realize outer synchronization are not yet clear [45].

An important result of this article is the discovery of a promising road to settle the problem of information transmission from one to another complex network. In fact, if the inner synchronization corresponds to a criticality condition and criticality generates crucial events with a power-law index μ fulfilling the inequality of Eq. (2), then a complex network at criticality is a generator of $1/f$ noise, with $S(f) \propto 1/f^{3-\mu}$. Thus the problem of information transmission from one to another complex network, in this regime, becomes equivalent to the phenomenon of $1/f$ resonance illustrated in this article. This is essentially the main result of earlier work [29]. The main conclusions of Ref. [29] are illustrated by Fig. 1, which is obtained using the phenomenological LRT.

What are the limits of this earlier result? The experiments [12,13] yield support to the dynamical rather than the phenomenological LRT, thereby generating doubt that the results of Ref. [27], although very attractive, may not completely reflect reality. It is important to stress that phenomenological LRT is a natural consequence of adopting the asymptotic time perspective replacing the waiting-times PDF $\psi(\tau)$ of Eq. (1) with $\psi(\tau) \propto 1/\tau^{\mu}$. This way of proceeding, although generating the elegant mathematics of fractional derivatives, has as an ultimate effect the misleading discovery of the death of linear response [19–24,28]. We do not adopt the asymptotic time perspective but the special form of Eq. (1). This is not a unique way of connecting the long-time to the short-time regime. However, whatever form we adopt we are convinced that there will be a parameter playing the same microscopc role of the parameter T of Eq. (1). It is reasonable to assume that an external perturbation may perturb either T or μ, or both of them. However, the perturbation of μ is incompatible with the assumption of a weak stimulus. In fact, μ is a consequence of the cooperation among the units of the network, and a perturbation may affect μ only if its strength is large enough to influence the interaction among the units of the network. Thus, an external weak perturbation can only have an effect on T, thereby making the dynamical LRT become the proper way to study the response of a complex network to a weak external stimulus, in accordance with the experimental results [12,13].

For these reasons, we can conclude that Fig. 2 is the original, and important, result of this paper. We hope that it may open the road to the dynamical solution of the problem of information transmission [48] from one to another complex network, a research topic that is still in its infancy.

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APPENDIX A

In this Appendix we record more details about the derivation of both the average response $\langle \sigma(t) \rangle$ and the input-output correlation function $\Phi(t)$ in the case of the phenomenological approach.

When we adopt the phenomenological theory, we obtain for the average response to external perturbation, in the nonstationary case [cf. Eq. (27)] the following asymptotic expression:

$$\frac{\langle\sigma(t)\rangle}{\epsilon} = \frac{k_1(\mu_S,\mu_P)}{t^{\mu_P-1}} + \frac{k_2(\mu_S,\mu_P)}{t^{\mu_S-1}}, \tag{A1}$$

where the coefficient of the first term is

$$k_1(\mu_S,\mu_P) = \frac{\Gamma(\mu_P-\mu_S)}{\Gamma(1-\mu_P)\Gamma(\mu_S)} - \frac{\Gamma(1-\mu_P+\mu_S)}{\Gamma(2-\mu_P)\Gamma(\mu_S)}, \tag{A2}$$

and the coefficient of the second term

$$k_2(\mu_S,\mu_P) = \frac{\Gamma(\mu_P-\mu_S)}{\Gamma(1-\mu_S)\Gamma(\mu_P)}. \tag{A3}$$

Note the logarithmic corrections corresponding to $\mu_S = \mu_P$, with

$$\frac{\langle\sigma(t)\rangle}{\epsilon} \approx \frac{\sin(\pi\mu_S)}{\pi}\frac{\log t}{t^{\mu_S-1}} + \frac{A(\mu_S)}{t^{\mu_S-1}}, \tag{A4}$$

and

$$A(\mu_S) = \frac{\sin(\pi\mu_S)}{\pi}\left[\frac{1}{\mu_S-1} - 2\gamma - 2\psi_L(\mu_S)\right] - \cos(\pi\mu_S), \tag{A5}$$

where γ is the Euler's constant and $\psi_L(z)$ the logarithm derivative of the Γ function. These predictions are qualitatively equivalent to the dynamical theory predictions with the assumption that the system S has been prepared in the very distant past.

The general expression of the correlation function in the phenomenological approach is given by Eq. (28).

Nonstationary case I: $\mu_S < 2,\mu_P < 2$.

In this range the asymptotic approximation for the function $R(t)$ defined in Eq. (11) for both S and P is

$$R(t) \simeq \frac{\sin\pi\mu}{\pi}t^{\mu-2}. \tag{A6}$$

The power-law properties of $\Psi_S(t)$, in the long-time limit, shift the dominant contribution to the integral of Eq. (28) to the range $t' \sim t$, this allows to adopt the approximation (A6) inside the integral. It follows

$$\Phi(t) \simeq \frac{\sin\pi\mu_S}{\pi}\frac{\sin\pi\mu_P}{\pi} \times \int_0^t \tau^{\mu_S-2}\Psi_S(t-\tau)d\tau \int_0^\tau x^{\mu_P-2}\Psi_P(t-x)dx. \tag{A7}$$

Using a generalized Newton binomial expansion of the power-law form of the functions $\Psi_P(t)$ we obtain the following expression:

$$\Phi(t) \simeq \frac{\sin\pi\mu_S}{\pi}\frac{\sin\pi\mu_P}{\pi}\sum_{n=0}^{\infty}\binom{1-\mu_P}{n}\frac{(-1)^n}{(t+T_P)^{n+\mu_P-1}} \int_0^t t'^{n+\mu_P+\mu_S-3}\Psi_S(t-t')dt', \tag{A8}$$

while the convolution with $\Psi_S(t)$ in the integral, leads to

$$\Phi(t) \simeq \frac{\sin\pi\mu_S}{\pi}\frac{\sin\pi\mu_P}{\pi}\sum_{n=0}^{\infty}\binom{1-\mu_P}{n}\frac{(-1)^n}{(t+T_P)^{n+\mu_P-1}} \times \frac{t^{n+\mu_P-1}}{n+\mu_P-1}\frac{\Gamma(n+\mu_P+\mu_S-2)\Gamma(2-\mu_S)}{\Gamma(n+\mu_P)}, \tag{A9}$$

where we have used the fact that for t large the main contribution to the integral comes from the range of values $x \lesssim \tau \lesssim t$ and therefore the asymptotic approximations for $R_S(t)$ an $R_P(t)$ are justified. The limit for $t \to \infty$ cancels the time dependence and leaves a sum which leads to the expression of Eq. (47) in the text.

Nonstationary case II: $\mu_S < 2,\mu_P > 2$.

In this case the function $R_P(t)$ tends asymptotically to a constant value and therefore it easy to see, following analogous procedure to the previous case, that the correlation tends to zero.

Stationary case I: $\mu_S > 2,\mu_P < 2$.

As mentioned in the text, from Eq. (30) it is enough to show that the third term on the right-hand side is zero. Such terms read as

$$\frac{\sin\pi\mu_P}{\pi}\int_0^t \tilde{\Psi}_S(t-t')dt't'^{\mu_P-2}\Psi_P(t-t'). \tag{A10}$$

Using a generalized binomial expansion for both $\tilde{\Psi}_S(t)$ and $\Psi_P(t)$ we get

$$\Phi(t) \simeq \frac{\sin\pi\mu_P}{\pi}\sum_{n,m}\binom{1-\mu_P}{n}\binom{1-\mu_S}{m}\frac{(-1)^n}{(t+T_P)^{n+\mu_P-1}} \times \frac{(-1)^m}{(t+T_S)^{m+\mu_S-2}}\int_0^t t'^{n+m+\mu_P-2}dt'. \tag{A11}$$

Carrying out the integration shows that the leading term in t vanishes for $t \to \infty$ Such a demonstration is valid also for the dynamical case in the same range of parameters [see Eqs. (51) and (52)].

APPENDIX B

This Appendix is devoted to detailed calculations involved the evaluation of the asymptotic limits of the function $\Phi(t)$ and $\langle\sigma(t)\rangle$ in the dynamic approach.

Herein only the parameter range $\mu_S < 2,\mu_P < 2$ is considered, the other ranges being analyzed in detail in the main text and in Appendix A.

$\Phi(t)$ is given by the following integral:

$$\Phi(t) = \int_0^t \psi_S(t,t')\Psi_P(t,t')dt', \tag{B1}$$

where $\psi_S(t,t')$ is defined in Eq. (10) and $\Psi_P(t,t')$ in Eq. (12), each with the appropriate labeling. $\Phi(t)$ can be decomposed as sum of four contributions

$$\Phi(t) = \sum_{i=1}^{4}\Phi_i(t), \tag{B2}$$

where

$$\Phi_1(t) = t\psi_S(t)\Psi_P(t), \tag{B3}$$

$$\begin{aligned}\Phi_2(t) &= \psi_S(t)\int_0^t\int_0^{t'} R_P(t'')\Psi_P(t-t'')dt''dt' \\ &= \psi_S(t)\int_0^t (t-t'')R_P(t'')\Psi_P(t-t'')dt'',\end{aligned} \tag{B4}$$

$$\begin{aligned}\Phi_3(t) &= \Psi_P(t)\int_0^t\int_0^{t'} R_S(t'')\psi_S(t-t'')dt''dt' \\ &= \Psi_P(t)\int_0^t (t-t'')R_S(t'')\psi_S(t-t'')dt'', \qquad \text{(B5)}\end{aligned}$$

$$\Phi_4(t) = \int_0^t\int_0^{t'}\int_0^{t'} R_S(t'')\psi_S(t-t'')R_P(\tau)\Psi_P(t-\tau)dt'dt''d\tau. \qquad \text{(B6)}$$

Here $\psi_S(t)$ and $\Psi_P(t)$ are defined in Eqs. (13) and (36). Using the methods of Ref. [49], Eqs. (B4), (B5) can be evaluated in the asymptotic limit to yield

$$\Phi_1(t) \approx \frac{c_1}{t^{\mu_S+\mu_P-2}}, \quad \Phi_2(t) \approx \frac{c_2}{t^{\mu_S-1}}, \quad \Phi_3(t) \approx \frac{c_3}{t^{\mu_P-1}}.$$

Since in the limit $t \to \infty$ the three contribute vanish, it is not important the evaluation of the constants c_1, c_2, c_3. To evaluate $\Phi_4(t)$ we first need an analytical expression for $R_S(t)$ and $R_P(t)$. It can be shown that [49]

$$R_S(t) \approx -\frac{\sin(\pi\mu_S)}{\pi T_S^{\mu_S-1}}\frac{1}{t^{2-\mu_S}}, \qquad \text{(B7)}$$

with the same expression, with the respective parameters being valid for $R_P(t)$. Then we have

$$\int_0^{t'} R_S(\tau)\psi_S(t-\tau)d\tau \approx -\frac{(t+T_S-t')^{1-\mu_S}t'^{\mu_S-1}\sin(\pi\mu_S)}{\pi(t+T_S)}.$$

On the other hand

$$\begin{aligned}&\int_0^{t'} R_P(\tau)\Psi_P(t-\tau)d\tau \\ &\quad = \frac{T_P^{\mu_P-1}}{(t+T_P)^{\mu_P-1}}\sum_{n=0}^{\infty}\binom{1-\mu_P}{n}\frac{(-)^n}{(t+T_P)^n} \\ &\quad \times \int_0^{t'} R_P(\tau)\tau^n d\tau. \qquad \text{(B8)}\end{aligned}$$

Using for $R_P(t)$ the same approximation of $R_S(t)$ Eq. (B2), we have

$$\begin{aligned}&\int_0^{t'} R_P(\tau)\Psi_P(t-\tau)d\tau \\ &\quad \approx -\frac{\sin\pi\mu_P}{\pi(t+T_P)^{\mu_P-1}}\sum_{n=0}^{\infty}\binom{1-\mu_P}{n}\frac{(-)^n}{(t+T_P)^n}\frac{t'^{n+\mu_P-1}}{n+\mu_P-1}. \qquad \text{(B9)}\end{aligned}$$

The function $\Phi_4(t)$ is

$$\begin{aligned}\Phi_4(t) \approx{}& \frac{\sin\pi\mu_P\sin\pi\mu_S}{\pi^2(t+T_P)^{\mu_P-1}}\frac{t^{\mu_S+\mu_P-1}}{(t+T_S)^{\mu_S}}\sum_{n=0}^{\infty}\left(-\frac{t}{t+T_P}\right)^n \\ &\times\binom{1-\mu_P}{n}\frac{F\left(n+\mu_P+\mu_S-1,\mu_S-1,n+\mu_P+\mu_S,\frac{t}{t+T_S}\right)}{(n+\mu_P-1)(n+\mu_P+\mu_S-1)},\end{aligned}$$

where $F(a,b,c,z)$ is the hypergeometric function. Finally

$$\begin{aligned}\Phi_4(t) \approx{}& \frac{\sin\pi\mu_P\Gamma(\mu_P+\mu_S-1)(t+T_P)^{1-\mu_P}t^{\mu_S}}{(1-\mu_P)\Gamma(\mu_P+1)\Gamma(\mu_S-1)(t+T_S)^{\mu_S}t^{1-\mu_P}} \\ &\times F\left[\{\mu_P-1,\mu_P-1,\mu_P+\mu_S-1\},\{\mu_P,\mu_P+1\},\frac{t}{t+T_S}\right],\end{aligned}$$

where ${}_pF_q(\{a\},\{b\},z)$ is the generalized hypergeometric function. In the limit for $t \to \infty$

$$\begin{aligned}\Phi_\infty = {}& -\frac{\sin\pi\mu_P\Gamma(\mu_P+\mu_S-1)}{\pi(\mu_P-1)\Gamma(\mu_P+1)\Gamma(\mu_S-1)} \\ &\times F\left[\{\mu_P-1,\mu_P-1,\mu_P+\mu_S-1\},\{\mu_P,\mu_P+1\},1\right]. \qquad \text{(B10)}\end{aligned}$$

As far as the evaluation of the function $\langle\sigma(t)\rangle$ is concerned, in the dynamic approach it is given by Eq. (35)

$$\langle\sigma(t)\rangle = \epsilon\int_0^t \psi_S(t,t')\Psi_P(t')dt', \qquad \text{(B11)}$$

which is to be evaluated in the parameter range $\mu_S < 2, \mu_P < 2$. Let $P_S(t) = R_S(t) + \delta(t)$, Eq. (B11) reads

$$\langle\sigma(t)\rangle = \epsilon\int_0^t dt'\Psi_P(t')\int_0^{t'} dx P_S(x)\psi_S(t-x), \qquad \text{(B12)}$$

which, after a transformation on the integration domain, turns into

$$\langle\sigma(t)\rangle = \epsilon\int_0^t dx\psi_S(t-x)P_S(x)\int_x^t dt'\Psi_P(t'). \qquad \text{(B13)}$$

The long-time limit shifts the main weight in the integral on the terms such that $x \lesssim t$. Therefore the approximation Eq. (B7), valid also for $P_S(t)$ in the long-time limit, can be adopted. Inserting such approximation in Eq. (B13) leads to

$$\langle\sigma(t)\rangle \simeq -\epsilon\frac{\sin\pi\mu_S}{\pi T_S^{\mu_S-1}}\int_0^t dx\psi_S(t-x)x^{\mu_S-2}\int_x^t dt'\Psi_P(t'). \qquad \text{(B14)}$$

After getting rid of the integral in t' by direct integration of $\Psi_P(t')$, one is left with a simple convolution which is easy to analyze in Laplace transform. After some straightforward algebra, extracting the two asymptotic leading terms, leads to

$$\langle\sigma(t)\rangle \simeq k_1 t^{1-\mu_P} + k_2 t^{\mu_S-\mu_P-1}, \tag{B15}$$

that is, the asymptotic expression of Eq. (41), with the coefficients given by Eqs. (42) and (43).

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