# Percolation in networks composed of connectivity and dependency links

Amir Bashan, Roni Parshani, and Shlomo Havlin
*Department of Physics, Bar-Ilan University, Ramat Gan, Israel*
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Networks composed from both connectivity and dependency links were found to be more vulnerable compared to classical networks with only connectivity links. Their percolation transition is usually of a first order compared to the second-order transition found in classical networks. We analytically analyze the effect of different distributions of dependencies links on the robustness of networks. For a random Erdös-Rényi (ER) network with average degree $k$ that is divided into dependency clusters of size $s$, the fraction of nodes that belong to the giant component $P_\infty$ is given by $P_\infty = p^{s-1}[1 - \exp(-kpP_\infty)]^s$, where $1 - p$ is the initial fraction of removed nodes. Our general result coincides with the known Erdös-Rényi equation for random networks for $s = 1$. For networks with Poissonian distribution of dependency links we find that $P_\infty$ is given by $P_\infty = f_{k,p}(P_\infty)e^{(\langle s\rangle-1)[pf_{k,p}(P_\infty)-1]}$, where $f_{k,p}(P_\infty) \equiv 1 - \exp(-kpP_\infty)$ and $\langle s\rangle$ is the mean value of the size of dependency clusters. For networks with Gaussian distribution of dependency links we show how the average and width of the distribution affect the robustness of the networks.

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## I. INTRODUCTION

Many systems can be efficiently modeled using a network structure where the system entities are the network nodes and the relations between the entities are the network links [1–14]. However, many systems are also characterized by small subgroups in which the entities belonging to a group strongly depend on each other. We coin the relation between each of the two nodes in such a group as dependency links [15]. For example, consider a financial network: Each company has trading and sales connections with other companies (connectivity links). These connections enable the companies to interact with others and function together as a global financial market. In addition, companies that belong to the same owner strongly depend on one another (dependency links). If one company fails, the owner might not be able to finance the other companies, which will fail too. Another example is an online social network (Facebook or Twitter): Each individual communicates with his friends (connectivity links), thus forming a social network through which information and rumors can spread. However, many individuals will only participate in a social network if other individuals with common interests also participate in that social network, thereby forming dependency groups.

Previous studies focused on network models containing only a single type of link, either connectivity links [10,16–22] or dependency links [23–27]. The main feature of connectivity links is to enable nodes to function cooperatively as a network. A node can function as long as it is connected to the majority of the network and fails only when it becomes completely disconnected from the network. Thus, the connectedness is a global property that depends on the structure of the whole network. In contrast, the dependency links represent local relations in the sense that, when a node fails, his direct dependency neighbors also fail, independent of the structure of the network. A network model containing both connectivity and dependency links was first introduced for two interdependent networks [28,29].

A recent paper [15] introduced for the first time a single network model containing both connectivity and dependency links. In this network model the initial failure of nodes may trigger an iterative process of cascading failures that has a devastating effect on the network stability. The cascading failures are a result of the synergy between two different effects: (a) a percolation process governed by connectivity links and (b) the failure of an entire dependency group due to a failure of one member within the group. For a high density of dependency links the network disintegrates in the form of a first-order phase transition, while for a low density of dependency links the network disintegrates in a second-order transition.

However, the combined model presented in [15] was based on an unrealistic assumption that all dependency groups are of size 2, i.e., only a pair of nodes depends on each other. In reality, as the examples above suggest, groups of several elements may depend on each other.

In this paper we analyze both analytically and numerically the general case of a network with different sizes of dependency clusters, as illustrated in Fig. 1. We study networks with three different types of dependency groups: (a) fixed size $s$ dependency groups, (b) normally distributed sizes of dependency groups, and (c) Poisson distributed sizes of dependency groups. We find that, for random networks with an average degree $k$ that are divided into dependency groups (clusters) of size $s$, the fraction of nodes belonging to the giant component $P_\infty$ is given by $P_\infty = p^{s-1}[1 - \exp(-kpP_\infty)]^s$, where $1 - p$ is the initial fraction of removed nodes. The critical threshold $p_c$ below which the network collapses ($P_\infty = 0$) is given by Eq. (18). Our result for $s = 1$ (a node depends only on itself) coincides with the known Erdös-Rényi (ER) equation, $P_\infty = 1 - \exp(-kpP_\infty)$, for a network without dependency relations [18–20]. We also show that for $s \geqslant 2$ a process of cascading failures occurs and the percolation transition is of first order [30].

For normally distributed dependency groups with an average size $\langle s\rangle$ and width $\sigma$, we find that the system becomes

 

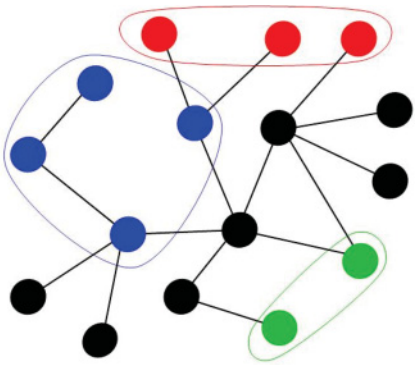

FIG. 1. (Color online) Connectivity network with dependency clusters. The edges represent connectivity relations, while the (blue, red, and green) groups surrounded by curves represent dependency relations between all the nodes of the same group (color). The dependency relations can be between very "$far$" nodes in the connectivity network. In the general case, the sizes of each dependency clusters follow a given distribution.

more stable (smaller $p_c$) for a broader size distribution. When $\sigma \to 0$, the results are the same as the case of fixed size dependency groups with $\langle s \rangle = s$. We also analyze both analytically and numerically the case of a Poisson distribution of dependency cluster sizes and obtain analytical equations for both $P_\infty(\langle s \rangle)$ and $p_c$ [Eqs. (27), (28), and (29)].

## II. GENERAL FORMALISM

When nodes fail in a network containing both connectivity links and dependency clusters, two different processes occur. (i) Connectivity links connected to these nodes fail, causing other nodes to disconnect from the network (percolation step). (ii) A failing node causes the failure of all the other nodes of its dependency cluster, even though they are still connected via connectivity links (dependency step). Thus, a node that fails in the percolation step leads to the failure of its entire dependency cluster, which in turn leads to a new percolation step, which further leads to a dependency step, and so on. Once the cascade process is triggered, it will only stop if nodes that fail in one step do not cause additional failure in the next step.

We start by presenting the formalism describing the iterative process of cascading failures. On each step we apply the two processes, a percolation process followed by the removal of relevant dependency groups. Before each percolation stage the accumulated cascades are described as equivalent to a single random removal, $1 - \psi_n^p$, where $\psi_n^p$ is the effective functional part of the network prior to the percolation stage at step $n$. Similarly, before each dependency stage the accumulated cascades are equivalent to a single random removal, $1 - \psi_n^D$, where $\psi_n^D$ is the effective functional part of the network prior to the dependency stage at step $n$.

We define two functions, $g_p(T)$ and $g_D(T)$, to evaluate the effect of the percolation process and the effect of the dependency clusters on the network, respectively. After random removal of $(1 - T)$ of the nodes, the size of the giant component is given by $g_p(T)$, and equivalently, the part of the network that is not dependent on the removed nodes (and, thus, remains functional) is given by $g_D(T)$. Thus, when applying the percolation process at stage $n$ on a network of size $\psi_n^p$, the remaining giant component consists of a fraction $g_p(\psi_n^p)$, which is a fraction $\phi_n^p = \psi_n^p g_p(\psi_n^p)$ from the original network. Similarly, applying the dependency process at stage $n$ on a network of size $\psi_n^D$ results in a remaining functional nodes consisting of a fraction $g_D(\psi_n^D)$, which is a fraction $\phi_n^D = \psi_n^D g_D(\psi_n^D)$ from the original network.

The iterative process is initiated by the removal of a fraction $1 - p$ of the network nodes. The remaining part of the network is $\psi_1^p \equiv p$. This initial removal will cause additional nodes to disconnect from the giant cluster due to the percolation process. The fraction of nodes that remain functional after the percolation process is $\phi_1^p = \psi_1^p g_P(\psi_1^p)$. Before the dependency step we describe the accumulated cascades of the previous steps. The fractions of nodes that fail due to the initial removal and due to first percolation step are $1 - \psi_1^p$ and $\psi_1^p - \phi_1^p$, respectively, and the accumulated cascades are equivalent to a single random removal of $1 - \phi_1^p = (1 - \psi_1^p) + (\psi_1^p - \phi_1^p)$. Thus, we denote the remaining functional part before the dependency step as $\psi_1^D \equiv \phi_1^p$. Each node from the nonfunctional part will cause all the other nodes of its dependency cluster to also fail (dependency process). The remaining functional part of the network after the dependency step is $\phi_1^D = \psi_1^D g_D(\psi_1^D)$.

Let us now calculate the accumulated failure up to this step. The sum of the previous steps, the initial removal of $(1 - p)$, the removal due to the percolation step $(\psi_1^p - \phi_1^p)$, and the removal due to the dependency step $(\psi_1^D - \phi_1^D)$, is equivalent to a single random removal of $[1 - pg^D(\psi_1^D)]$ from the original network ([15]). After such removal the remaining part of the network before the second percolation step is $\psi_2^p \equiv pg^D(\psi_1^D)$, and the size of the giant cluster is then $\phi_2^p = \psi_2^p g_p(\psi_2^p)$.

Following this approach, we can construct the sequences $\psi_n^p$ and $\psi_n^D$ of the remaining fraction of nodes and the sequences $\phi_n^p$ and $\phi_n^D$ of functional nodes at each stage of the cascade of failures. The general form is given by

$$\begin{aligned}
&\psi_1^p \equiv p, \quad \phi_1^p = \psi_1^p g_p\left(\psi_1^p\right), \\
&\psi_1^D = g_p\left(\psi_1^p\right)p, \quad \phi_1^D = \psi_1^D g_D\left(\psi_1^D\right), \\
&\psi_2^p = g_D\left(\psi_1^D\right)p, \quad \phi_2^p = \psi_2^p g_p\left(\psi_2^p\right), \\
&\qquad\vdots \\
&\psi_n^p = g_D\left(\psi_{n-1}^D\right)p, \quad \phi_n^p = \psi_n^p g_p\left(\psi_n^p\right), \\
&\psi_n^D = g_p\left(\psi_n^p\right)p, \quad \phi_n^D = \psi_n^D g_D\left(\psi_n^D\right).
\end{aligned} \tag{1}$$

To determine the state of the system at the end of the cascade process, we look at $\psi_m^p$ and $\psi_m^D$ at the limit of $m \to \infty$. This limit must satisfy the equation $\psi_m^p = \psi_{m+1}^p$ (or $\psi_m^D = \psi_{m+1}^D$) since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step $m$ and $m + 1$ are equal. Thus, at steady state the system satisfies the set of two equations

$$\begin{aligned}
\psi_\infty^p &= g_D\left(\psi_\infty^D\right)p, \\
\psi_\infty^D &= g_p\left(\psi_\infty^p\right)p.
\end{aligned} \tag{2}$$

Denoting $x \equiv \psi_\infty^D$ and $y \equiv \psi_\infty^p$, we arrive at a system of two equations with two unknowns: $x = pg_p(y)$, $y = pg_D(x)$, which can be reduced to

$$x = pg_p[pg_D(x)]. \tag{3}$$

Solving equation (3), we obtain the size of the network at the end of a cascade initiated by random removal of $1 - p$ of the nodes.

Next, we calculate explicitly $g_D(T)$ and $g_p(T)$. In the general case, each node belongs to a dependency group of size $s$ with a probability $q(s)$ so that the number of groups of size $s$ is equal to $q(s)N/s$. Since, after random removal of $1 - T$ of the nodes, each group of size $s$ remains functional with a probability $T^s$, the total number of nodes that remain functional is given by $\sum_{s=1}^{\infty} q(s)NT^s$. Thus, we define the function $g_D(T)$ as the fraction of nodes that remain functional out of the $TN$ nodes that were not removed,

$$g_D(T) \equiv \sum_{s=1}^{\infty} q(s)T^{s-1}. \tag{4}$$

Analogous to $g_D(T)$, $g_p(T)$ is defined as the fraction of nodes belonging to the giant cluster of the connectivity network after random removal of $1 - T$ of the nodes. The percolation process can be solved analytically by using the apparatus of generating functions. As in Refs. [31–33], we will introduce the generating function of the degree distributions $G_0(\xi) = \sum_k P(k)\xi^k$. Analogously, we will introduce the generating function of the underlining branching processes, $G_1(\xi) = G_0'(\xi)/G_0'(1)$. Random removal of fraction $1 - T$ of nodes will change the degree distribution of the remaining nodes, so the generating function of the new distribution is equal to the generating function of the original distribution with the argument equal to $1 - T(1 - \xi)$ [31]. The fraction of nodes that belong to the giant component after the removal of $1 - T$ nodes is [32,33]

$$g_p(T) = 1 - G_0[1 - T(1 - u)], \tag{5}$$

where $u = u(T)$ satisfies the self-consistency relation

$$u = G_1[1 - T(1 - u)]. \tag{6}$$

## III. ER NETWORKS

The formalism presented in Sec. II is general for a random network having any degree distribution. In the case of an ER network, whose degrees are Poisson distributed [18–20], the problem can be solved explicitly. Suppose that the average degree of the network is $k$. Then, $G_1(\xi) = G_0(\xi) = \exp[k(\xi - 1)]$. Thus, $g_p(x) = 1 - u$, and therefore, Eq. (3) becomes

$$x = p[1 - u], \tag{7}$$

where $u$ is defined according to (6) by

$$u = \exp[-kpg_D(x)(1 - u)]. \tag{8}$$

Using the definition of $g_D(x)$, Eq. (4), together with Eq. (8), we get the general solution for the steady state of the network at the end of the cascade failure process,

$$u = e^{-k\sum_{s=1}^{\infty} q(s)p^s(1-u)^s}. \tag{9}$$

In order to present $u$, obtained from Eq. (9), in terms of $P_\infty$, recall that, at steady state, the size of the giant cluster $\phi_\infty \equiv \phi_n^p = \phi_n^D$, and, according to (1),

$$\phi_\infty = xg_D(x) = x\sum_{s=1}^{\infty} q(s)x^{s-1} = \sum_{s=1}^{\infty} q(s)x^s. \tag{10}$$

Since $P_\infty \equiv \phi_\infty/p$, we get the relation

$$P_\infty p = \sum_{s=1}^{\infty} q(s)p^s(1 - u)^s, \tag{11}$$

and using (9), a simple equation for $P_\infty$ is obtained,

$$P_\infty = -\frac{\ln u}{kp}, \tag{12}$$

where $u$ is the solution of Eq. (9).Up to this point, we obtained the size of the network at each step of the cascade process, Eq. (1), and, in particular, its size, $P_\infty$, Eqs. (9) and (12), at the end of the cascade for the general case of a given distribution $q(s)$ of sizes of dependency clusters.

## IV. FIXED SIZE OF DEPENDENCY CLUSTERS

Using the general solution described above, we analyze the case of a fixed size $s$ of dependency clusters. In particular, we find the size of the giant component and the critical fraction of the network, $1 - p_c$, that, if removed, leads to complete fragmentation of the network. In this case, $g_D$, given in (4), becomes $g_D(T) = T^{s-1}$, and Eqs. (9) and (11) become, respectively,

$$u = e^{-kp^s(1-u)^s}, \tag{13}$$

$$P_\infty = p^{s-1}(1 - u)^s, \tag{14}$$

which can be combined into a single equation:

$$P_\infty = p^{s-1}(1 - e^{-kpP_\infty})^s. \tag{15}$$

Eq. (15) coincides for $s = 1$ (a node depends only on itself) with the known Erdös-Rényi equation [18–20], $P_\infty = 1 - \exp(-kpP_\infty)$, for a network without dependency relations. Moreover, for $s = 2$, Eq. (15) yields the result obtained in [15] for the case of dependency pairs.

Figure 2 shows the size of the giant cluster $P_\infty$ versus the fraction of nodes $p$ remaining after an initial random removal of $1 - p$ for the case of an ER network with fixed size of dependency clusters $s$. The case of $s = 1$, each node depend only on itself, is the regular second-order percolation transition. For any $s \geqslant 2$, a first-order phase transition characterizes the percolation process. Both the regular and the new first-order percolation obey Eq. (15).

Finding the transition point via simulations is always a difficult task that requires high precision. In the case of $s \geqslant 2$, where first-order transition occurs, we are able to calculate the transition point with good precision by identifying the special behavior characterizing the number of iterations (NOI) in the cascading process [15]. This number sharply drops as the distance from the transition point is increased. Thus, plotting the NOI as a function of $p$ provides a useful and precise

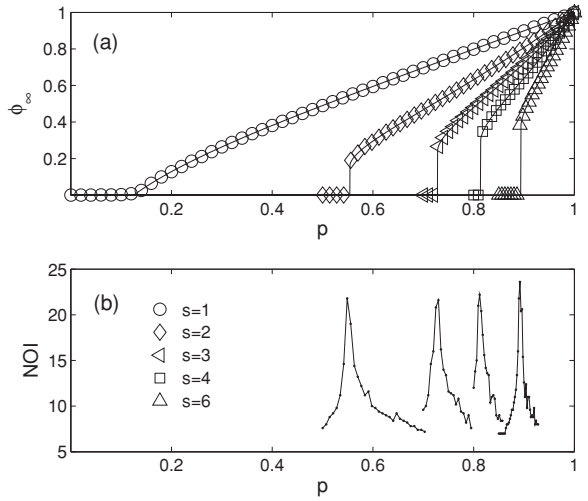


FIG. 2. (a) The size of the giant cluster, $\phi_\infty \equiv P_\infty p$, vs $p$, the fraction of nodes that remain after random removal, for ER networks ($k = 8$) for different fixed sizes of dependency clusters $s$. The symbols represent simulation results of systems of 50000 nodes, and the solid lines show the theoretical predictions. For the case of $s = 1$, there are no dependency clusters, and the regular percolation process leads to second-order phase transition. For $s \geqslant 2$, a first-order phase transition characterizes the percolation process represented by discontinuity of $P_\infty$ at $p_c$. Both the regular and the new first-order percolation obey Eq. (15). (b) The number of iterative failures (NOI) sharply increases when approaching the critical threshold $p_c$ for the first-order transitions, and thus, they represent a useful method for identifying accurately the value of $p_c$ [15]. Each curve is maximal as its related curve in (a) approaches the critical threshold from both sides.

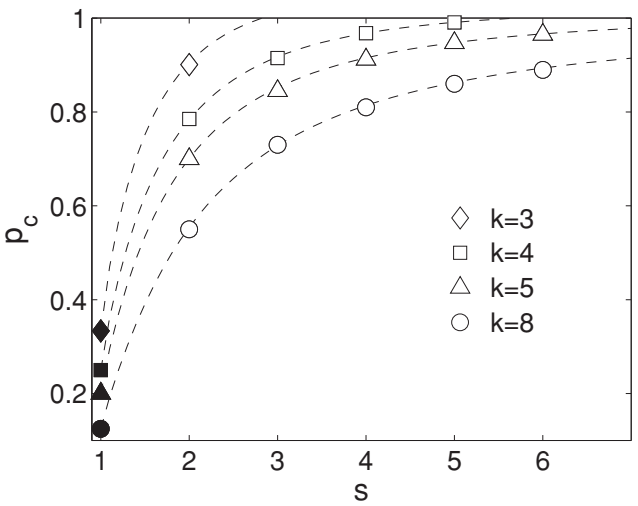


FIG. 3. Theory [Eq. (18), dashed lines] and simulation results (symbols) are compared for the values of $p_c$ for ER networks with different average degree $k$ and different fixed sizes of dependency clusters $s$. The solid symbols, for $s = 1$, represent the known Erdös-Rényi second-order phase transition threshold for a network without dependency relations, while the open symbols represent first-order phase transition thresholds.

method for identifying the transition point $p_c$ in the first-order region. Figure 2(b) presents the NOI of the simulation results of Fig. 2(a). The transition point $p_c$ can easily be identified by the sharp peak characterizing the percolation threshold. The results shown in Fig. 2(b) are in excellent agreement with theory.

Next, we find analytically the percolation threshold $p_c$ for the case of a fixed size of dependency clusters. Equation (13), which is the condition for a steady state, has a trivial solution at $u = 1$, which corresponds, by (12), to a complete fragmentation of the network. For large $p$ there is another solution of $0 < u < 1$, corresponding to a finite fraction of the network. Therefore, the critical case corresponds to satisfying both the tangential condition for Eq. (13),

$$1 = u[kp^s s(1-u)^{s-1}], \tag{16}$$

as well as Eq. (13). Thus, combining Eqs. (16) and (13), we get a closed-form expression for the critical value $u_c$,

$$u_c = \exp\left(\frac{u_c - 1}{s u_c}\right). \tag{17}$$

Once $u_c$ is known, we obtain $p_c$ by substituting it into Eq. (16):

$$p_c = [k s u_c (1-u_c)^{s-1}]^{-1/s}. \tag{18}$$

For $s = 1$ we obtain the known result $p_c = 1/k$ of Erdös-Rényi [18–20]. Substituting $s = 2$ in Eqs. (17) and (18), one obtains ${p_c}^2 = 1/[2ku_c(1-u_c)]$, which coincides with the exact result found in [15]. In Fig. 3 we plot the values of $p_c$ as a function of $s$ for several $k$ values. Note the dramatic effect of the dependencies on the vulnerability of the system. Even for high values of $k$, $p_c$ approach rapidly to 1, even for relatively small $s$ values.

Next, we show that the difference between the continuous second-order percolation transition and the first-order transition is characterized not only by the abrupt jump in the size of the giant cluster at the critical point $p_c$ but also by a difference in the scaling behavior of the giant component $\phi_\infty$ near $p_c$. The scaling near $p_c$ is defined by the exponent $\beta$,

$$\phi_\infty(p) - \phi_\infty(p_c) \sim (p - p_c)^\beta. \tag{19}$$

Eq. (15) can be written in terms of $\phi_\infty (\equiv P_\infty p)$, the size of the giant component,

$$\phi_\infty = p^s (1 - e^{-k\phi_\infty})^s. \tag{20}$$

For the case of $s = 1$ (ER), $\phi_\infty(p) - \phi_\infty(p_c)$ changes (as well known [17]) linearly with $p - p_c$ and $\beta = 1$. For $s \geqslant 2$, we calculate the behavior close to (and above) the critical point,

$$p \equiv p_c + \epsilon,$$
$$\phi_\infty \equiv \phi_\infty^c + \delta,$$

when $\delta, \epsilon \to 0$ and $\phi_\infty^c \equiv \phi_\infty(p_c)$. For this case, Eq. (20) can be written as

$$p_c + \epsilon = \frac{\left(\phi_\infty^c + \delta\right)^{1/s}}{1 - e^{-k(\phi_\infty^c + \delta)}} = A[1 + C_1\delta + C_2\delta^2 + \cdots], \tag{21}$$

where $A \equiv \frac{(\phi_\infty^c)^{1/s}}{1 - e^{-k\phi_\infty^c}} = p_c$ and the linear coefficient is given by $C_1 \equiv (\frac{1}{s\phi_\infty^c} - \frac{k}{e^{k\phi_\infty^c} - 1})$. However, using Eqs. (12) and (17),

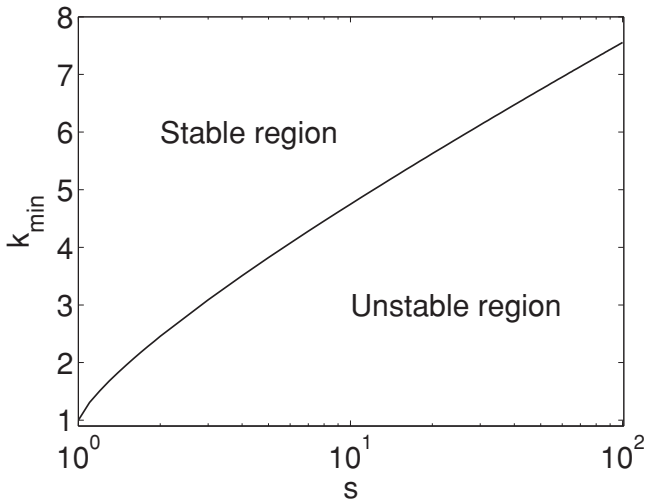


FIG. 4. The minimum averaged degree $k_{\min}$ as a function of the (fixed) size of dependency clusters $s$. The solid line shows the theoretical results, obtained from Eqs. (17) and (22). The region of $s$ and $k$ values below the line represents an unstable network that will collapse after any single node failure. For $k$ and $s$ values above the line, the network is stable, and there exists $p_c < 1$.

we obtain that $C_1 = 0$, so near the critical point, $\epsilon \sim \delta^2$ and $\delta \sim \epsilon^{1/2}$. Thus, the scaling behavior of the giant component near the first-order transition, Eq. (19), is characterized by the critical exponent $\beta = 1/2$.

For a fixed $s$, when $k$ is smaller than a critical number $k_{\min}(s)$, $p_c \geqslant 1$, meaning that for $k < k_{\min}(s)$, the network will collapse for any finite number of nodes failure. From Eq. (18) we get the minimum of $k$ as a function of $s$,

$$k_{\min}(s) = [su_c(1-u_c)^{s-1}]^{-1}. \tag{22}$$

Fig. 4 shows the minimum averaged degree $k_{\min}$ as a function of the size of dependency clusters $s$.

## V. GAUSSIAN DISTRIBUTION OF DEPENDENCY GROUPS

Using the general solution, given in Eqs. (9) and (11), one can calculate $P_\infty$ after initial removal of $1-p$ of the nodes and get $p_c$ for every distribution of sizes of dependency clusters $q(s)$. Here we calculate $p_c$ in the case of a normal Gaussian distribution for the size of the dependency clusters with average size $\langle s\rangle$ and variance $\sigma^2$. In this case, the probability of a random node to belong to a dependency cluster of size $s$ is given by

$$q(s) = \begin{cases} Ae^{-(s-\langle s\rangle)^2/2\sigma^2} & 1 < s < 2\langle s\rangle - 1, \\ 0 & \text{otherwise}, \end{cases} \tag{23}$$

where $A$ is a normalization constant. Note that $q(s) \neq 0$ only for $1 < s < 2\langle s\rangle - 1$ in order to have a symmetrical distribution around $\langle s\rangle$.

This case generalizes our results of dependency clusters (for $\sigma \to 0$) having a single size $s$ and shows the deviations from these results as the distribution becomes broader. In this case there are nodes that belong to dependency clusters that are larger than $\langle s\rangle$ and thus have higher probability of becoming nonfunctional, while the same number of nodes

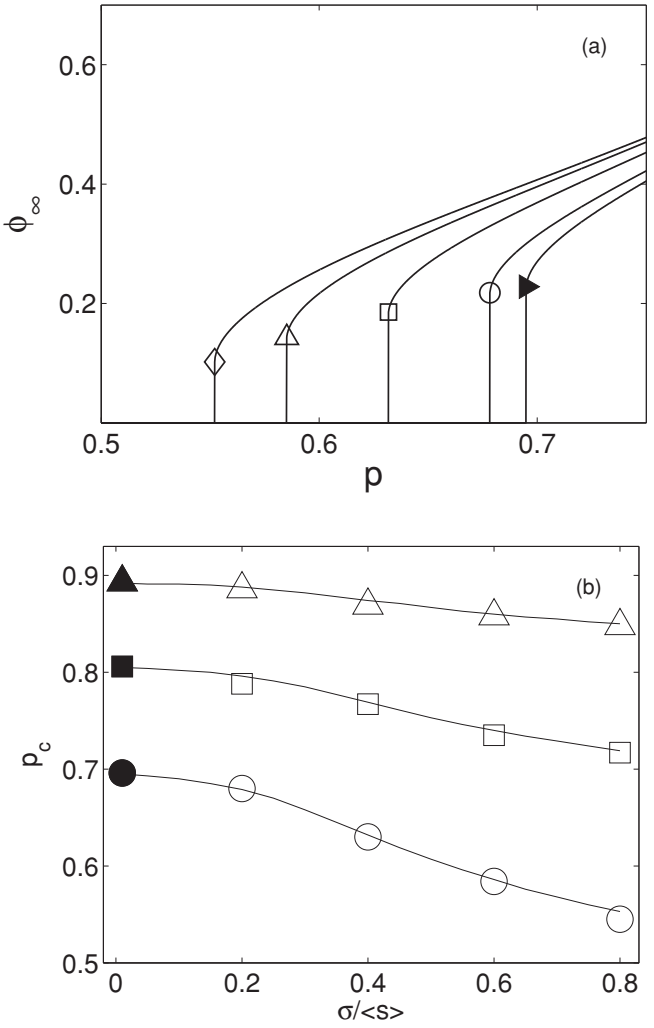


FIG. 5. (a) The size of the giant cluster, $\phi_\infty \equiv P_\infty p$, vs $p$ (solid lines) and $\phi_\infty(p_c)$ (symbols) for ER networks with average degree $k = 15$ and dependency clusters normally distributed around averaged size $\langle s\rangle = 4$. The different curves represent different standard deviations $\sigma$ [$\sigma = 0$ (solid triangle), $\sigma = 0.8$ (circle), $\sigma = 1.6$ (square), $\sigma = 2.4$ (open triangle), and $\sigma = 3.2$ (diamond)]. (b) Theory (solid lines) and simulation (symbols) values of $p_c$ for ER networks with average degree $k = 15$ and dependency clusters normally distributed around averaged size $\langle s\rangle$ vs the width of the distribution $\sigma$. The three curves represent different values of $\langle s\rangle$: $\langle s\rangle = 4$ (circles), $\langle s\rangle = 6$ (squares), and $\langle s\rangle = 10$ (triangles). For $\sigma \to 0$ (solid symbols) the Gaussian distribution becomes a $\delta$ function with fixed size of dependency clusters, and thus, $p_c$ is identical to those obtained by Eqs. (17) and (18) for groups of single size $s$.

belong to dependency clusters that are smaller than $\langle s\rangle$ have a smaller probability of becoming nonfunctional. We find that the first-order transition threshold decreases as the distribution of dependency groups becomes broader, and thus, the network is more stable, as shown in Fig. 5(a). However, this effect becomes weaker for larger $\langle s\rangle$, as shown in Fig. 5(b).

## VI. POISSON DISTRIBUTION OF DEPENDENCY GROUPS

Next, we study the case of a Poisson distribution of dependency cluster sizes. In this case, the probability that a

random node depends on $s'$ other nodes (and thus the size of the cluster $s$ equal to $s'+1$) is

$$p(s') = \frac{\lambda^{s'} e^{-\lambda}}{s'!} \equiv \lim_{n\to\infty} \binom{n}{s'} \left(\frac{\lambda}{n}\right)^{s'} \left(1-\frac{\lambda}{n}\right)^{n-s'} \quad (24)$$

when $\lambda \equiv \langle s\rangle - 1$ is the average number of other nodes that depend on a random node. The dependency process can be calculated, using Eq. (4) and considering that $q(s) = p(s-1)$, as

$$g_D(T) = \sum_{s=1}^{\infty} p(s-1)T^{s-1} = \sum_{s'=0}^{\infty} p(s')T^{s'}, \quad (25)$$

with $s' \equiv s-1$. Using (24), we obtain

$$\begin{aligned} g_D(T) &= \sum_{s'=0}^{\infty} \lim_{n\to\infty} \binom{n}{s'} \left(\frac{\lambda}{n}\right)^{s'} \left(1-\frac{\lambda}{n}\right)^{n-s'} T^{s'} \\ &= \sum_{s'=0}^{\infty} \binom{n}{s'} \left(\frac{\lambda}{n}T\right)^{s'} \left(1-\frac{\lambda}{n}\right)^{n-s'} \\ &= \lim_{n\to\infty} \left[\frac{\lambda(T-1)}{n} + 1\right]^n \\ &= e^{\lambda(T-1)} = e^{(\langle s\rangle -1)(T-1)}. \end{aligned}$$

Following (7) and (8), we get an equation for $u$ for the case of Poisson distribution:

$$\ln u = -kp(1-u)e^{(\langle s\rangle -1)[p(1-u)-1]}. \quad (26)$$

Finally, using (12), $P_\infty$ is obtained in a closed form,

$$P_\infty = f_{k,p}(P_\infty)e^{(\langle s\rangle -1)[pf_{k,p}(P_\infty)-1]}, \quad (27)$$

where $f_{k,p}(P_\infty) \equiv 1 - \exp(-kpP_\infty)$.

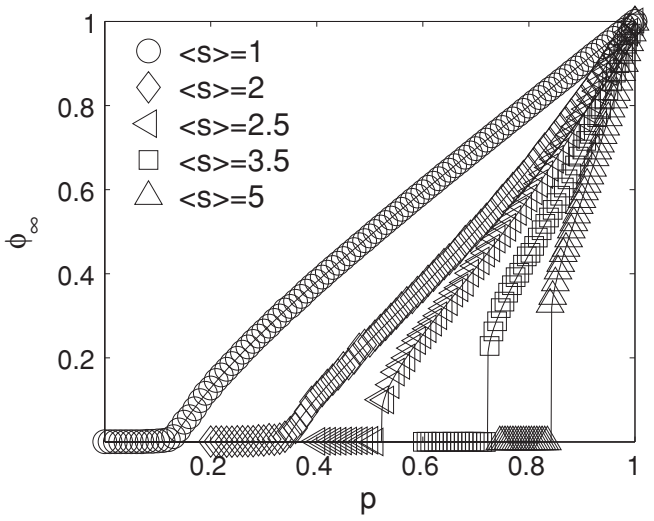


FIG. 6. The size of the giant cluster, $\phi_\infty \equiv P_\infty p$, vs $p$ for ER networks ($k=8$) and Poisson distribution of dependency clusters with different average sizes $\langle s\rangle$. The symbols represent simulation results of systems of 50 000 nodes, and the solid lines show the theoretical predictions. For $\langle s\rangle = 1$ and 2 the network undergoes a second-order transition, while for $\langle s\rangle = 2.5, 3.5$, and 5 the network undergoes a first-order transition [see Fig. 7, where the exact transition point from first- to second-order transition is shown] .

Fig. 6 shows the size of the giant cluster at steady state versus $p$ for different values of $\langle s\rangle$. For small $\langle s\rangle$, many nodes do not depend on other nodes, so the effect of the dependency clusters is rather weak, and thus, the percolation transition

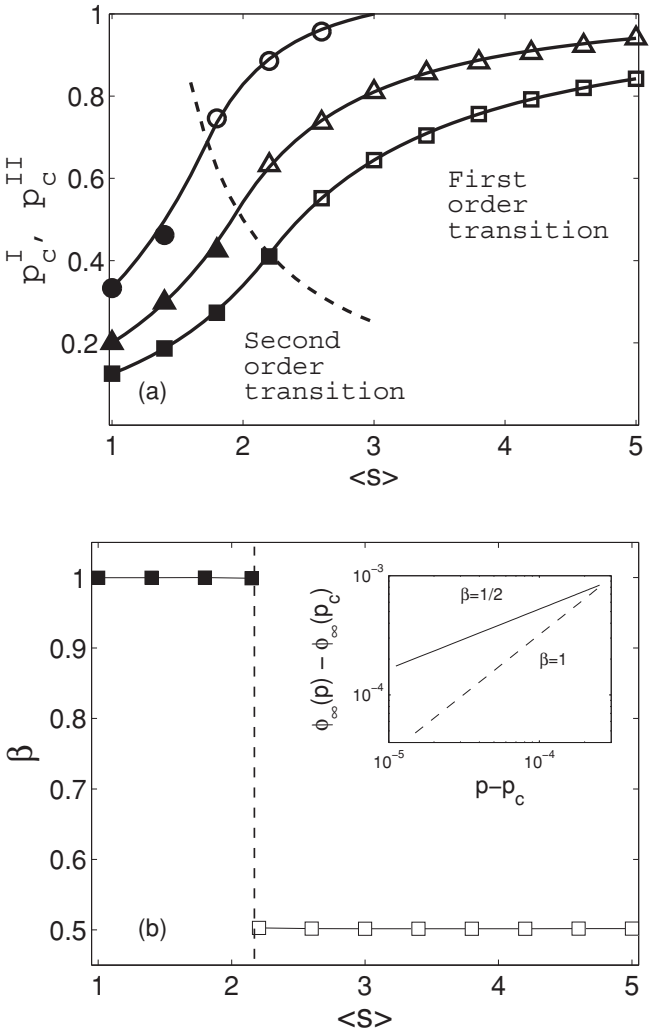


FIG. 7. (a) Theory (lines) and simulations (symbols) are compared for the values of $p_c^I(\langle s\rangle)$ and $p_c^{II}(\langle s\rangle)$ for ER networks with different average degree $k$ (circles for $k=3$, triangles for $k=5$, and squares for $k=8$). For $\langle s\rangle > \langle s\rangle_c$ (dashed line) the network undergoes a first-order transition. The theoretical values of the transition point $p_c^I(\langle s\rangle)$ that are calculated according to Eq. (29) are compared with simulations (open symbols) performed using the NOI method (explained in text). For $\langle s\rangle < \langle s\rangle_c$ the network undergoes a second-order transition, and the theoretical values of the transition point $p_c^{II}(\langle s\rangle)$ that are calculated according to Eq. (28) are compared with simulations (solid symbols) performed using the second-largest cluster method. The dashed line separating the first and second orders is obtained according to Eq. (32). (b) The size of the giant cluster $\phi_\infty$ above the critical point $p_c$ is described by $\phi_\infty(p) - \phi_\infty(p_c) \sim (p-p_c)^\beta$, as shown in the inset. The critical exponent $\beta$ above the transition point $p_c$ is plotted vs the average size of the dependency clusters $\langle s\rangle$ for a network with $k=8$ [compare to (a); squares]. For the region of the second-order transition, $\langle s\rangle < \langle s\rangle_c$, we find $\beta = 1$ ($\langle s\rangle_c$ is marked by the vertical dashed line), while for the region of the first-order transition, $\langle s\rangle > \langle s\rangle_c$, $\beta = 1/2$.

is of second order, meaning that $P_\infty$ continuously decreases from a finite value to zero at a specific transition point $p_c^{II}$. However, for large $\langle s\rangle$ the network undergoes a first-order transition, meaning that the size of the giant cluster abruptly jumps discontinuously from finite size for $p > p_c^I$ to zero for $p < p_c^I$. Such a network is qualitatively more vulnerable than a network that undergoes a second-order transition due to the cascading failures, leading to high vulnerability of the network around $p_c^I$.

Next, we find explicitly, by analyzing Eq. (26), the first-order transition point $p_c^I$ and the second-order transition point $p_c^{II}$. Equation (26) has a trivial solution for $u = 1$, which means that the network is completely fragmented. The second-order transition point $p_c^{II}$ corresponds to the solution of Eq. (26), where $u \to 1$. This condition gives $p_c^{II}$,

$$p_c^{II} = \frac{e^{\langle s\rangle - 1}}{k}. \tag{28}$$

Note that for the case of $\langle s\rangle = 1$, meaning that all the nodes are not dependent, $p_c^{II} = 1/k$ as for regular Erdös-Rényi networks.

The first-order transition point $p_c^I$ corresponds to the tangential intersection of the left and right terms of Eq. (26), meaning that the derivatives of both with respect to $u$ are equal. This yields

$$p_c^I(\langle s\rangle - 1) = \frac{1}{u-1} - \frac{1}{u\ln u}, \tag{29}$$

where $u$ is the solution of Eq. (26).

Fig. 7(a) shows $p_c^I$ and $p_c^{II}$ versus the average size of the dependency clusters $\langle s\rangle$. At critical values $p = p_c^*$ and $\langle s\rangle = \langle s\rangle_c$ the phase transition changes from first order to second order. The values of $p_c^*$ and $\langle s\rangle_c$ are obtained when the conditions for both the first- and second-order transitions are satisfied simultaneously. Applying both conditions, we obtain

$$2(\langle s\rangle_c - 1) = ke^{-(\langle s\rangle_c - 1)}, \tag{30}$$

$$k = \frac{1}{p_c^*}e^{1/2p_c^*}. \tag{31}$$

For a given ER network with average degree $k$, Eq. (30) provides the critical average size of dependency clusters $\langle s\rangle_c$. A network with $\langle s\rangle < \langle s\rangle_c$ undergoes a second-order phase transition, while for $\langle s\rangle > \langle s\rangle_c$ the network undergoes a first-order transition. Therefore, $p_c^*$, obtained from Eq. (31) for the case of $\langle s\rangle = \langle s\rangle_c$, characterizes the stability of a network with maximal $\langle s\rangle$ under the constraint of undergoing second-order transition. The critical case, shown in Fig. 7(a), of transition from first-order to second-order transition (dashed line) is given by

$$p_c^* = \frac{1}{2(\langle s\rangle_c - 1)}. \tag{32}$$

As shown in Figs. 6 and 7, increasing the size of the dependency clusters increases the network vulnerability, and $p_c^I$ becomes larger. A critical average size of dependency clusters $\langle s\rangle_{\max}$ corresponds to $p_c^I = 1$, meaning that the network completely fragments as a result of removal of even a single node. Such a network can be regarded as unstable. The value of $\langle s\rangle_{\max}$ is given by Eqs. (29) and (26) under the condition of $p_c^I = 1$.

Thus, the stability of a random network with dependency clusters with average size $\langle s\rangle$ with Poissonian distribution can be summarized as

$$\langle s\rangle < \langle s\rangle_c,$$
$$\langle s\rangle_c < \langle s\rangle < \langle s\rangle_{\max},$$
$$\langle s\rangle \geqslant \langle s\rangle_{\max},$$

for second-order transition, first-order transition, and an unstable network, respectively.

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