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Traffic flow in the Biham-Middleton-Levine model with random update rule

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A stochastic version of the Biham-Middleton-Levine model with random update rule is studied. It is shown that under periodic boundary condition, the system exhibits a sharp transition from moving phase to jamming phase. Under open boundary condition, the coexistence of moving phase and jamming phase can be observed. We have presented a mean-field analysis for the moving phase, which successfully takes into account the correlation and produces good agreement with simulation results.

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Traffic flow has attracted a lot of attention in recent years [1–3]. The Biham-Middleton-Levine (BML) model was the first cellular automaton model to simulate urban traffic flow [4] and has served as a theoretical underpinning for physicists to model urban traffic.

In the original BML model, the cars are updated in parallel (hereafter BML-P model) [4]. It has been generally believed that there exists a sharp transition from free-flow to jamming phase in the model, separated by a critical density ρ_c. In the free-flow phase, the cars self-organize into a pattern with ordered stripes of alternating eastbound and northbound cars (see Fig. 1(a) in Ref. [5]), and the average velocity of cars is $v = 1$. Efforts have been devoted to investigating ρ_c analytically [6]. However, recent study shows that the model exhibits an intermediate stable phase apart from the free-flow phase and jamming phase [5]. Benyoussef *et al.* have studied a version of the BML model with the random sequential update rule [7]. It is shown that in the free-flow phase, the average velocity of cars is still $v = 1$ as in the BML-P model.

This Brief Report studies another stochastic version of the BML model with random update rule (hereafter BML-R model). It is shown that different from the previous two BML models, the average velocity v decreases with the increase of density in the moving phase in the BML-R model.

On the other hand, recently the asymmetric exclusion process (ASEP) has become a paradigm in nonequilibrium statistical physics [8]. In the 1D ASEP with random update rule, the correlation is absent so that the average velocity $v = 1 - \rho$ (ρ is global density). The BML-R model is actually a version of 2D ASEP. However, it is different from the 1D ASEP in that the correlation is nontrivial. We have presented a mean-field analysis for the moving phase which considers the correlation and gives very good agreement with the simulation results.

The initial settings of the BML-R model are the same as in the BML-P model. There are two species of cars, eastbound and northbound, initially distributed randomly on a two-dimensional square lattice with size $L \times L$. The eastbound (northbound) cars are not allowed to change either their direction or row (column).

Under periodic boundary conditions, the following steps are repeated L^2 times in one Monte Carlo step (MCS): (i) One site is selected randomly; (ii) if the selected site is empty, nothing happens; (iii) otherwise, if the selected site is occupied, the car moves to the next site unless the target site is occupied. Under open boundary conditions, the eastbound (northbound) cars are injected with probability α on the west (south) boundary and removed with probability β on the east (north) boundary. At the southwest corner, both eastbound cars and northbound cars are injected with probability $\alpha/2$.

Figure 1 shows the average velocity $\langle v \rangle$ versus the car density ρ under periodic boundary conditions. The results are averaged over 100 runs. In each run, v is obtained after discarding the first 10^6 MCSs (as transient time) and then averaged in the next 10^6 MCSs. It can be seen that the system exhibits sharp transition between two phases. When the density $\rho < \rho_c$, the system is in a moving phase. Figure 2 shows a typical configuration of this phase, where the self-organized space arrangement of the BML-P model disappears and the cars are distributed quite homogenously in the space. When the density $\rho > \rho_c$, the system is in a jamming phase and the average velocity $\langle v \rangle = 0$.

We would like to point out that in the BML-R model, the intermediate stable phase identified in the BML-P model does not exist. This is because, as pointed in the last paragraph in Ref. [5], the dynamics in the BML-R model is not fully deterministic. As a result, it is believed that the phase transition from moving phase to jamming phase is of first order in the BML-R model. With the increase of system size, ρ_c decreases. Nevertheless, as in the BML-P model, currently we are not able to determine whether ρ_c converges to finite value or to zero in the infinite system limit.

Next we present mean-field analysis for the average velocity in the moving phase. The key point of the mean-field analysis is as follows. If we randomly select a site, then the probability that the site is empty is $1 - \rho$. However, if we have selected a site occupied by an eastbound (northbound) car, then the probability that its east (north) site is empty equals to the average velocity v rather than $1 - \rho$. In this way, the correlation has been naturally taken into account.

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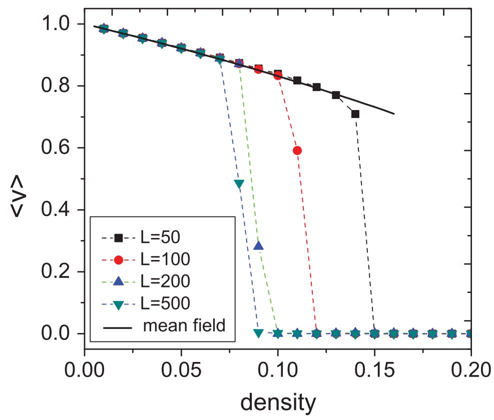

FIG. 1. (Color online) The average velocity $\langle v \rangle$ against the car density ρ for different lattice sizes L under periodic boundary condition. The solid line is the mean-field result [Eq. (2)].

Assuming an eastbound car is chosen at the current time step, we study the probability p that it could move when the car is chosen again. Obviously, this probability should equal the average velocity v.

To calculate the probability p, we need to consider the nine situations as shown in Fig. 3. The $\rightarrow$, $\uparrow$, filled box, and empty box represent, respectively, an eastbound car, a northbound car, a car, and an empty site. The dashed arrow $\dashrightarrow$ is the target eastbound car. The left side of each subfigure is the existence probability of the corresponding situation at the current time. The right side is the moving probability of the eastbound car when it is chosen again.

We explain the subfigures A1, A3, and B3 in detail. Other situations could be obtained similarly. On the left side of subfigure A1, $(1-v)$ is the probability that the east site of $\dashrightarrow$ is occupied so that $\dashrightarrow$ could not move in the current time step, and $(1-\rho)$ is the probability that the southeast site of $\dashrightarrow$ is empty. On the right side of subfigure A1, $1/2$ is the probability that the car in the east site is chosen before $\dashrightarrow$ is chosen again, and v is the probability that the car in the east site could move.

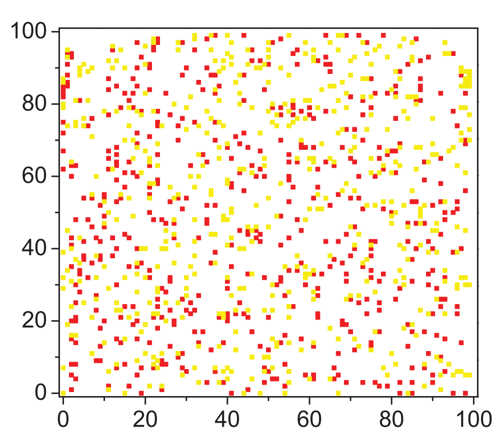

FIG. 2. (Color online) A typical configuration of moving phase below the transition. The system size is 100×100 and $\rho = 0.1$. The northbound cars are indicated by red (dark gray) and the eastbound cars by yellow (light gray).

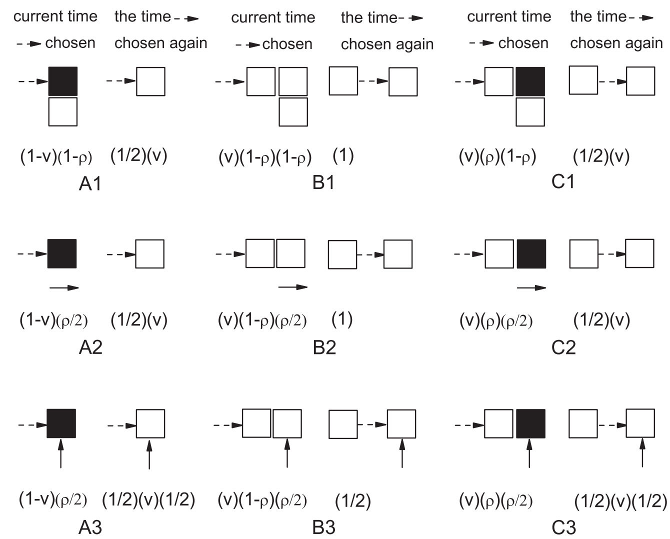

FIG. 3. The illustration of the mean-field method.

On the left side of subfigure A3, $(1-v)$ is the same as in A1, and $\rho/2$ is the probability that the southeast site of $\dashrightarrow$ is occupied by a northbound car. On the right side of subfigure A3, $1/2$ and v are the same as in A1, the other $1/2$ is the probability that the northbound car in the southeast site is not chosen between the car in the east site is chosen and $\dashrightarrow$ is chosen again.[1]

On the left side of subfigure B3, v is the probability that the east site of $\dashrightarrow$ is empty so that $\dashrightarrow$ could move in the current time step, $(1-\rho)$ is the probability that the east-east site of $\dashrightarrow$ is empty, and $\rho/2$ is the probability that the southeast-east site of $\dashrightarrow$ is occupied by a northbound car. In the right side of subfigure B3, $1/2$ is the probability that the northbound car is not chosen before $\dashrightarrow$ is chosen again.

The moving probability p equals the sum of the product of the probabilities in the nine subfigures, which yields

$$\begin{aligned} p &= v \\ &= [1-v][1-\rho]\left[\frac{1}{2}\right][v] + [1-v]\left[\frac{\rho}{2}\right]\left[\frac{1}{2}\right][v] \\ &\quad + [1-v]\left[\frac{\rho}{2}\right]\left[\frac{1}{2}\right][v]\left[\frac{1}{2}\right] + [v][1-\rho][1-\rho][1] \\ &\quad + [v][1-\rho]\left[\frac{\rho}{2}\right][1] + [v][1-\rho]\left[\frac{\rho}{2}\right]\left[\frac{1}{2}\right] \\ &\quad + [v][\rho][1-\rho]\left[\frac{1}{2}\right][v] + [v][\rho]\left[\frac{\rho}{2}\right]\left[\frac{1}{2}\right][v] \\ &\quad + [v][\rho]\left[\frac{\rho}{2}\right]\left[\frac{1}{2}\right][v]\left[\frac{1}{2}\right]. \end{aligned} \tag{1}$$

By solving Eq. (1), we obtain the average velocity:

$$v = \frac{1 - 2.75\rho + 0.5\rho^2}{1 - 1.25\rho + 0.25\rho^2}. \tag{2}$$

[1] It is probably that the northbound car could move twice after the car in the east site moves forward and before $\dashrightarrow$ is chosen again. This probability is relatively small and is ignored in our mean-field analysis. Other similar probabilities are also ignored.

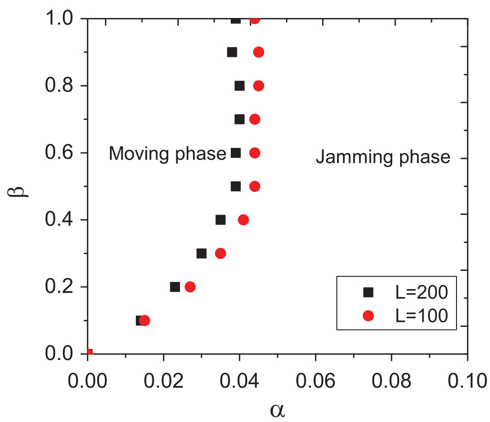

FIG. 4. (Color online) Phase diagram of the model under open boundary condition for the lattice size 100 × 100 and 200 × 200.

Figure 1 compares the simulation result with the mean-field result. One can see that the analytical result is in good agreement with the simulation result, which validates our analysis.

Now we investigate the BML-R model under open boundary conditions. Figure 4 shows the phase diagram of the system. Roughly speaking, there are still two phases: the moving phase and jamming phase. Here by "roughly," we means that there is a coexistence of the jamming phase and moving phase in the vicinity of the boundary, see the following text for details.

The phase diagram is different from that of the ASEP model [8], in which the transition from low density to high density occurs at $\alpha > \beta$ and $\alpha < 1/2$. Moreover, the maximum current appears when $\alpha > 1/2$ and $\beta > 1/2$. This is obviously due to the mutual blockage of cars moving in different directions in the BML-R model.

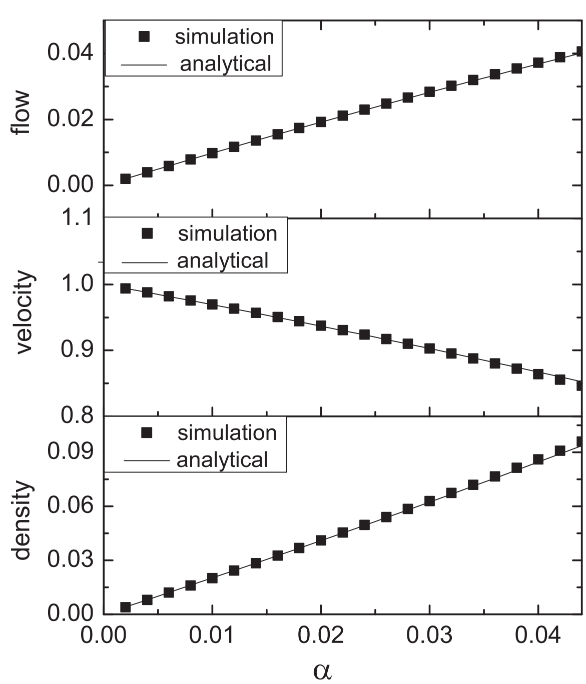

FIG. 5. The global density, the average velocity, and the flow in the moving phase.

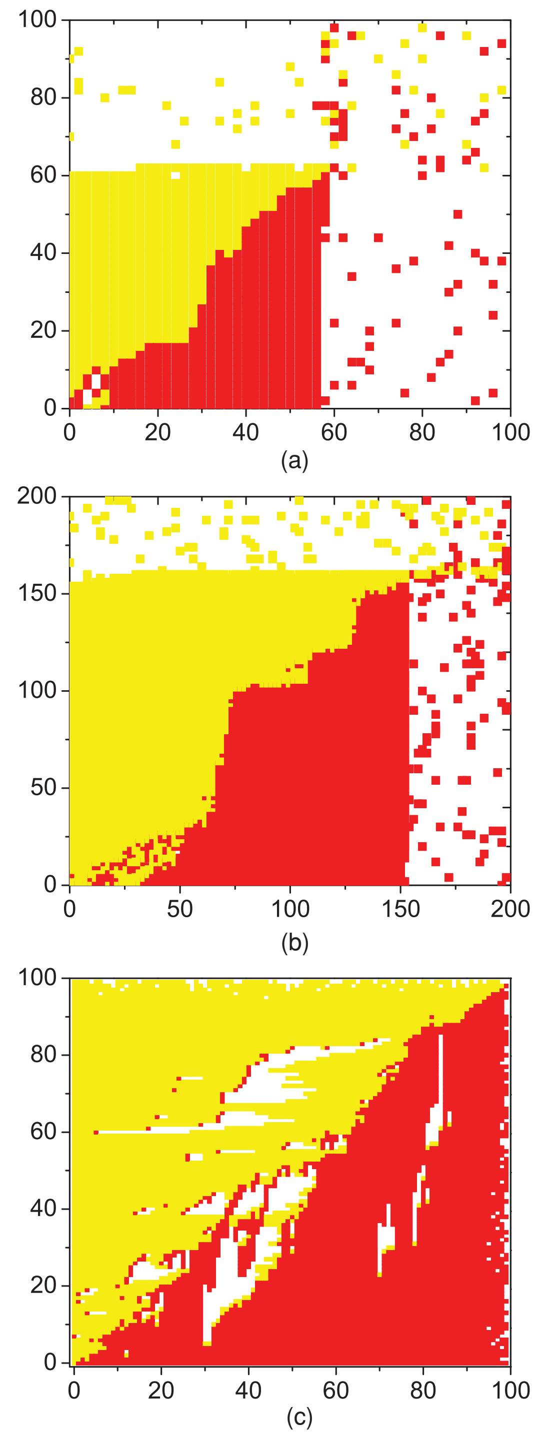

FIG. 6. (Color online) (a), (b) Coexistence patterns under different system sizes; the parameters $\alpha = 0.046, \beta = 1.0$. (c) Schematic configuration of the system in the jamming phase for $\alpha = 1, \beta = 1$.

We study the global density ρ, the average velocity v, and the flow J in the moving phase. Because of the conservation of flow, the flow in the bulk equals the inflow; i.e.,

$$J = \frac{\rho}{2} v = \alpha(1 - \rho). \tag{3}$$

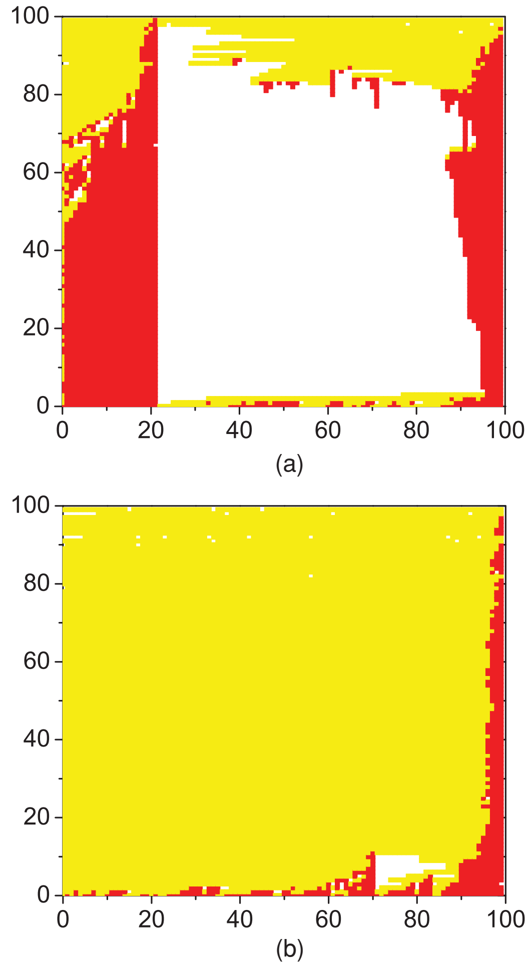

FIG. 7. (Color online) Two unexpected patterns (a) $\alpha = 0.03, \beta = 0.01$; (b) $\alpha = 0.03, \beta = 0.03$.

Plugging Eq. (2) into Eq. (3), we obtain

$$\rho \approx 2\alpha(1 + \alpha + 12\alpha^2) \tag{4}$$

by ignoring higher order terms. Substituting Eq. (4) into Eq. (2), the average velocity can also be obtained by ignoring higher order terms:

$$v \approx 1 - 3\alpha - 9\alpha^2. \tag{5}$$

The flow thus can be calculated:

$$J = \frac{\rho}{2} v \approx \alpha(1 - 2\alpha). \tag{6}$$

The analytical results are shown in Fig. 5 and are in good agreement with the simulation results.

Next we study the jamming phase. In the vicinity of the phase boundary, the coexistence of jamming phase and moving phase is observed as shown in Figs. 6(a) and 6(b). Depending on the random seed, the moving phase appears as a rectangle or square, and it remains dynamically stationary [9]. The size of the moving phase is roughly the same under different system sizes [Figs. 6(a) and 6(b)]. When the system size is close to or smaller than the size of the moving phase, coexistence cannot be maintained and the system transits into the moving phase. This helps to understand why the phase boundary shifts toward the right with the decrease of system size in the phase diagram in Fig. 4.

With the increase of α, the size of the moving phase in the coexistence pattern decreases. When $\alpha = 1$, only cars very near the exit boundaries can move [Fig. 6(c)] [9].

Finally, we would like to mention that when both α and β are small, some unexpected patterns might appear. For example, Fig. 7(a) shows a pattern with a vast blank in the center area;[2] Fig. 7(b) shows a pattern with eastbound cars dominating [9].

To summarize, this Brief Report has studied a version of the BML model with random update rule. A sharp transition from moving phase to jamming phase is observed under periodic boundary conditions, which is believed to be of first order. The intermediate stable phase observed in the BML-P model is absent in the BML-R model since the dynamics is not fully deterministic in the latter model. Under open boundary conditions, the coexistence of moving phase and jamming phase can be observed. Another contribution of this report is that we have developed a mean-field analysis for the moving phase, which successfully takes into account the correlation and produces good agreement with simulation results.

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[2]It might be that after a sufficiently long time, the vast blank disappears and one type of car dominates as shown in Fig. 7(b).

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[9] See supplemental material at [http://link.aps.org/supplemental/10.1103/PhysRevE.83.047101] for the evolution process.