Numerical simulation of asymptotic states of the damped Kuramoto-Sivashinsky equation

Hector Gomez^{*} and José París

University of A Coruña, Campus de Elviña s/n, 15071, A Coruña, Spain (Received 13 October 2010; revised manuscript received 21 February 2011; published 4 April 2011)

The damped Kuramoto-Sivashinsky equation has emerged as a fundamental tool for the understanding of the onset and evolution of secondary instabilities in a wide range of physical phenomena. Most existing studies about this equation deal with its asymptotic states on one-dimensional settings or on periodic square domains. We utilize a large-scale numerical simulation to investigate the asymptotic states of the damped Kuramoto-Sivashinsky equation on annular two-dimensional geometries and three-dimensional domains. To this end, we propose an accurate, efficient, and robust algorithm based on a recently introduced numerical methodology, namely, isogeometric analysis. We compared our two-dimensional results with several experiments of directed percolation on square and annular geometries, and found qualitative agreement.

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I. INTRODUCTION

A thermodynamical system far from equilibrium may exhibit primary instabilities which drive it into a inhomogeneous asymptotic state. In many relevant systems, these asymptotic states consist of spatially and temporally ordered cellular structures. In the past decades, there has been increasing interest in the so-called secondary instabilities [1], which may destroy the ordered cellular state, giving rise to disordered states both in space and time. Prime examples of phenomena where secondary instabilities exist are, for instance, the Rayleigh-Benard convection [2,3], directional solidification [4,5], Faraday waves [6,7], or directed percolation [8–13]. The transition from a homogeneous stationary state to asymptotic cellular states through primary instabilities may be successfully analyzed by using a linear stability analysis, but more sophisticated methods are necessary to understand secondary instabilities [14]. Remarkably, one of the most successful tools for the understanding of secondary instabilities has turned out to be the study of the asymptotic states of the damped Kuramoto-Sivashinsky equation [15], which has emerged as a fundamental universal model describing the onset and evolution of secondary instabilities [14]. As a consequence, there is significant interest in the study of the asymptotic states of the damped Kuramoto-Sivashinsky equation. The main difficulty to achieve this goal is that the damped Kuramoto-Sivashinsky equation is not a gradient system [16]. Thus, there is no known Lyapunov functional for the equation. This fact significantly limits our capacity to study its asymptotic states using analytical techniques, so a numerical simulation appears as a very attractive alternative. At this point, the one-dimensional equation is fairly well understood [15,17]. In past years, significant progress has been made in the understanding of the two-dimensional equation [18], but the results are limited to periodic square domains. Given the strong dependence of the asymptotic states on the geometry and the dimensionality of the domain [18,19], the understanding of the late-time states on nonsquare two- and three-dimensional domains is considered a very relevant research topic. This is precisely one of the objectives of this work. To investigate the

asymptotic states we use a numerical simulation. Thus, we propose an effective, accurate, and robust numerical scheme for the damped Kuramoto-Sivashinsky equation, which is another contribution of this work.

The numerical simulation of the damped Kuramoto-Sivashinsky equation presents several challenges. This is the reason why most calculations available in the literature are restricted to one-dimensional settings [8,20-24] and only very recently were two-dimensional simulations on square domains available [18,25-27]. We do not know of any threedimensional simulation nor we are aware of two-dimensional calculations on nonsquare domains (although we know of numerical solutions to a modified Kuramoto-Sivashinsky equation on a disk [28–30]). We feel that one of the main reasons for this is that the damped Kuramoto-Sivashinsky equation includes a fourth-order partial-differential operator. The numerical resolution of higher-order partial-differential equations is significantly less developed than that of secondorder problems. For example, in the context of finite-element methods, the use of conforming discretizations for fourth-order partial-differential spatial operators requires utilizing globally C^1 -continuous basis functions. There exist some threedimensional finite elements possessing global C^1 continuity, but they introduce a number of additional degrees of freedom and severely restrict the geometrical complexity of the domain. Thus, in the finite-element context, the standard approach is to use a mixed method, which for a fourth-order problem doubles the number of global degrees of freedom compared to the primal variational formulation. As a consequence, the most widely used numerical methodologies for fourth-order partial-differential equations are either finite differences or pseudospectral collocation methods, whose applicability to complicated three-dimensional geometries is limited. Thus, we feel that there is no totally satisfactory solution to the higher-order operator problem, yet fourth-order equations are becoming ubiquitous, primarily due to the fast development of phase-field modeling [31,32].

This work proposes a numerical formulation based on isogeometric analysis [33], which is a generalization of finiteelement analysis with several advantages [34-41]. Isogeometric analysis is based on developments of computational geometry and consists of using nonuniform rational *B* splines (NURBS) as basis functions in a variational formulation. For

^{*}hgomez@udc.es



FIG. 1. (Color online) Numerical solution to the damped Kuramoto-Sivashinsky equation on the domain $[0,100]^2$ at time $t = 15\,000$ for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Boundary conditions are periodic. The computational mesh is composed of $128^2 C^1$ -quadratic elements. For $\alpha = 0.225$ we observe a hexagonal pattern. The solution for $\alpha = 0.210$ corresponds to the so-called breathing hexagonal state. For $\alpha = 0.195$ we observe a disordered state consistent with a chaotic behavior.

an introduction to NURBS, the reader is referred to Refs. [42] and [43]. Among the advantages of isogeometric analysis over finiteelement analysis [44], we mention precise geometrical modeling, simplified mesh refinement, superior approximation capabilities, and, most importantly for the present work, C^1 continuity or higher on nontrivial geometries.

In addition to space discretization, the other key difficulty in the simulation of the damped Kuramoto-Sivashinsky equation is time integration. Since the Kuramoto-Sivashinsky equation is nonlinear, noncontractive, and essentially a nongradient system, a fundamental question emerges. What would be an adequate notion of stability for the time integrator to satisfy? In the absence of a clear notion of stability, we use an *A*-stable method that has proved very effective for the Kuramoto-Sivashinsky equation in our numerical simulations, namely, the generalized- α method [45,46]. We also make use



FIG. 2. (Color online) Time evolution of the \mathcal{L}^2 energy for the damped Kuramoto-Sivashinsky equation on the square $[0, 100]^2$. Boundary conditions are periodic. The black, red, and blue lines correspond to $\alpha = 0.195$, $\alpha = 0.210$, and $\alpha = 0.225$, respectively (the lines are also labeled with the value of α to which they correspond). We represent three subplots that show to the evolution of the \mathcal{L}^2 energy in the time period $t \in [12000, 13000]$.



FIG. 3. (Color online) \mathcal{L}^2 energy phase planes for the damped Kuramoto-Sivashinsky equation on the square $[0,100]^2$ for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Note that the vertical scales of the three subplots are different. The solid squares in the plots indicate where the phase planes start.

of an adaptive time-stepping algorithm to impose control over local errors [47].

Our space and time discretization schemes render an effective, accurate, and robust methodology. We present twodimensional numerical examples on nonsquare geometries and three-dimensional simulations.

II. THE DAMPED KURAMOTO-SIVASHINSKY EQUATION

Here we state an initial and boundary-value problem for the damped Kuramoto-Sivashinsky equation over the time interval [0,T]. Let $\Omega \subset \mathbb{R}^3$ be an open set. We denote Γ the boundary of Ω , which is assumed to have a continuous unit outward normal vector \boldsymbol{n} . The problem is stated as follows: Given $u_0 : \overline{\Omega} \mapsto \mathbb{R}$, find $u : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} = -\Delta u - \Delta^2 u - \alpha u + |\nabla u|^2 \quad \text{in} \quad \Omega \times (0, T), \quad (1)$$

$$u(\mathbf{x},0) = u_0(\mathbf{x})$$
 in $\overline{\Omega}$, (2)

with adequate boundary conditions. Periodic boundary conditions are the standard choice in square domains. On more complex geometries the boundary conditions,

$$\nabla(u + \Delta u) \cdot \boldsymbol{n} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{0},\tag{4}$$

may be utilized. In a variational formulation, Eqs. (3) and (4) may be thought of as natural boundary conditions for the damped Kuramoto-Sivashinsky equation.

Although the dynamics of the Kuramoto-Sivashinsky equation [48,49] is very complex, the four terms on the right-hand side of Eq. (1) have a clear meaning in their own right. The first term is destabilizing in the sense that it increases the \mathcal{L}^2 energy in the system. Using the same terminology, we would qualify the second and third terms as stabilizing. Finally, the last term is an energy-transfer operator. It transfers energy from lower to higher frequencies [50].

Additional insight about primary instabilities of the equation may be obtained by using a linear stability analysis



FIG. 4. (Color online) Numerical solution to the damped Kuramoto-Sivashinsky equation on the square $[0,512]^2$ at time t = 15000 for (a) $\alpha = 0.225$ and (b) detail of the solution. The plot on the right-hand side corresponds to the area marked with a box on the left-hand side. The plots highlight the appearance of the penta-hepta defects experimentally observed in a pattern of two-dimensional jets [9].



FIG. 5. (Color online) On the left-hand side we plot the evolution of the \mathcal{L}^2 energy from a chaotic initial state to a stable state of constant energy for seven values of α and different chaotic initial conditions. On the right-hand side we plot α vs the reciprocal of the average time before the system achieves a stable state of constant energy. The data fits a straight line with a coefficient of determination 0.9982.

[14]. It may be shown that the homogeneous constant state u = 0 is linearly stable for $\alpha > 0.25$. Additionally, it is known that for $\alpha = 0$ the Kuramoto-Sivashinsky equation leads to spatiotemporal chaotic states [51,52]. For intermediate values of α , the asymptotic states may be different. One of the most significant developments in past years is due to Paniconi and Elder [18], who identified three asymptotic states of the damped Kuramoto-Sivashinsky equation on a square domain for different values of α , namely, the hexagonally ordered (0.2176 < α < 0.2500), the so-called breathing hexagonal state (0.2070 < α < 0.2176), and the spatiotemporally chaotic (weakly turbulent) state (0 $\leq \alpha < 0.2070$). Given the strong dependence of late-time states

on the dimensionality and topology of the domain, we aim at generalizing those results by performing numerical simulations on two-dimensional nonsquare geometries and three-dimensional calculations. The next section shows our proposed numerical formulation to achieve this goal.

III. NUMERICAL FORMULATION

In this section we present our numerical formulation for the damped Kuramoto-Sivashinsky equation. We first derive a semidiscrete formulation and then use an adaptive timestepping method to advance the solution in time.



FIG. 6. (Color online) Numerical solution to the damped Kuramoto-Sivashinsky equation on an annular surface at t = 15000 for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Boundary conditions are defined in Eqs. (3) and (4). The computational mesh is composed of 256 elements in the circumferential direction and 64 in the radial direction. For $\alpha = 0.225$ we observe a hexagonal pattern. The solution for $\alpha = 0.210$ corresponds to the so-called breathing hexagonal state. For $\alpha = 0.195$ we observe a disordered state consistent with a chaotic behavior.



FIG. 7. (Color online) Time evolution of the \mathcal{L}^2 energy for the damped Kuramoto-Sivashinsky equation on an annular surface. The black, red, and blue lines correspond to $\alpha = 0.195$, $\alpha = 0.210$, and $\alpha = 0.225$, respectively (the lines are also labeled with the value of α to which they correspond). We represent three subplots corresponding to the evolution of the \mathcal{L}^2 energy in the time interval $t \in [12\ 000, 13\ 000]$.

A. Semidiscrete formulation

Our starting point is the weak formulation of the continuous problem. At this point we assume periodic boundary conditions in all directions. Let us call \mathcal{V} the space of trial and weighting functions which are assumed to be the same. We suppose $\mathcal{V} \subset \mathcal{H}^2$, where \mathcal{H}^2 is the Sobolev space of square integrable functions with square integrable first and second derivatives. The problem may be stated as follows: Find $u \in \mathcal{V}$ such that for all $w \in \mathcal{V}$

$$B(w,u) = 0, (5)$$

where

$$B(w,u) = \left(w, \frac{\partial u}{\partial t}\right) - (\nabla w, \nabla u) + (\Delta w, \Delta u) + (w, \alpha u - |\nabla u|^2), \quad (6)$$

and (\cdot, \cdot) is the \mathcal{L}^2 -inner product with respect to the domain Ω .

To perform the space discretization of (5) we make use of the Galerkin method. We approximate (5) by the following



FIG. 8. (Color online) \mathcal{L}^2 energy phase planes for the damped Kuramoto-Sivashinsky equation on an annular surface for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Note that the vertical scales of the three subplots are different. The solid squares in the plots indicate where the phase planes start.



FIG. 9. (Color online) Numerical solution to the damped Kuramoto-Sivashinsky equation on an annular domain. Boundary conditions are defined in Eqs. (3) and (4). This example shows qualitative agreement with an experiment of directed percolation presented in Ref. [11].

finite-dimensional problem over the finite-element space $\mathcal{V}^h \subset \mathcal{V}$: find $u^h \in \mathcal{V}^h$ such that, for all $w^h \in \mathcal{V}^h$,

$$B(w^{h}, u^{h}) = 0. (7)$$

In Eq. (7) u^h takes on the form

$$u^{h}(\mathbf{x},t) = \sum_{A=1}^{n_{b}} u_{A}(t) N_{A}(\mathbf{x}),$$
(8)

where n_b is the dimension of the discrete space, the N_A 's are the basis functions of the discrete space, and the u_A 's are the coordinates of u^h on \mathcal{V}^h .

Note that the condition $\mathcal{V}^h \subset \mathcal{V}$ mandates our discrete space to be \mathcal{H}^2 conforming. This condition is satisfied by \mathcal{C}^1 -continuous NURBS basis functions.

B. Time integration

The time integration of the Kuramoto-Sivashinsky equation constitutes a significant challenge. Explicit methods have to face severe limitations on the time step due to the fourth-order spatial derivatives. Semi-implicit methods may be attractive because the fourth-order term is linear. Thus, treating implicitly this term and explicitly the rest may permit taking somewhat larger time steps, while avoiding the use of a nonlinear solver. However, we favor the use of a fully implicit algorithm, namely, the generalized- α method. The reason for this is that using the fully implicit algorithm we were able to take significantly larger time steps compared to the semi-implicit scheme. This is due to the fact that the most straightforward semi-implicit methods are only first-order accurate and, as a consequence, may be inaccurate for large time steps. Another reason to use the fully implicit method is that our numerical examples showed a very fast convergence of the nonlinear solver.

To define our time integration scheme, we introduce the following residual vector,

$$\boldsymbol{R} = \{R_A\},\tag{9}$$

$$R_A = B(N_A, u^h). (10)$$

Let us call U and \dot{U} the vector of global degrees of freedom of the scalar field u and its time derivative, respectively. The time-stepping algorithm may be described as follows: Given U_n , \dot{U}_n , find U_{n+1} , \dot{U}_{n+1} such that

$$\boldsymbol{R}(\boldsymbol{U}_{n+\alpha_m}, \boldsymbol{U}_{n+\alpha_f}) = 0, \tag{11}$$

where

$$\dot{\boldsymbol{U}}_{n+\alpha_m} = \dot{\boldsymbol{U}}_n + \alpha_m (\dot{\boldsymbol{U}}_{n+1} - \dot{\boldsymbol{U}}_n), \tag{12}$$

$$\boldsymbol{U}_{n+\alpha_f} = \boldsymbol{U}_n + \alpha_f (\boldsymbol{U}_{n+1} - \boldsymbol{U}_n), \tag{13}$$

$$\boldsymbol{U}_{n+1} = \boldsymbol{U}_n + \Delta t \dot{\boldsymbol{U}}_n + \gamma \Delta t (\dot{\boldsymbol{U}}_{n+1} - \dot{\boldsymbol{U}}_n).$$
(14)

Note that, although U and \dot{U} are independently treated in the algorithm, U_{n+1} and \dot{U}_{n+1} are related through Eq. (14) and thus they are not independent unknowns.



FIG. 10. (Color online) Numerical solution to the Kuramoto-Sivashinsky equation at time t = 3500 for $\alpha = 0.225$. The computational domain is $[0, 128]^3$ and boundary conditions are periodic in all directions. The computational mesh is composed of $128^3 C^1$ -quadratic elements. The left-hand side shows isosurfaces of the solution, while the right-hand side presents slices along several planes.



FIG. 11. (Color online) Numerical solution to the Kuramoto-Sivashinsky equation at time t = 3500 for $\alpha = 0.210$. The computational domain is $[0, 128]^3$ and boundary conditions are periodic in all directions. The computational mesh is composed of $128^3 C^1$ -quadratic elements. The left-hand side shows isosurfaces of the solution, while the right-hand side presents slices along several planes.

To complete the description of the method, it remains to define α_m , α_f , and γ . These are real-valued parameters that define the accuracy and the stability properties of the algorithm. Jansen *et al.* [46] proved that, for a linear model problem, second-order accuracy is attained if

$$\gamma = \frac{1}{2} + \alpha_m - \alpha_f, \tag{15}$$

while unconditional A stability requires

$$\alpha_m \geqslant \alpha_f \geqslant 1/2. \tag{16}$$

We are interested in second-order accurate unconditionally *A*-stable methods, so we will take values of α_m , α_f , and γ that satisfy Eqs. (15) and (16) simultaneously. One of the key features of the generalized- α method is that α_m and α_f can be parametrized in terms of ρ_{∞} , the spectral radius of the amplification matrix that controls high-frequency dissipation [46]. Thus,

$$\alpha_m = \frac{1}{2} \left(\frac{3 - \rho_\infty}{1 + \rho_\infty} \right), \quad \alpha_f = \frac{1}{1 + \rho_\infty}.$$
 (17)

As a consequence, if we set ρ_{∞} , and then select α_m and α_f using (17) and calculate γ utilizing (15), we have a family of second-order accurate unconditionally A-stable methods with optimal control over high-frequency dissipation. The details of the implementation of the generalized- α method for a nonlinear problem may be found in Ref. [47].

C. Time-step adaptivity

The undamped Kuramoto-Sivashinsky equation is known to amplify exponentially small perturbations in finite time intervals. For small values of the linear stabilizing term, the damped equation retains this feature. Thus, accurate time integration is key to perform reliable long-time computations. We feel that these arguments recommend the use of adaptive time-step control. We employ a recently proposed adaptive algorithm that can be used in conjunction with the generalized- α method. The details of this algorithm may be found in Ref. [47].

IV. NUMERICAL SIMULATIONS

In this section we present some two- and three-dimensional numerical examples. The purpose of these examples is threefold: First, we aim at illustrating the effectiveness and robustness of our numerical formulation; second, we investigate the asymptotic states of the damped Kuramoto-Sivashinsky equation on nonsquare domains in two dimensions and on cubic three-dimensional domains; third, we compare our simulations with directed percolation experiments.

Throughout this paper, for the computation of the asymptotic states, we take as initial condition a random perturbation of the homogeneous state u = 0. The perturbations are directly applied to control variables and are uniformly distributed on [-0.05, 0.05]. For the comparison with experiments we may take different initial conditions that will be specified in each case.

For the space discretization we employ C^1 quadratic NURBS for all the numerical examples.

A. Numerical simulations on a periodic square

1. Asymptotic states

Here we present the numerical solution to the damped Kuramoto-Sivashinsky equation on the domain $\Omega = [0, 100]^2$. We use periodic boundary conditions and a computational mesh composed of $128^2 C^1$ -quadratic elements. Following Refs. [18] and [25], we present simulations for $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$. Our results show the hexagonal ($\alpha = 0.225$), breathing hexagonal ($\alpha = 0.210$), and disordered $(\alpha = 0.195)$ states found by Paniconi and Elder [18]. The hexagonal state is a spatially ordered stationary solution of the damped Kuramoto-Sivashinsky equation found for relatively large values of α . The breathing hexagonal state is an unsteady solution characterized by a quasiperiodic oscillation of the hexagonal pattern in which each cell oscillates out of phase with its closest neighbor. In Fig. 1 we plot the numerical solutions at time $t = 15\,000$ for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. To further illustrate the difference between



FIG. 12. (Color online) Numerical solution to the Kuramoto-Sivashinsky equation at time t = 3500 for $\alpha = 0.195$. The computational domain is $[0, 128]^3$ and boundary conditions are periodic in all directions. The computational mesh is composed of $128^3 C^1$ -quadratic elements. The left-hand side shows isosurfaces of the solution, while the right-hand side presents slices along several planes.

these three types of asymptotic states, we make use of the \mathcal{L}^2 energy, defined as

$$\mathcal{E} = ||u||_2 = (u,u)^{1/2}.$$
 (18)

The \mathcal{L}^2 energy has been identified as a fundamental quantity to understand the dynamics of the Kuramoto-Sivashinsky equation [52-54]. In Fig. 2 we plot the time evolution of the \mathcal{L}^2 energy for $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$. In the same figure, we also depict three subplots corresponding to the evolution of the \mathcal{L}^2 energy during the time interval $t \in [12000, 13000]$. These subplots show the complexity of the evolution of the \mathcal{L}^2 energy and illustrate the difference between the steady state ($\alpha = 0.225$), the breathing hexagonal state ($\alpha = 0.210$), and the chaotic state ($\alpha = 0.195$). Thus, for $\alpha = 0.225$, the \mathcal{L}^2 energy is constant in time, which is consistent with a stationary solution. For $\alpha = 0.210$, the \mathcal{L}^2 energy exhibits a two-scale behavior. The small-scale component is quasiperiodic and its frequency approximately coincides with the temporal frequency of the oscillating pattern. Finally, for $\alpha = 0.195$, we observe a complex evolution consistent with a chaotic state.

Additional insight may be obtained by plotting the \mathcal{L}^2 energy phase plane, which is the set of points $(\mathcal{E}, \dot{\mathcal{E}})$ for a given time interval (here $\dot{\mathcal{E}}$ denotes the time derivative of \mathcal{E}). The \mathcal{L}^2 energy phase plane is regarded as a fundamental tool for the understanding of the damped Kuramoto-Sivashinsky equation [52,53]. In Fig. 3 we plot the \mathcal{L}^2 energy phase plane for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Note that the vertical scales of the three subfigures are different. For $\alpha = 0.225$ we observe a typical phase plane of a stationary solution. For $\alpha = 0.210$ and $\alpha = 0.195$ we plot the phase planes during the time interval $t \in [14900, 15000]$, which corresponds to the last 100 units of time of the simulation. We observe that for $\alpha = 0.195$, $\dot{\mathcal{E}}$ takes values one order of magnitude larger than those achieved for $\alpha = 0.210$. This is a consequence of wilder and rougher variations of the \mathcal{L}^2 energy, which are consistent with a more chaotic behavior. We also observe that the phase plane for $\alpha = 0.210$ reveals a quasiperiodic structure that manifests itself through ordered

loops. This behavior is consistent with the breathing hexagonal state, and thus it is not present in the phase plane that corresponds to $\alpha = 0.195$.

2. Comparison with experiments

The breathing hexagonal state has been experimentally observed in a pattern of two-dimensional jets [9], obtained by way of a directed percolation experiment. In the same work, the authors observe a stationary hexagonal state with several topological defects. They identify the so-called penta-hepta defect which consists of a spot surrounded by seven (rather than six) dots and a neighbor of it enclosed by five. Here we aim to show that our simulations of the damped Kuramoto-Sivashinsky equation reproduce this topological defect. For this purpose, we performed a simulation for $\alpha = 0.225$ on a significantly larger domain, namely, $\overline{\Omega} = [0,512]^2$. Figure 4 shows a snapshot of the solution with several penta-hepta defects marked (a) and a detailed view of the area where the penta-hepta defects are located (b).

As an additional comparison, we perform a statistical study of the average time before the system reaches a stable state of constant energy from a chaotic initial condition. The initial conditions correspond to different realizations of a chaotic asymptotic state calculated using $\alpha = 0.195$. Then, we suddenly increase the value of α and measure the time before the system reaches a stable state of constant energy. We sampled seven values of α , which lead to stationary solutions from $\alpha = 0.225$ to $\alpha = 0.240$. For each value of α , we performed 11 calculations corresponding to different chaotic initial conditions. Figure 5(a) shows the evolution of the \mathcal{L}^2 energy from a chaotic state to a stable state for several values of α and different initial conditions. From these curves we calculated the average time before stabilization $\langle \tau \rangle$. In Fig. 5(b) we plot α vs $\langle \tau \rangle^{-1}$. The data fits a straight line with a coefficient of determination 0.9982. The behavior is the qualitatively the same as that found in a experiment of directed percolation [13], where the same scaling has been measured.



FIG. 13. (Color online) Time evolution of the \mathcal{L}^2 energy for the damped Kuramoto-Sivashinsky equation on the cube $[0, 100]^3$. Boundary conditions are periodic in all directions. The black, red, and blue lines correspond to $\alpha = 0.195$, $\alpha = 0.210$, and $\alpha = 0.225$, respectively (the lines are also labeled with the value of α to which they correspond). We represent three subplots corresponding to the evolution of the \mathcal{L}^2 energy in the time period $t \in [2500, 2800]$.

We remark that this scaling has also been observed in statistical studies of relaminarization in pipe or shear flows [55].

B. Numerical simulations on an annular surface

1. Asymptotic states

In this example we calculate the numerical solution to the damped Kuramoto-Sivashinsky equation on an annular surface. This geometrical setting has been very recently analyzed under the assumption of radial symmetry [19]. Here we remove this hypothesis. This example also shows that our numerical formulation can be applied to nonsquare geometries, while maintaining its accuracy, stability, and robustness. Our study suggests that the hexagonal, breathing hexagonal, and disordered states found in the periodic square also exist in this geometrical setting.

The exterior radius of the annular surface is $r_e = 50.0$, while the interior is $r_i = 12.5$. On the boundary, we impose the conditions (3) and (4). We construct the computational mesh joining four NURBS patches. Each of these patches corresponds to a quarter of the annular surface and is composed of C^1 quadratic NURBS elements. The patches are joint as to maintain C^1 continuity of the solution over the whole domain. This can be accomplished by applying linear restriction operators to the solution and weighting functions spaces. The resulting mesh (comprising four patches) is composed of a total of 256 elements in the circumferential direction and 64 in the radial direction. We note that our formulation achieves exact geometrical modeling of this problem. The reason for this is that NURBS can represent all conic sections exactly [33].

In Fig. 6 we plot the solution for $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$ at time t = 15000. To better understand the asymptotic states we make use again of the \mathcal{L}^2 energy time evolution (see Fig. 7). The blue, red, and black lines correspond to $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$, respectively (see also the labels appended to the lines). The dynamics of the \mathcal{L}^2 energy is qualitatively similar to the behavior exhibited in the last example. In the subplots displayed in Fig. 7 we observe that for $\alpha = 0.225$ the \mathcal{L}^2 energy is fairly constant, which is consistent with a stationary solution. For $\alpha = 0.210$ we observe a quasiperiodic behavior, which is the manifestation of the breathing structure in the \mathcal{L}^2 energy. Finally, for $\alpha = 0.195$, the plot shows a complex behavior consistent with a chaotic state. This is again confirmed by the \mathcal{L}^2 energy phase planes, which are shown in Fig. 8. The phase plane for $\alpha = 0.225$ clearly corresponds to a stationary solution. For $\alpha = 0.210$ and $\alpha = 0.195$ we plot the phase planes in the time interval $t \in [14\,900, 15\,000]$, which corresponds to the last 100 units of time of the simulation. Observe that the vertical scales are different for each subfigure. We note that for $\alpha = 0.195$, $\hat{\mathcal{E}}$ takes values an order of magnitude larger than those taken for $\alpha = 0.210$. Also, for $\alpha = 0.210$ there is a quasiperiodic structure in the phase plane, which is not present for $\alpha =$ 0.195. We conclude that the snapshots of the solution (Fig. 6)



FIG. 14. (Color online) \mathcal{L}^2 energy phase planes for the Kuramoto-Sivashinsky equation on cube at the time interval $t \in [3400, 3500]$ for (a) $\alpha = 0.225$, (b) $\alpha = 0.210$, and (c) $\alpha = 0.195$. Note that the vertical scale is the same for all subplots. The solid squares in the plots indicate where the phase planes start.

and the phase planes (Fig. 8) suggest that the asymptotic states in the annular surface are the same as those found on the periodic square, although this result may not hold if we change the ratio of the exterior to the interior radii.

2. Comparison with experiments

The annular geometry has recently received the attention from experimentalists both in the context of Rayleigh-Benard convection [56] and directed percolation [11]. Here we compare our numerical simulations with the experiements presented in Ref. [11]. In particular, we show that the damped Kuramoto-Sivashinsky equation reproduces the transition from a stationary liquid curtain to a pattern of columns, as predicted by the experiments. The annular geometry is defined by an exterior radius $r_e = 61$ and a interior radius $r_i = 51$. This geometry corresponds to one of the experiments presented in Ref. [11]. The computational mesh is composed of 256 C^1 -quadratic elements in the circumferential direction and 16 elements in the radial direction. Boundary conditions are defined by Eqs. (3) and (4). We simulate the stationary liquid curtain by taking a constant initial condition $u_0(\mathbf{x}) = 10$. Then, we let the solution evolve until a pattern of columns develops. Figure 9 shows the equilibrium arrangement, which is in agreement with the cellular pattern found in Ref. [11].

C. Asymptotic states on three-dimensional domains

In this section we present the three-dimensional counterpart of the simulation presented in Sec. IV A 1. The computational domain is $\overline{\Omega} = [0,100]^3$ and we employ a uniform mesh composed of $128^3 C^1$ -quadratic elements. In this example we will not be able to run the calculations until such long times as in the two-dimensional simulations due to excessive computational cost. We ran the examples up to t = 3500, which required ~20000 time steps. We assume that the asymptotic states are reached before this time. In Figs. 10, 11, and 12 we plot the numerical solutions at time t = 3500for $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$, respectively. On the left-hand side of each figure we plot isosurfaces of the solution, while on the right-hand side we present slices along several planes, which clearly show disordered states. We also note that there are no qualitative differences between the solutions for different values of α . Figure 13 shows the evolution of the \mathcal{L}^2 energy for $\alpha = 0.225$, $\alpha = 0.210$, and $\alpha = 0.195$. In all cases we observe a complex evolution without a clear structure. Additionally, Fig. 13 shows that the trend in the evolution of the \mathcal{L}^2 energy is fairly constant from t = 300 until the end of the computation. This supports our hypothesis about the asymptotic states being reached before t = 3500. To further analyze the evolution of the \mathcal{L}^2 energy we make use again of the phase planes, which are shown in Fig. 14. The phase planes correspond to the last 100 units of time of the simulation. Unlike in the two-dimensional examples, the vertical scale is the same for all subplots. We do not observe qualitative differences between the phase planes for the three values of α . We conclude that, although further study is warranted, the snapshots of the solution (Figs. 10–12) and the \mathcal{L}^2 energy phase planes (Fig. 14) suggest that the hexagonal and the breathing hexagonal states may not exist on three-dimensional domains.

V. CONCLUSION

We presented a computational approach to investigate the asymptotic states of the damped Kuramoto-Sivashinsky equation. We applied our numerical technique to problems on nonsquare two-dimensional geometries and three-dimensional domains. Thus, our work extends previous studies on the topic which were almost invariably restricted to one-dimensional settings or square domains. Our study suggests that for the two-dimensional domain that we analyze the asymptotic states are the same as those found on a periodic square. However, in three-dimensional domains we consistently found chaotic asymptotic states. Since the damped Kuramoto-Sivashinsky equation is a fundamental model that describes the onset and evolution of secondary instabilities, our study may contribute to a better understanding of physical phenomena exhibiting this behavior. We presented several comparisons of our numerical simulations with experiments of directed percolation. We conclude that the damped Kuramoto-Sivashinsky equation reproduces the hexagonal and breathing hexagonal states found on directed percolation experiments on squares, and it also exhibits the penta-hepta defects found in experiments. We have also computed numerically the scaling of the stabilization time of chaotic solutions with respect to the control parameter α . Our study agrees qualitatively with an experiment of directed percolation. We have also presented a qualitative comparison of our simulations with a directed percolation experiment on an annular geometry.

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