Enhancement of "logical" responses by noise in a bistable optical system

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(Received 29 September 2010; published 25 April 2011)

We verify numerically the phenomenon of logical stochastic resonance in a polarization bistable laser. Namely, we show that when one presents two weak binary inputs to the laser system, the response mirrors a logical OR(NOR) output. The reliability of the logic operation is dependent on the noise intensity. As one increases the noise, the probability of the output reflecting the desired OR(NOR) operation increases to nearly unity and then decreases. We also demonstrate that changing the bias morphs the output into another logic operation, AND(NAND), whose probability displays analogous behavior. Furthermore, we highlight the possibility of processing two logic gates in parallel in our laser system by exploiting two coupled orthogonal polarizations that can be detected simultaneously. This suggests that the computational power of the optical system may be enhanced by this additional potential for parallel processing.

DOI: 10.1103/PhysRevE.83.046219 PACS number(s): 05.45.−a, 42.60.Mi, 42.65.Pc, 02.50.Ey

I. INTRODUCTION

Over the past few years, it has become increasingly obvious that understanding how noise and nonlinearity cooperate to produce different effects is critical in understanding how complex systems behave and evolve. Stochastic resonance (SR) provides one such example wherein the cooperative interplay between noise and dynamics produces interesting, often counterintuitive, physical phenomena. SR has received much attention over the past two decades [1] and it has been demonstrated over a variety of physical systems on a large span of time scales [1]. The basic signature of SR consists of an enhanced optimal response to otherwise weak input signals through the delicate interplay between a coherent signal, noise, and nonlinearity.

Recently it was found that the response of a simple threshold detector to input signals consisting of two random square waves shows a remarkable feature. In an optimal band of noise, the output is a logical combination of the two input signals. This phenomenon has been termed logical stochastic resonance (LSR) [2]. The motivation to study LSR in further detail stems from an issue that is currently receiving considerable attention. As computational devices continue to shrink in size, we are increasingly encountering fundamental, performance-degrading circuit noise that cannot be suppressed or eliminated. Hence an understanding of the cooperative behavior between a device's noise floor and its nonlinearity plays a more crucial role in the design and development of computational devices. Analogous to SR, the necessary ingredients to observe logical SR are fairly generic, suggesting that this noise-induced effect could be observed in many diverse systems [2–7].

In this paper we provide theoretical results on LSR in an optical bistable system, namely, a polarization bistable laser (a polarization rotor), to demonstrate the generality of the idea over a range of systems and operating conditions. The optical system considered here has been used previously to demonstrate one-noise and two-noise SR effects [8,9]. The laser system is very sensitive to various external perturbations of optical, magnetic, and spontaneous emission fluctuations. This sensitivity of the system is exploited to study the LSR effect under various conditions.

Furthermore, by changing the intracavity birefringence of the laser another distinct regime of the polarization bistability is realized [10]. In the so-called inhibition regime, the laser dynamics is governed by two coupled polarization modes. This has considerable implications for LSR, in the sense that one can process two logic gates in parallel. Thus our proposal has the potential to enhance the power of the concept of LSR, as it suggests how higher-dimensional systems may provide additional parallel processing capability.

The outline of the paper is as follows. In Sec. II we introduce our model of a bistable polarization rotor and demonstrate LSR by multiplicative and additive noises and signals that are optical and magnetic in nature. The LSR is quantified and fundamental logic gates OR(NOR) and AND(NAND) are shown for weak input bits in the presence of noise. In Sec. III we consider another regime of the laser system and introduce the underlying coupled equations for the orthogonal polarizations. It is important to note that in this regime we show the possibility of processing two complementary logic gates in parallel by exploiting the coupled polarization dynamics. Section IV provides concluding remarks and discussion.

II. LSR IN A BISTABLE OPTICAL SYSTEM

A. Model of a bistable polarization rotor

Here we investigate LSR in the polarization dynamics of the two-dimensional vectorial bistable laser. We consider the laser system in the rotational stochastic resonance regime [9] whereby the polarization flip takes place by rotation of the linear polarization. In this regime, the laser dynamics can be described by a single variable θ characterizing the angle of the polarization vector with respect to the x axis. The time evolution of such a polarization rotor under the multiple inputs

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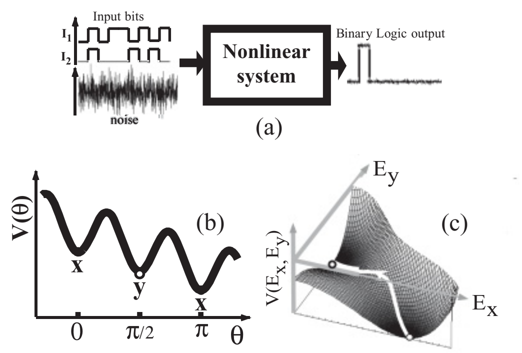

FIG. 1. (a) Schematic of LSR: The logic gate is comprised of a noisy nonlinear system driven by an input signal $I_{\text{input}} = I_1 + I_2$. (b) Optical potential $V(\theta)$ for the polarization vector. The minima along $\theta = 0,\pi$ (x axis) and $\theta = \pi/2, 3\pi/2$ (y axis) are the stable polarization states. The laser polarization can hop by external optical or magnetic perturbations (see the text for details). (c) Two-parameter potential $V(E_x, E_y)$ for the laser in the inhibition regime. Two minima along the x and y axes are visible. The two potentials in (b) and (c) correspond to two different regimes of polarization bistability.

is then given by the Langevin-type equation

$$\frac{d\theta}{dt} = -V_0 \sin 4\theta + M_0[I_{\text{in}}(t) + B + \eta(t)] \sin 2\theta + \zeta(t) + B_0, \tag{1}$$

where V_0, M_0, and B_0 are parameters characterizing the strength of laser potential, optical feedback, and magnetic field, respectively. $I_{\text{in}}(t)$ denotes an external low-amplitude input signal that drives the system.

In the absence of external perturbations, one can associate a multistable potential $V(\theta) = -(V_0/4)\cos(4\theta) + B_0\theta$. This potential, plotted in Fig. 1(b), exhibits minima along the x and y directions. The constant term B applied to the optical input is the dc bias to one stable minimum compared to the other, which plays the role of the lever, as it is described in Ref. [9]. Noises of different physical origins can be modeled by multiplicative noise $\eta(t)$ and additive noise $\zeta(t)$, both being δ-correlated white Gaussian noises of different amplitudes,

$$\langle \eta(t)\eta(t')\rangle = 2D_1\, \delta(t - t'), \tag{2}$$

$$\langle \zeta(t)\zeta(t')\rangle = 2D_2\, \delta(t - t'). \tag{3}$$

The source of the multiplicative noise $\eta(t)$ is optical due to feedback, while the source of the additive noise $\zeta(t)$ is magnetic. Note that these two noises are completely independent and only one of them is present at a time.

The laser output is the intensity detected via a polarizer, e.g., aligned along the x axis as

$$I_x(t) = I_{\max} \cos^2(\theta - \theta_{\text{polarizer}}), \tag{4}$$

with $\theta_{\text{polarizer}} = 0$. Similarly, by changing $\theta_{\text{polarizer}} = \pi/2$ one can detect a complimentary intensity $I_y(t)$. It is this intensity $I_{x,y}(t)$ that will be used to obtain the logic output.

B. Basic signatures of logical stochastic resonance

Now we would like to achieve in this system the input-output association corresponding to a two-input logic function, such as that displayed in Table I. In order to encode the external logic inputs we drive the system with an external low-amplitude signal $I_{\text{in}}(t)$, which is taken to be the sum of two trains of aperiodic pulses, namely,

$$I_{\text{in}}(t) = I_1(t) + I_2(t),$$

where $I_1(t)$ and $I_2(t)$ reflect the two logic inputs. The logic inputs can be either 0 or 1, giving rise to four distinct logic input sets (I_1, I_2): (0,0), (0,1), (1,0), and (1,1). The input sets (0,1) and (1,0) give rise to the same I_{in}, thus the four distinct input conditions (I_1, I_2) reduce to three distinct values of I_{in}. Hence the input signal $I_{\text{in}}(t)$, which is generated by adding two independent input signals, is a three-level aperiodic wave form.

The intensity, given by Eq. (4), encodes the logic output. Specifically if I_x is in the higher state it is taken to encode a logical 1 and if it is in the lower state it encodes a logical 0. Complementary gates are obtained by taking the output determination to be the opposite, namely, if I_x is in the lower state it encodes 1; otherwise it encodes 0 (see Fig. 1).

It is evident from Figs. 2 and 3 that, for a given set of inputs (I_1, I_2), a logical output from the bistable optical system is observable, in accordance with the truth tables of the basic logic operations shown in Table I. It is crucial to note that this occurs consistently and robustly only in an optimal window of noise. For very small or very large noise the system does not yield any consistent logic output; however, in a reasonably wide band of moderate noise, the system produces the desired logical outputs very reliably.

The logic response of the system can be changed by changing the dc bias (lever) in the system. Namely, changing the bias B can morph the response of the system from a robust OR(NOR) gate (middle panel of Fig. 3) to an AND(NAND) gate (bottom panel of Fig. 3).

TABLE I. Relationship between the two inputs and the output of the fundamental OR, AND, NOR, and NAND logic operations. Note that the four distinct possible input sets (0,0), (0,1), (1,0), and (1,1) reduce to three conditions since (0,1) and (1,0) are symmetric. Any logical circuit can be constructed by combining the NOR (OR the NAND) gates [11].

Input set (I_1, I_2)	OR	AND	NOR	NAND
(0,0)	0	0	1	1
(0,1) or (1,0)	1	0	0	1
(1,1)	1	1	0	0

C. Quantifying reliability

We quantify the consistency (or reliability) of obtaining a given logic output by calculating the probability of obtaining the desired output for different input sets. Specifically we calculate the probability

$$P(\text{logic}) = \frac{R_{\text{good}}}{R_{\text{total}}}, \tag{5}$$

where R_{good} is the number of successful runs and R_{total} is the total number of runs. For each run, one presents the input sets (0,0), (0,1), (1,0), or (1,1) in random order and the run is deemed successful if and only if all four input sets yield the correct logic output (within some acceptable small tolerance).

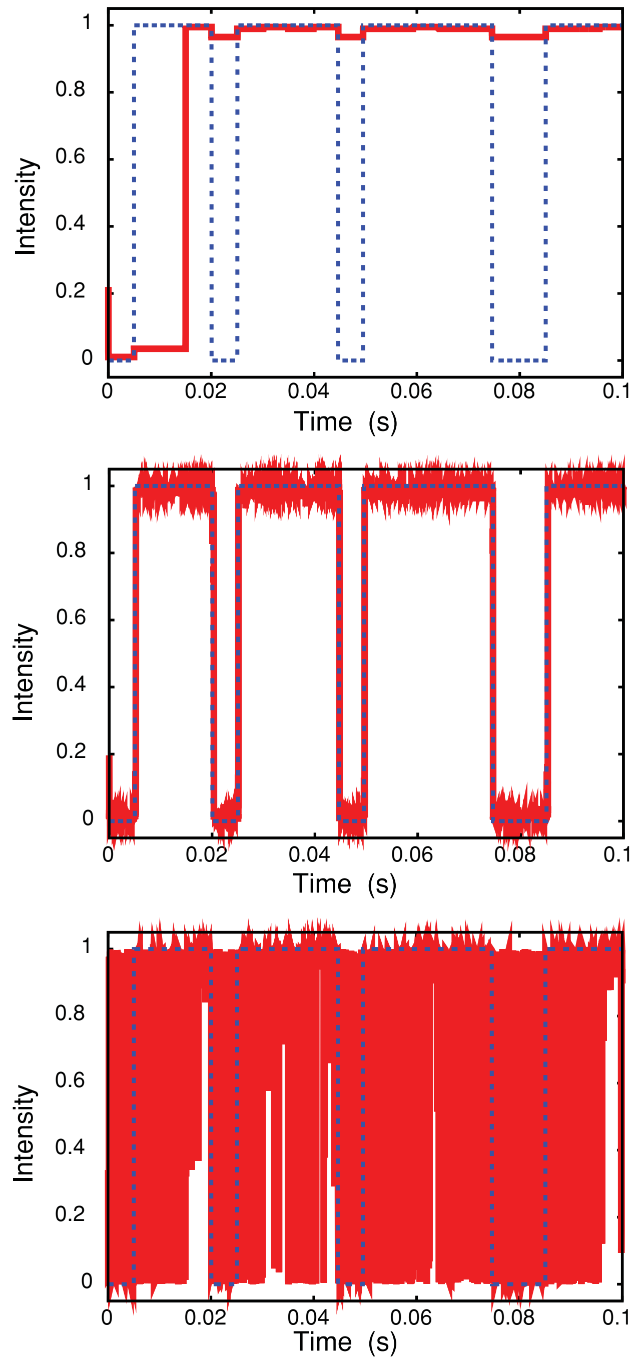

FIG. 2. (Color online) Intensity $I_x(t)$ [see Eq. (4)] for different multiplicative noise intensities (from top to bottom): 0, 5×10^{-7}, and 5×10^{-6}. The dashed line shows the desired or (OR NOR) logic output. The parameters in Eq. (1) are $B = 0.5$, $V_0 = 10^5, M_0 = 10^5$, $\theta_{\text{polarizer}} = -\pi/2$, and $I_{\max} = 1.0$ and the additive noise amplitude is 0. Specifically we take the input level to be 0.5 for logic input 1 and -0.5 for logic input 0.

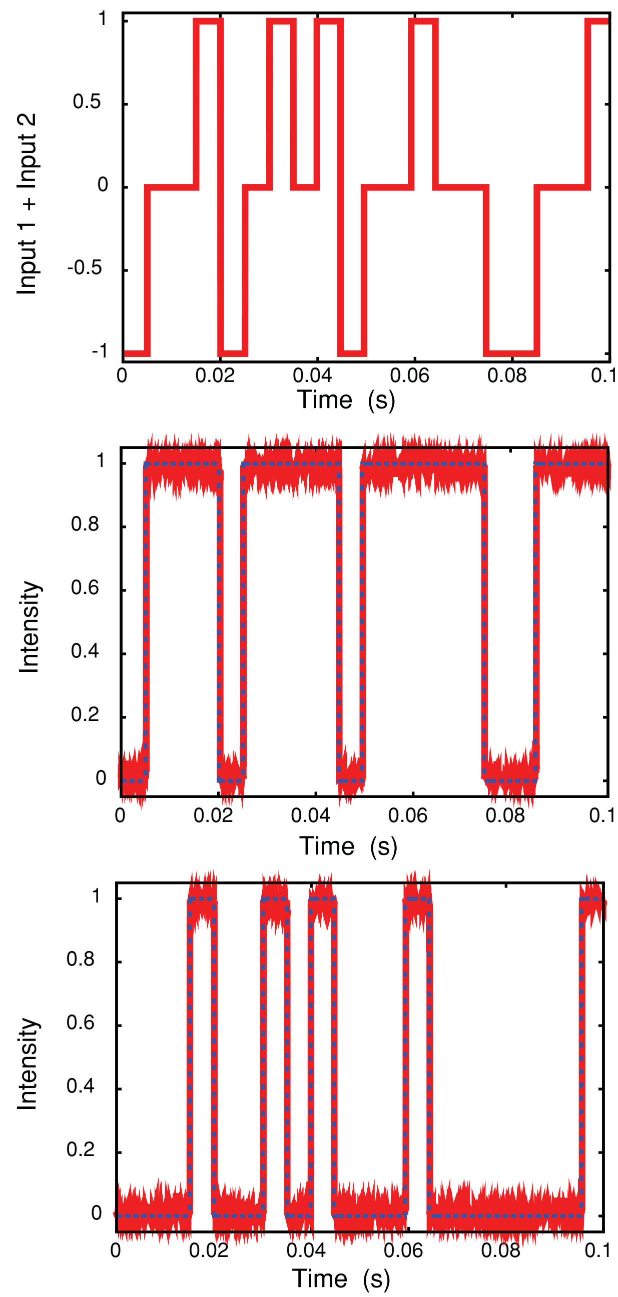

FIG. 3. (Color online) From top to bottom the panels show logic inputs $I_1 + I_2$, intensity mirroring the OR(NOR) logic response, and intensity mirroring the AND(NAND) logic response. The bias B is 0.5 for the middle panel and -0.5 for the bottom panel. The additive noise amplitude is 5×10^{-7} and the multiplicative noise amplitude is 5×10^{-7}. The parameters in Eq. (1) are $V_0 = 10^5, M_0 = 10^5$, $\theta_{\text{polarizer}} = -\pi/2$, and $I_{\max} = 1.0$ and we take the input level to be 0.5 for logic input 1 and -0.5 for logic input 0.

When $P(\text{logic})$ is 1 the logic operation is obtained completely reliably, namely, the system always yields the correct output.

Figure 4 shows this quantity obtained from numerical simulations over 10^5 different runs [12]. It is evident that the fundamental logic operation AND(NAND) (analogously, OR(NOR)) is realized consistently in an optimal band of moderate noise. The remarkable thing here is that these stable logic operations are realized (for subthreshold input signals) only in the presence of noise. More specifically, in relatively wide windows of moderate noise, the system yields logic operations with near certain probability, i.e., $P(\text{logic}) \sim 1$. Furthermore, we have also verified that analogous curves are obtained if one employs magnetic noise $\zeta(t)$ instead of optical noise $\eta(t)$, which suggests the robustness of LSR for noises of different origins.

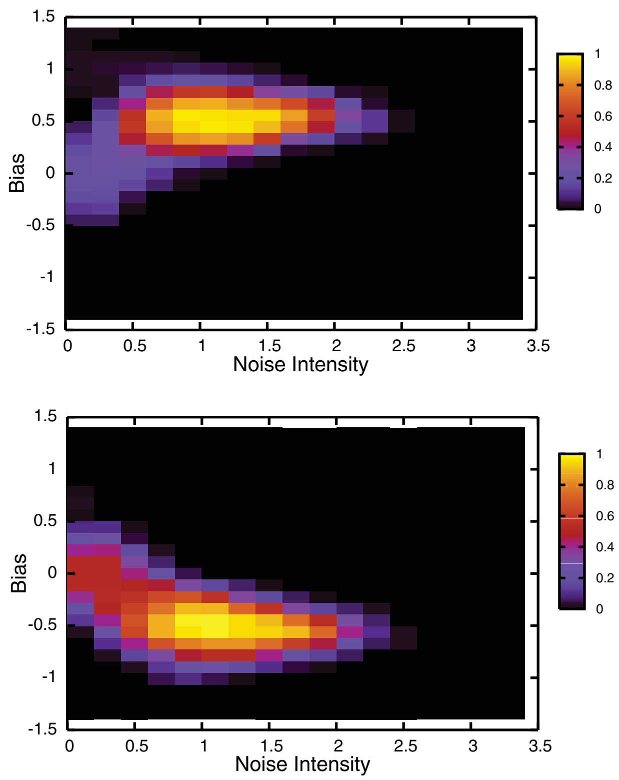

FIG. 4. (Color online) Probability of obtaining the correct logic response P(logic) for logic functions OR(NOR) (top panel) AND(NAND) (bottom panel). The x axis displays the multiplicative noise levels (in units of 10^{-6}) and the y axis displays the bias B. Specifically we take the input level to be 0.5 for logic input 1 and -0.5 for logic input 0. The logic output is 1 if $x > 0.5$; otherwise it is 0 (or vice versa for the complementary logic operation). Here the additive noise amplitude is 0. Note that very similar results are obtained for small (nonzero) additive noise amplitudes as well.

It is clear that noise plays a constructive role in obtaining a large robust response to input signals, i.e., different levels of input pulses yield a 0 or 1 output, determined by the system being in either one of the two widely separated wells. This kind of response is necessary for logic operations, as it allows one to consistently map different distinct inputs to a binary output.

The effect of the bias B in Eq. (1) (over a temporal interval longer than the noise correlation time) is also evident in Fig. 4: As the value of the bias changes the response of the system switches from OR(NOR) to AND(NAND) logic, or vice versa. This effect arises from the change in the symmetry and the depths of the potential wells due to changing B, leading to different responses.

Thus Fig. 4 shows that for a given noise intensity in the optimal range adjusting bias B will yield the desired logic behavior. It should be noted that the plateaus of enhanced performance (in the two panels of Fig. 4) overlap for the OR(NOR) and AND(NAND) performance. Hence, for a noise intensity somewhere in the plateau we can switch from NOR to NAND (and vice versa) by simply adjusting the bias B. This is equivalent to using the bias signal as a knob to tune the system to select different logic truth tables.

It should be emphasized that while the logic responses are switched by changing the bias, the desired output is obtained only for optimal noise intensities (Fig. 4), without which one would not be able to extract any significantly consistent logic response. Alternately, we can view this as a strategy to optimize the desired logic response, given a specific noise floor. This demonstration of LSR in an optical system highlights the range of applicability and extent of the concept of LSR and provides another example of the constructive role of noise in wide-ranging applications [13–16].

III. TWO-DIMENSIONAL MODEL: COMPLEMENTARY GATES IN PARALLEL

A. Coupled dynamics for orthogonal polarizations

In order to highlight the possibility of processing two logic gates in parallel in our laser system, we consider a so-called inhibition regime of the polarization switching [10]. In this regime, two frequency nondegenerate modes with orthogonal polarizations are strongly coupled such that the onset of one polarization inhibits the other one during polarization switching. The optical system is now a two-dimensional system that is subject to stochastic fluctuations by the dynamical equations

$$\frac{dE_x}{dt} = E_x\left[\alpha_x(t) - \beta_x - \theta_{xy}E_y^2\right], \quad (6)$$

$$\frac{dE_y}{dt} = E_y\left[\alpha_y - \beta_y - \theta_{xy}E_x^2\right], \quad (7)$$

where α_x and α_y include losses and characterize the net gain of the two eigenstates, β_x and β_y their self-saturation, and θ_{xy} is their cross coupling. The Langevin term $\alpha_x(t)$ represent the optical injection via the feedback and contains the two input bit streams along with the white-noise term:

$$\alpha_x(t) = \alpha_0[1 + I_1(t) + I_2(t) + B + \eta(t)].$$

The steady-state solution (without any externally injected signals) for the amplitude of the two competing eigenstates can be associated with a two-parameter potential

$$V(E_x,E_y) = -\tfrac{1}{2}\alpha_x E_x^2 - \tfrac{1}{2}\alpha_y E_y^2 + \tfrac{1}{4}\alpha_x E_x^4 + \tfrac{1}{4}\alpha_y E_y^4 + \theta_{xy}E_x^2E_y^2. \quad (8)$$

Such a potential is plotted in Fig. 1(c), which clearly exhibits two minima corresponding to two orthogonal eigenstates along the x and y directions. By varying the differential gain coefficient $\alpha_x(t)$ by means of injecting a fraction of laser light back into the cavity, switching between the two stable polarization states of the laser can be induced. Note that in this case it is possible to detect both of the complementary outputs intensities $I_x(t)$ and $I_y(t)$ simultaneously. (In fact, one can observe two complementary bistability cycles, as shown in the inset of Fig. 6.)

B. Two parallel logic gates

Analogous to the previous case, when only two bit streams $I_1(t)$ and $I_2(t)$ are present (without noise), the laser output remains locked to only one state. Such a weak signal $I_1(t) + I_2(t)$ is a three-level signal, as shown in Fig. 5(a). By adding an optimum amount of optical noise $\eta(t)$, the laser output detected through a polarizer is aligned along the x axis as shown in Fig. 5(b); it is obvious that the output signal is an OR

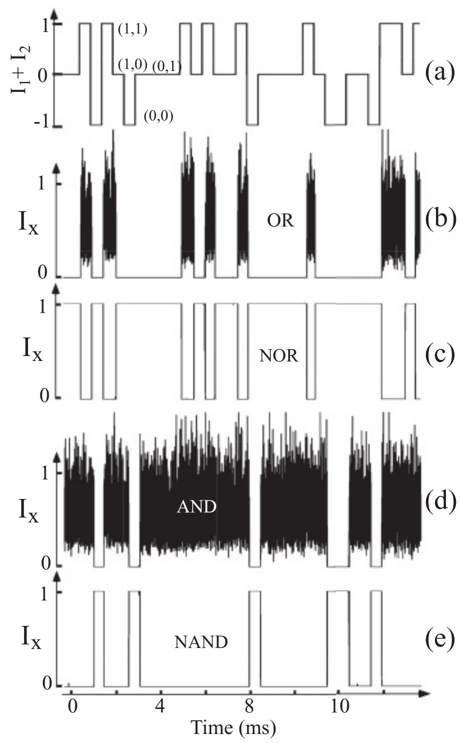

FIG. 5. Time series showing two parallel complementary gates for (a) a three-level input signal $I_1 + I_2$; (b) OR gates detected on I_x and (c) NOR gates detected on I_y, both for bias $B = 0.5$ and optical noise amplitude 4.1×10^{-5}; and (d) AND gates on I_x and (e) NAND gates on I_y, both in response to input signals in (a) for $B = -0.5$ and the same noise amplitude. The panels display intensity vs time. The noise intensity is taken at the optimum point.

gate. When the same output is simultaneously detected along the y direction a complementary logic output NOR is produced [Fig. 5(c)]. Note that according to the truth table of Table I, the system now flips only when the logic input enters the $(1,1)$ state or leaves it, which means that the system is synchronized to the upper level in the three-level input signal $I_1 + I_2$. We have quantified the reliability of the logic gate using a slightly different indicator, $P(\text{logic}) = C_{\text{good}}/C_{\text{total}}$, where C_{total} is the total number of $(1,1)$ bits in a long run and C_{good} is the number of output bits that respond to it as per the truth table. Such a definition is used to eliminate a $\frac{3}{4}$ success probability offset that would always be there if the system remains stuck to one state. Such an indicator is plotted in Fig. 6 versus the noise amplitude; it exhibits an optimum close to 1 for a given noise level.

Similarly, when one flips the sign of the bias $B = -0.5$, the output state processes two complementary gates AND and NAND when the output intensities are detected through the polarizers aligned along the x and y directions, as shown in Figs. 5(d) and 5(e). In contrast to the OR(NOR) gate, the laser output now synchronizes to the lower level of the three-level input signal. With the use of the above-mentioned definition for the reliability of a gate a similar curve for the logic gates is seen versus the noise level (Fig. 6). The optimum noise levels for the two gates are identical, thereby the control to switch from OR to AND is only by means of varying the bias of the

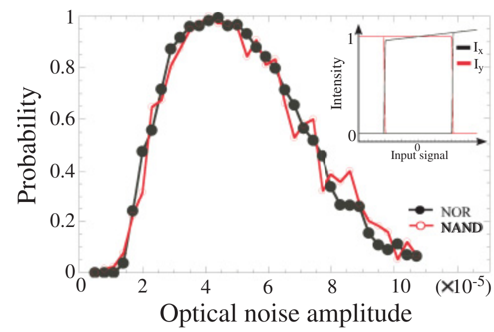

FIG. 6. (Color online) Probability of obtaining a successful gate operation (see the text for definition) vs optical noise amplitude. NOR gate, solid circles; NAND gate, open circle. Inset: Two complementary bistability cycles when the laser outputs I_x and I_y are detected simultaneously.

system. These results not only demonstrate the generality of LSR features in two completely different regimes of the laser operation but also suggest the possibility of processing two complementary gates simultaneously.

IV. CONCLUSION

In this paper we have verified the phenomenon of logical stochastic resonance, i.e., we have shown that the interplay between the noise floor and nonlinearity can indeed be exploited for the design of robust logic gates in a bistable optical system. Specifically we have shown the direct and flexible implementation of the fundamental logic gates OR(NOR) and AND(NAND) in an optimal band of noise, from which any universal computing photonic device can be constructed. Furthermore, we have demonstrated the switching of logic functions, namely, AND(NAND) to OR(NOR) and vice versa, by using a bias signal as a logic response controller. The sensitivity of the polarization rotor allows us to demonstrate the LSR using noises of various physical origins.

The plasticity and sensitivity associated with the optical system considered makes it an excellent system to explore various features of LSR and demonstrate its generic nature. Furthermore, we proposed the exploitation of the inhibition regime of the laser system in order to implement two parallel complementary logic gates simultaneously. This considerably enhances the power of the LSR concept, as it indicates how higher-dimensional systems may provide additional parallel processing capability.

In summary, we expect that our theoretical study will motivate experimental demonstrations. Further, it has the potential to lead to useful photonic devices that exploit the cooperative effects of noise and nonlinearity to optimize performance in the presence of a noise floor. Finally, we have suggested ideas for parallel logic operations that expand the scope of LSR.

ACKNOWLEDGMENT

K.P.S. would like to acknowledge financial support from the Department of Science and Technology, India.

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