Paucity of attractors in nonlinear systems driven with complex signals

Shawn D. Pethel and Jonathan N. Blakely
U.S. Army RDECOM, RDMR-WS, Redstone Arsenal, Alabama 35898, USA

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We study the probability of multistability in a quadratic map driven repeatedly by a random signal of length N, where N is taken as a measure of the signal complexity. We first establish analytically that the number of coexisting attractors is bounded above by N. We then numerically estimate the probability p of a randomly chosen signal resulting in a multistable response as a function of N. Interestingly, with increasing drive signal complexity the system exhibits a paucity of attractors. That is, almost any drive signal beyond a certain complexity level will result in a single attractor response ($p = 0$). This mechanism may play a role in allowing sensitive multistable systems to respond consistently to external influences.

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Nonlinear dynamical systems can respond in a variety of ways to repeated external input. One possibility is that the driven system is conditionally stable with respect to the external signal and the resulting response is consistent regardless of the initial state. Reproducible behavior is possible even when the undriven dynamical system is chaotic, such as in the case of generalized synchronization [1]. Consistent response has been suggested as the common element behind information transmission in biological and physiological systems and has recently been studied in the context of neuronal models and laser dynamics [2,3]. Another possibility is that the dynamical system is not stable with respect to the external signal, but is instead driven toward a chaotic attractor. Such systems are consistent in a statistical sense because, although each response depends sensitively on the initial state, all have identical long-term statistics due to ergodicity.

The situation we wish to consider here is that the driven system exhibits multistability; the phase space is divided into more than one basin of attraction. Long-term statistics will depend on the initial condition and this dependence can be highly sensitive if the basin boundaries are fractal [4]. In the context of driven systems multistability has been reported in lasers [5] and in biological processes [6]. In some cases external modulation induces multistability [7], whereas in other cases it is quenched [8]. Recently, a rigorous study of a driven logistic map reported that sinusiodal modulation tends to destroy multistability as the modulation period is increased; however, for dichotomous driving multistability is persistent in some regions [9].

In this paper we analyze multiattractors in a logistic map under generic modulation. We find that the probability of multistability depends on the complexity of the drive signal, where complexity is quantified by the length, N, of the repeated portion of the drive signal. For very simple signals coexisting attractors are common, but for more complex signals we observe a paucity of attractors. That is, the probability $p(N)$ that a randomly chosen signal results in multistability vanishes exponentially with the complexity of the drive signal. The mechanism behind the paucity of attractor phenomenon may play a role in allowing multistable dynamical subsystems to cooperate in complex systems.

We begin with a quadratic map,

$$x_{n+1} = f_n(x_n) = r x_n(1 - x_n) + \delta_n, \qquad (1)$$

to which we have added an extra term δ_n. We wish to investigate the number of attractors supported by the map with respect to a repeated driving signal of period N such that $\delta_n = \delta_{n \bmod N}$. Figure 1(a) shows an example where $N = 3$. Our notation subsumes the driving signal into the definition of the map so that the original picture of an externally driven system can be expressed as a sequence of different map functions [Fig. 1(b)]. We have found that multiplicative modulation such as in [9] does not qualitatively change the results presented here.

We recast this problem into studying the stability of the Nth composition [Fig. 1(c)]:

$$y_{n+1} = F(y_n) = f_{N-1} \circ f_{N-2} \circ \cdots \circ f_0(y_n). \qquad (2)$$

A period one orbit of (2) corresponds to a periodic solution of (1) of the same length as the drive signal. The existence and stability of such solutions relates to the issue of consistency [2]; however, in this article we focus on the general question of whether coexisting attractors are supported by (2). Our definition of an attracting set is stated in the Appendix. One exception will be attractors at infinity associated with trajectories leaving the interval. Accordingly, we restrict the motion of (2) to the [0,1] interval by enforcing $0 \leqslant \delta_n \leqslant 1 - r/4$.

By fixing r and restricting δ as described, we are defining a family of maps [Fig. 2(a)]. The driven system we are studying is equivalent to a composition of N random selections from this family. A constant drive signal, or $N = 1$, is just a single one of these maps. So long as a drive signal of length N is also of prime period N—that is, it is not decomposable into repeating parts shorter than N—the number of attractors of (1) and (2) are equivalent (see also [9]). Such repetitions have almost no probability of happening in a randomly generated drive signal.

Each map has only one attractor, but compositions of them may support more, including ones with fractal basin boundaries [10]. Figure 2(b) is a sampling of the possible responses to a drive signal consisting of $N = 2$ repeated (δ_0, δ_1) pairs. The two grayscale ranges indicate different attracting sets. Where only one grayscale is seen there is a single limit set with the basin of attraction covering the entire interval. Roughly half the parameter space is occupied by two-attractor systems, being either periodic or chaotic with any combination

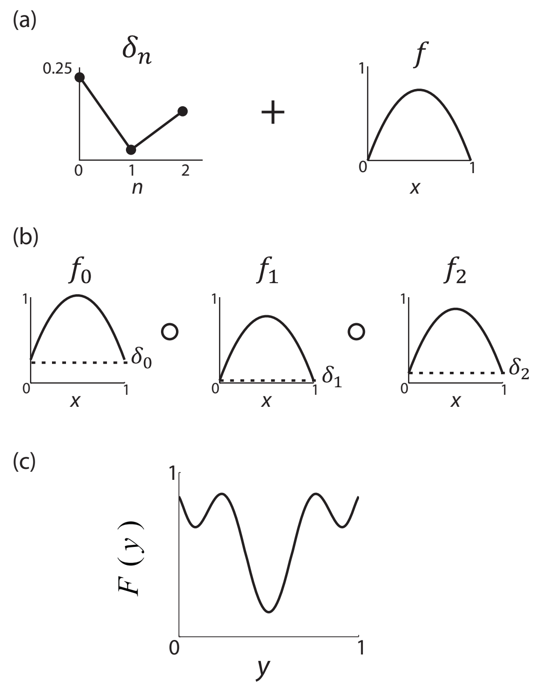

FIG. 1. Three equivalent systems. (a) Quadratic map driven by additive signal δ_n, (b) consecutive applications of quadratic maps each shifted by δ_n, and (c) the composition of all the individual maps.

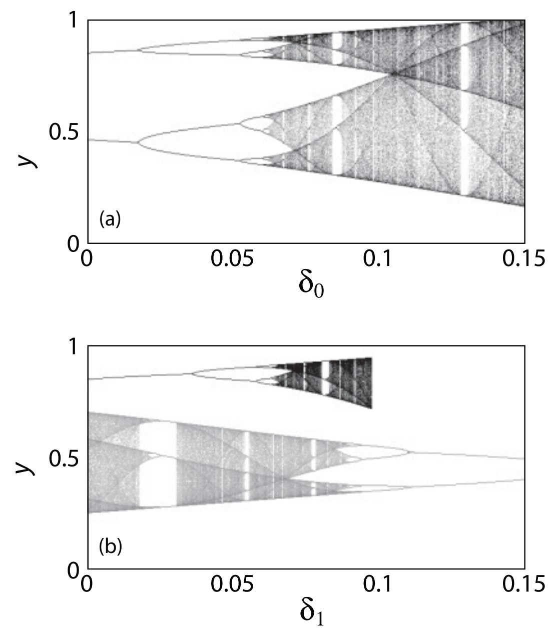

FIG. 2. Attractors as a function of driving signal amplitudes for $r = 3.4$. (a) For $N = 1$ the map has a single attractor for each drive signal level δ_0. (b) $N = 2$, δ_0 is fixed at 0.07 and δ_1 is scanned over its full range. The two grayscales indicate coexisting attractors. For $\delta_1 > 0.1$ there is only one attractor.

possible. The situation of multiple attractors has also been noted in a driven Lorenz system [11].

What bounds are there on the number of attractors versus N? Also, what is the probability of encountering multiattractors given a random choice of drive signal of length N? This second topic is equivalent to calculating the fill fraction of multiattractor systems in parameter space. Since the parameter space is N-dimensional we take a statistical approach to this question.

For maps with negative Schwarzian derivative, the first question has an analytic answer in terms of critical points. The property of everywhere negative Schwarzian derivative

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0 \tag{3}$$

is satisfied for the family of quadratic maps studied here and for all compositions of them. Almost all smooth one-dimensional maps reported in the literature share this property. Under this assumption, Singer [12] showed that there are a finite number of stable periodic attractors, each of which attracts the iterates of some critical point. As a result, the number of periodic attractors cannot exceed the number of critical points. In the Appendix we show that the trapping region of a chaotic attractor contains at least one critical point as well. Together, these results imply that the total number of attractors of F is bounded by the number of critical points.

A critical point x^* marks the location of an extremum: $F'(x^*) = 0$. Using the chain rule we can expand $F'(x)$ as a product of the derivatives of its constituent maps:

$$\frac{dF}{dx} = \left.\frac{df_{N-1}}{dx}\right|_{f_{N-2}\circ\cdots\circ f_0(x)} \cdots \left.\frac{df_1}{dx}\right|_{f_0(x)} \cdot \left.\frac{df_0}{dx}\right|_x. \tag{4}$$

In our case, each f_n has a single extremum at 1/2. From this we can infer that the extrema of $F(x)$ take on one of N different values: $f_{N-1}(1/2)$, $f_{N-1} \circ f_{N-2}(1/2), \ldots, f_{N-1} \circ \cdots \circ f_0(1/2)$. Since all the critical points map to at most N different critical *values* (peak/valley heights), the total number of attractors is bounded above by N. We can trivially construct a system that achieves this upper bound by taking the Nth composition of a map with a stable period N orbit. The resulting composition has N stable fixed points. In general, the number of attractors of a stable period P map composed N times with itself is the greatest common denominator of N and P. Note that this construction is consistent with our stated upper bound because $\gcd(N,P) \leqslant N$. This connection between attractors and critical values makes possible a drastic simplification in the upcoming attractor counting algorithm.

Although the bound allows for an increasing degree of multistability as $N \to \infty$, what we find in practice is the opposite behavior: The likelihood p that a randomly chosen system (2) hosts multiple attractors drops rapidly with increasing signal complexity N. In what follows we use the relationship between critical points and attractors as the basis of an algorithm to estimate p as a function of the signal complexity.

We take a statistical approach to estimating the probability of finding multiple attractors versus N. To isolate the effect of increasing the complexity, N, we use a large ensemble of uniformly random signals, each with the same rms power as that expected for an infinite random sequence. For each random signal we construct the composition map $F(x)$ and compute its critical values. The next task requires testing each member of the ensemble for the presence of multiple attractors. Based on the prior analysis, we need only follow the N critical values, since all attractors attract at least one member of this set. About each critical value we place an interval of near machine precision width (10^{-15}) and propagate each interval for 4000 iterations of F. The intervals are not allowed to contract below their initial width. The resulting N sets of 4000 intervals are intersected with one another to determine which belong to the same limit set. The use of interval arithmetic serves two purposes: It eliminates the problem of round-off error giving multiple values to the same limit set and it greatly facilitates the detection of chaotic sets as the intervals grow exponentially in that case. We find that this algorithm provides reliable detection of multiple attractors in most cases; however, it can overcount when convergence to the limit set is extremely slow (critical slowing). This issue can be minimized by increasing the number of iterations. The charts that follow were generated using several months of CPU time.

We denote the probability of the occurrence of multiattractors as p and define it to be the ratio of the number of confirmed multiattractor systems over the total number of systems tested. The size of each ensemble is adjusted to achieve reasonable confidence intervals about p. For the data presented here the ensembles range from a few hundred to several million.

In all cases we see the same qualitative result: For small N multiattractors are not uncommon, but as N increases $p(N)$ drops steeply. For $N \leqslant 15$ there is significant fluctuation in $p(N)$ depending on whether N is even or odd; the dominance of even period orbits causes enhanced multistability for even period driving signals over odd period ones. Beyond $N = 15$ this difference becomes small. On a log plot the p values fall on a fairly straight line for $N > 15$, implying an exponential relationship characterized by the slope Γ (Fig. 3). This parameter could be viewed as an extinction coefficient for multiattractors and when it is negative it serves as a mathematical definition of the paucity of attractors phenomenon. In Fig. 4 we show the results of repeating the procedure for estimating $\Gamma(r)$ for $2.8 \leqslant r \leqslant 3.6$ in increments of 0.01. We find $\Gamma < 0$ over this range of r, suggesting that the paucity phenomenon is universal in this system. Estimating $\Gamma(r)$ for $r > 3.6$ is numerically difficult due to the very small probability of multistability.

The magnitude of the paucity parameter Γ tends to be greatest when the families are dominated by chaotic maps (inset, Fig. 4) possibly because chaotic attractors are more space filling. Plotting the average Lyapunov exponent versus r for each ensemble bears out this general trend. Prior graph theoretic results show that random compositions of chaotic unimodal maps cannot support more than one attractor [13]; however, we do not reach this ideal due to the presence of a dense set of periodic windows in this system [14]. We note that most known chaotic physical systems are structurally unstable and display a similar dense set of periodic windows [15].

More complex drive signals tend to result in attractors with greater Lyapunov exponents. A comparison of the mean Lyapunov exponents for $r = 3.1$ and $r = 3.4$ versus N is shown in Fig. 5. Unlike Fig. 4, the mean exponent is not for the underlying map family, but for the attractors resulting

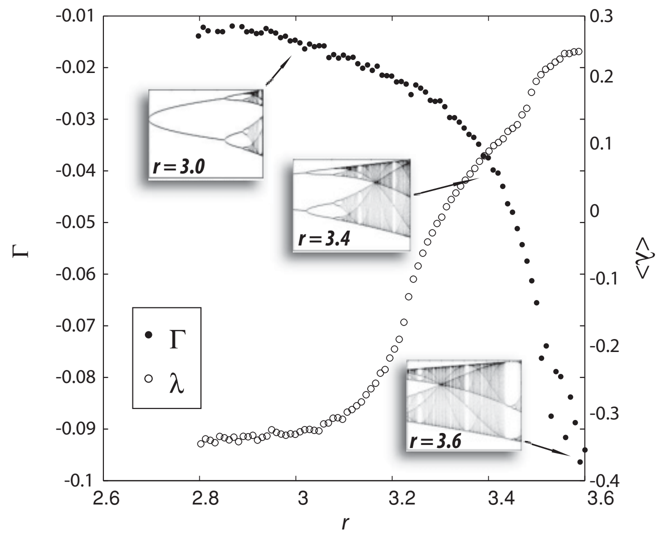

FIG. 4. The paucity parameter Γ and the mean Lyapunov exponent λ plotted versus r. Each value of r defines a family of maps (1) where $0 \leqslant \delta_n \leqslant 1 - r/4$. For each map family the probablity $p(N)$ of multiattractors for $15 < N \leqslant 45$ is computed and the corresponding paucity parameter Γ plotted (solid circles). The open circles show the mean Lyapunov exponent for that particular family of maps. The insets depict the family of maps ($N = 1$) just as in Fig. 2(a), but for the indicated r values.

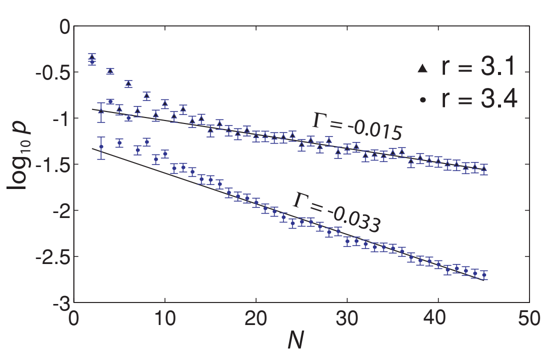

FIG. 3. (Color online) Log of the probability of multiattractors as a function of signal complexity N. Cases $r = 3.1$ and $r = 3.4$ are shown along with an estimate of the slope Γ for $N \geqslant 15$. The error bars indicate 95% confidence intervals.

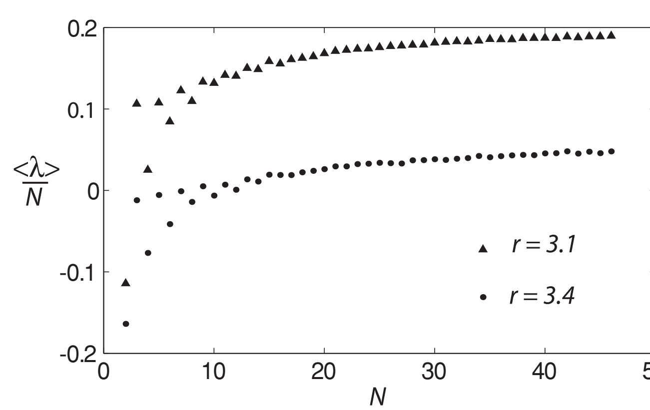

FIG. 5. The mean Lyapunov exponent $\langle\lambda\rangle$ normalized by N plotted versus N for $r = 3.1$ and $r = 3.4$. For each value of r and N 8000 instances of randomly driven maps were generated and the Lyapunov exponent of resulting attractors computed. The Lyanpunov exponent per time step increases with the drive signal length, indicating that more complex drive signals tend to push the dynamics more often into less stable regions of the underlying map.

from N random compositions of them averaged over an ensemble of 8000. The exponent is normalized by N to allow comparison of different length drive signals. Even when the map family is dominated by stable maps, the composition of them can be chaotic. Increasing N increases the probability of a space-filling chaotic response, indicating that more complex drive signals tend to push the dynamics into highly unstable regions of the underlying map.

In summary, there is a relationship between the existence of multistability and the complexity of the drive signal in driven one-dimensional maps. In particular, we show that smooth maps with negative Schwarzian derivative have a tight upper bound on the number of attractors they can support given the drive signal complexity. By following the iterates of critical values we are able to exhaustively explore this relationship in a driven quadratic map. It is possible to generalize the approach here to higher-dimensional maps; critical points become critical curves [16], although the analysis is likely to be difficult to automate. For continuous time systems different methods would need to be employed for the exhaustive detection of attractors.

Given the remarkable success of quadratic maps in capturing the rich dynamical behavior of a wide range of systems, we expect that the reported behavior also appears elsewhere. The predominant feature uncovered here is the paucity of attractors phenomenon characterized by an exponential decline in the probability of multistability with the complexity of the drive signal. The significance of this behavior is that it is a prerequisite for consistent response between dynamical systems, and therefore it is one mechanism that could allow highly complex systems—possibly rife with fractal basins of attraction—to signal each other in a reliable way.

APPENDIX

Definition. A closed, connected set $\mathbf{U}$ is called a *trapping region* if $\mathbf{F}^j(\mathbf{U}) \subset \mathbf{U}$, $j > 0$. Then

$$\mathbf{A} \overset{\text{def}}{=} \bigcap_{j \geqslant 0} \mathbf{F}^j(\mathbf{U})$$

is an *attracting set* [17].

Let $\mathbf{I}$ be an interval and $\mathbf{F} : \mathbf{I} \to \mathbf{I}$ be at least C^1 smooth. Consider the case that there is an attracting chaotic set $\mathbf{A}$ with an associated trapping region $\mathbf{U} \subset I$. From Devaney's definition of chaos [18] we know that unstable periodic orbits are dense in $\mathbf{A}$, or equivalently, for j large enough $\mathbf{F}^j(x)$, $x \subset \mathbf{U}$ has arbitrarily many fixed points where $|d\mathbf{F}^j/dx| > 1$.

Theorem. There is a critical point of $\mathbf{F}$ in $\mathbf{U}$.

Proof. Let $a < b$ be consecutive fixed points of $\mathbf{F}^j$ in $\mathbf{U}$. Continuity of the first derivative and the condition of instability requires $d\mathbf{F}^j/dx$ to have opposite signs at points a and b. Therefore, there is a point in $[a,b]$ where the derivative of $\mathbf{F}^j$ passes through zero (a critical point). It is straightforward to show that critical points of $\mathbf{F}^j$ are preimages of the critical points of $\mathbf{F}$. Since there is a preimage of a critical point of $\mathbf{F}$ in $\mathbf{U}$ and $\mathbf{U}$ is a trapping region, it follows that there is at least one critical point of $\mathbf{F}$ in $\mathbf{U}$.

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