Minimal model of financial stylized facts

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In this work we propose a statistical characterization of a linear stochastic volatility model featuring inverse-gamma stationary distribution for the instantaneous volatility. We detail the derivation of the moments of the return distribution, revealing the role of the inverse-gamma law in the emergence of fat tails and of the relevant correlation functions. We also propose a systematic methodology for estimating the parameters and we describe the empirical analysis of the Standard & Poor's 500 index daily returns, confirming the ability of the model to capture many of the established stylized facts as well as the scaling properties of empirical distributions over different time horizons.

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I. INTRODUCTION

A large number of empirical studies has shown that financial time series exhibit statistical features strongly departing from the Gaussian behavior. This finding dates back to the work of Mandelbrot [1] whose attention was mainly focused on recognizing realizations of stable processes, and to the analysis of Fama [2] concerning the long-tailed nature of the Dow Jones Industrial Average single components. Since these fundamental contributions, the modeling of financial returns has grown considerably and very heterogeneous models able to reproduce the degree of asymmetry and the excess of kurtosis of the measured distributions have been proposed. A nonexhaustive list includes approaches developing from specific distributional assumptions, as it is the case of the Lévy flights [3–5], the generalized Student-t, or Tsallis distributions [6–8] and the exponential one [9]. Past empirical analysis have also proved the existence of nontrivial scalings of higher order correlations between returns at different times, pointing toward the existence of a secondary stochastic process as fundamental as that of the price governing the volatility of returns. Many effective mechanisms allowing us to reproduce the observed correlation structures where the stochastic nature of the volatility plays a central role were proposed. Discrete time models include AutoRegressive Conditional Heteroskedasticity (ARCH) and Generalized ARCH (GARCH) processes [10,11] and multifractal models [12,13] inspired by the cascades originally introduced by Kolmogorov in the context of turbulent flows. As far as continuous time approaches are concerned, fractional Brownian motion and stochastic volatility models have been extensively analyzed. For a review of the latter approach we suggest [14] and the discussion in Sec. II. Focusing on the continuous time stochastic volatility framework, in this work we aim at reproducing many of the above mentioned facts which are generally accepted as universal evidences shared among different markets in different times.

The structure of the paper is the following. After introducing a general class of stochastic models driving the evolution of the volatility, in Sec. II we concentrate on a linear one able to reproduce an inverse-gamma distribution in the long run. In Sec. III we detail the derivation of the moments of the probability density function $p(x;t)$ of the returns over the time lag t, taking into account explicitly the time at which the secondary process has started and rigorously deriving the stationary limit of the volatility. We describe the mechanism through which the power law distribution of σ induces fat tails on $p(x;t)$ for all the finite time lags. In Secs. IV and V we derive the analytical expressions of the leverage correlation and the volatility autocorrelation functions, respectively. In Sec. VI we propose a systematic methodology for estimating the model parameters, and we apply it to the time series of the daily returns of the Standard & Poor's 500 index. The relevant conclusions, along with possible perspectives, will be summarized in Sec. VII.

II. THE MODEL

We consider a model where the asset price

$$S_t = S_0 \exp(\mu t + X_t)$$

is a function of the stochastic centered log-return X_t and μ is a constant drift coefficient. We assume that X_t can be modeled with the following stochastic differential equation (SDE):

$$dX_t = \sigma_t \, dW_{1,t}, \tag{1}$$

where σ_t is the instantaneous volatility of the price and $dW_{1,t}$ is the increment of a standard Wiener process. Since $X_0 = 0$ from the above assumption we have that $\langle X_t \rangle = 0$ and $\langle \ln(S_t/S_0) \rangle = \mu t$ for all t. In the context of stochastic volatility models (SVMs) the instantaneous volatility is assumed to be a function of an underlying driving process Y_t, that is, $\sigma_t = \sigma(Y_t)$. Typically the dynamics chosen for Y_t corresponds to a particular case of the following general multiplicative diffusion process:

$$dY_t = (aY_t + b)\, dt + \sqrt{c\, Y_t^2 + d\, Y_t + e}\, dW_{2,t}, \tag{2}$$

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with suitable constraints on the parameters in order to ensure the well definiteness of the process. Moreover, the two standard Wiener processes $W_{1,2}$ are possibly correlated:

$$\langle dW_{1,t_1}\, dW_{2,t_2}\rangle = \rho\, \delta(t_1 - t_2)\, dt, \tag{3}$$

with $\rho \in [-1,1]$, which is necessary to account for skewness effects and for the return-volatility correlation. For instance, in the Stein-Stein model [15,16] the volatility is linear, $\sigma_t \propto Y_t$, and Y_t follows a mean reverting Ornstein-Uhlenbeck dynamics corresponding to $a < 0$, $b > 0$, $c = d = 0$. Under the same Y dynamics but with $\sigma_t \propto \exp(Y_t)$ we obtain the exponential Ornstein-Uhlenbeck model [17,18]. In the Heston model [19,20] $\sigma_t = \sqrt{Y_t}$ and Y_t evolves according to a Cox-Ingersoll-Ross dynamics, stemming from (2) by taking $a < 0$, $b > 0$ with $c = e = 0$. Finally, in the Hull-White model the volatility has the same functional dependence as in Heston, but Y_t has a log-normal (nonmean reverting) dynamics corresponding to $a > 0$ and $b = d = e = 0$.

In the econophysics literature several studies have been devoted to assessing the statistical properties of the volatility (see for instance Chap. 7 in [6] and [21]), especially its distribution, and it has been recognized that the instantaneous volatiliy, measured by suitable proxies, distributes in good agreement with a log-normal or an inverse-gamma law. The best fit being obtained with the latter [6] which is able to better capture the heavy tail of the empirical distribution. None of the previously cited models feature an inverse-gamma probability density function (PDF) for σ_t, even though this distribution has been considered previously in different contexts. For instance, the inverse-gamma was introduced in the context of an ARCH-like evolution of the variance in [22], and in the statistical modeling of financial data the marginalization of normally distributed returns conditionally on inverse-gamma variance was widely exploited since it generates generalized Student-t distributions (see [23,24]). However, as clarified by the empirical analysis performed in [21], where intraday returns are used to estimate a proxy for the daily volatility, an inverse-gamma PDF for σ_t^2 leads to an overweighting of the tail region.

Here we propose the statistical characterization of the simplest linear SVM able to account for this stylized fact about the volatility. The process (2) has been extensively studied and characterized in [25] where exact solutions for the moments of the associated PDF have been obtained allowing us to study its relaxation modes toward a stationary distribution, if any. In particular, when $a < 0$ and $d = e = 0$, with $c > 0$, process (2) has indeed an inverse-gamma stationary distribution, whose support is $[0, +\infty)$ as long as $b > 0$. Thereby we consider the following SVM:

$$\begin{aligned} dX_t &= \sqrt{c}\, Y_t\, dW_{1,t}, \quad X_0 = 0,\\ dY_t &= (aY_t + b)\, dt + \sqrt{c}\, Y_t\, dW_{2,t}, \quad Y_{t_0} = y_{t_0}, \end{aligned} \tag{4}$$

where $t_0 \leqslant 0$, y_{t_0} may be a fixed constant or randomly sampled, and the constant factor in the expression of the instantaneous volatility $\sigma_t = \sqrt{c}\, Y_t$ has been added for later convenience. As explained in [25] the stationary PDF of σ_t is

$$\Pi_{\rm st}(\sigma) = \frac{\lambda^\nu}{\Gamma(\nu)} \frac{\exp(-\lambda/\sigma)}{\sigma^{\nu+1}}, \tag{5}$$

where the shape parameter ν and the scale parameter λ are given by

$$\nu = 1 - \frac{2a}{c} \quad \text{and} \quad \lambda = \frac{2b}{\sqrt{c}}. \tag{6}$$

III. EMERGENCE OF FAT TAILS

A major point to be discussed before presenting a detailed derivation of our results is the different role played by the initial time conditions for the X and Y processes. Since X_t represents the detrended logarithmic increment of the price over the time lag t, it can be directly measured from real time series, and in a natural way we can assume as starting point for this process the spot time $t = 0$. On the other hand, the secondary process cannot be observed directly but some of its statistical properties have been measured by means of suitable proxies. In particular, for intraday frequencies there is no clear evidence of mean reversion, that is, the high frequency volatility is very close to its asymptotic value [26,27]. In order to capture this evidence, we assume that the process Y, driving the returns from 0 to t, started in the past at $t_0 < 0$ and we will perform the limit $t_0 \to -\infty$ at the end. The assumption of stationarity for the σ_t process in (1) allows us also to consider the returns dX_t as identically distributed and uncorrelated, even though not independent variables, by virtue of the *i.i.d.* property of the Wiener increments.

The structure of the model (4) allows us to compute the moments of the PDF of X_t at all times t recursively. Application of the Itô Lemma to the function X_t^n readily provides

$$\langle X_t^n \rangle = \frac{1}{2} n(n-1) c \int_0^t \langle X_s^{n-2}\, Y_s^2 \rangle\, ds,$$

and the same Lemma proves that the correlation functions between X and Y satisfy the following differential equation:

$$\begin{aligned} \frac{d}{dt}\langle X_t^p Y_t^q\rangle &= F_q \langle X_t^p Y_t^q\rangle + A_q \langle X_t^p Y_t^{q-1}\rangle\\ &\quad + c\,\rho\, p\, q\, \langle X_t^{p-1} Y_t^{q+1}\rangle\\ &\quad + \frac{1}{2} p(p-1) c\, \langle X_t^{p-2} Y_t^{q+2}\rangle, \end{aligned} \tag{7}$$

where we defined $F_k = ka + k(k-1)c/2$, $A_k = kb$ for every $k \in \mathbb{N}$, and $p,q \in \mathbb{N}$. The previous equation is a linear ordinary differential equation (ODE) for every p and q, which can be solved recursively starting from the lowest order of p and q,[1] and whose solution involves integration of the moments $\langle Y_t^n \rangle \doteq \mu_n(t;t_0)$ of the Y process. For every n and every time t the latter can be expressed as a linear superposition of exponential functions

$$\mu_n(t;t_0) = \sum_{j=0}^{n} K_j^{(n)} \exp\left[F_j(t-t_0)\right]. \tag{8}$$

[1]It is worth mentioning that a similar equation holds for the more general dynamics (3) after defining the volatility as $\sigma_t = \sqrt{c\,Y_t^2 + d\,Y_t + e}$.

The explicit expressions of the coefficients in the above expansion can be computed as explained in [25], and it turns out that $K_j^{(n)}$ involves the values $\mu_k(t_0;t_0)$ for $k = 1,\ldots,j$, while $K_0^{(n)}$ does not. This implies that whenever the constants F_j are all negative, the only term surviving in the limit $t_0 \to -\infty$ is $K_0^{(n)}$ and the process loses all information about the distribution of y_{t_0}. It is worth noticing that even though the moments $\mu_n(t;t_0)$ are homogeneous functions of time when t_0 is finite this is not true for the solution of Eq. (7) which is obtained by integration from 0 to t with boundary condition $\langle X_0^p Y_0^q\rangle = 0$ for every $p > 0$.[2]

From the analysis of Eq. (7) it can be verified that the moments of X can always be expressed as a superposition of exponential functions of the starting time of the volatility as follows:

$$\langle X_t^n\rangle = \sum_{j=0}^{n} H_j^{(n)}(t)\,\exp(-F_j t_0). \tag{9}$$

The coefficients $H_j^{(n)}$ depend on the time lag t and, more precisely, by virtue of the linearity of the ODEs (7) they correspond to a combination of exponential terms weighted by polynomial functions of t. In Appendix A we report the explicit expressions of the coefficients $H_j^{(n)}(t)$ for the cases $n = 2$ and $n = 3$, from which it can be readily verified that the skewness of the PDF converges to zero asymptotically for $t \to +\infty$. A messy calculation would show that an analogous behavior holds for kurtosis. Thus the scaling of the lowest order moments is in full agreement with the one of the empirical distributions over long time horizons [3,6]. When t is finite the coefficients $H_j^{(n)}$ are finite quantities themselves, and all the relevant information about the behavior of $\langle X_t^n\rangle$ in the stationary limit of Y is retained by the t_0 exponentials in Eq. (9). Two cases are possible here: if all the F_j are negative ($j \neq 0$), $\langle X_t^n\rangle$ is finite in the stationary limit $t_0 \to -\infty$, otherwise it diverges[3] indicating the emergence of fat tails in the PDF of X_t. The latter case applies when $n > \nu = 1 - 2a/c$, as can be checked directly from the definition of F_n. Since $F_{n+1} > F_n$, when $F_n > 0$, the divergence of $\langle X_t^n\rangle$ implies the divergence of all the higher order moments.[4] The same condition is responsible for the divergence of the moments $\mu_n(t)$ of the volatility for $n > \nu$ [see Eq. (8)] in agreement with the fact that the stationary distribution of the volatility (5) is an inverse-gamma distribution with tail index ν. Here we see at work a mechanism in which the power law tail of the stationary distribution of the volatility induces fat tails in the return distribution for every time lag t, and its scaling for large $|x|$ is compatible with a power law assumption

$$p(x) \underset{x\to\pm\infty}{\sim} \frac{1}{|x|^{1+\beta}}\,.$$

[2]From now on we will drop the dependence on t_0 from the moments μ_n.

[3]Since $F_j \neq F_k$ for every $j,k > 1$ with $j \neq k$, no cancellation of the divergent terms can take place in the limit $t_0 \to -\infty$.

[4]The case $\rho = 0$ represents an exception since, due to symmetry arguments, all the odd moments vanish identically.

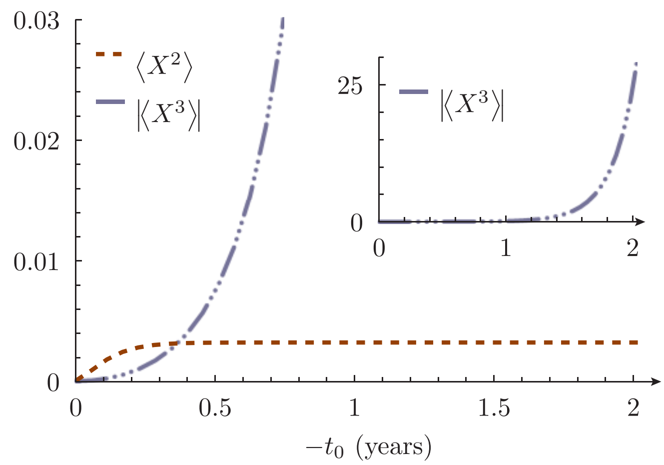

FIG. 1. (Color online) Scaling as a function of t_0 of the second and third moment of X at $t = 1$ day for $a = -16.06$ yr, $b = 0.86$ yr, $c = 17.84$ yr, and $\rho = -0.51$, $|a|/c = 0.6$. Yearly units (1 yr = 250 trading days).

This is in agreement with empirical studies about the distribution of returns over daily or intraday time scales [3,6,28–30], and from the previous considerations we are able to constrain the tail index in the following range:

$$n^* < \beta \leqslant n^* + 1, \tag{10}$$

where $n^* > 0$ is the largest integer satisfying $n^* < \nu$. As an example, in Fig. 1 it is shown the scaling of $\langle X_t^2\rangle$ and of the absolute value of $\langle X_t^3\rangle$ as a function of the starting time of the volatility for $t = 1$ day and for a choice of the parameters corresponding to $|a|/c = 0.6$. For this value of the ratio the tail index of the return distribution is $2 < \beta \leqslant 3$ and consequently the third moment of the stationary distribution of the volatility diverges as t_0 becomes more and more negative, while $\langle X_t^2\rangle$ approaches its finite stationary value.

IV. LEVERAGE EFFECT

For the linear model (4) the leverage measuring the correlation between returns and volatility can be computed exactly. Since the squared increment dX^2 provides an estimation of the instantaneous volatility, it can be defined through the following function:

$$\mathcal{L}(\tau;t) = \frac{\langle dX_t\, dX_{t+\tau}^2\rangle}{\langle dX_t^2\rangle^2}. \tag{11}$$

Empirically, for arbitrary t, $\mathcal{L}(\tau;t)$ is found to be exponentially decaying for positive τ and approximately zero otherwise, meaning that a correlation exists between past returns and the volatility in the future and not *vice versa*. Empirical analysis shows that it is a short-range correlation; more precisely, the decay time of $\mathcal{L}(\tau;t)$ is found to be of approximately 69 days for U.S. stocks and even smaller, about 10 days, for indexes [6].

The numerator (11) can be rewritten as

$$\langle dX_t\, dX_{t+\tau}^2\rangle = c^{3/2}\langle \zeta_{1,t}\, Y_t\, Y_{t+\tau}^2\rangle dt^2,$$

expressing the Wiener increment as $\zeta_t\, dt$, where ζ_t is a Gaussian noise with zero mean and $1/dt$ variance. The

Novikov theorem [31,32] allows us to compute the expectation involving $\zeta_{1,t}$, giving us

$$\langle dX_t\, dX_{t+\tau}^2\rangle = 2\rho\, c^2 H(\tau)\exp(a\tau) \times \langle Y_t^2\, Y_{t+\tau} \exp[\sqrt{c}\Delta_t W_2(\tau)]\rangle,$$

where we defined $\Delta_t W(\tau) \doteq \int_t^{t+\tau} dW_s$. We took into account the correlation structure (3) and we used the following expression of the functional derivative of Y:

$$\frac{\delta Y_{t+\tau}}{\delta\zeta_{1,t}} = \rho\, \frac{\delta Y_{t+\tau}}{\delta\zeta_{2,t}} = \rho\sqrt{c}H(\tau)\exp(a\tau)Y_t \exp[\sqrt{c}\Delta W_{2,t}(\tau)],$$

with the Heaviside step function $H(\tau)$ defined as zero if $\tau \leqslant 0$ and one otherwise. The expectation $f(\tau;t,Y) \doteq \langle Y_t^2\, Y_{t+\tau} \exp[\sqrt{c}\,\Delta_t W_2(\tau)]\rangle$ satisfies an integral Volterra equation of the second kind, whose derivation is detailed in Appendix B, and the final expression of the leverage correlation reads

$$\mathcal{L}(\tau;t) = \frac{2\,\rho\, H(\tau)}{\mu_2(t)^2}\left\{\left[\mu_3(t) + \frac{b}{a+c}\,\mu_2(t)\right] \times \exp\left[\left(2a + \frac{3}{2}c\right)\tau\right] - \frac{b}{a+c}\,\mu_2(t) \times \exp\left[\left(a + \frac{c}{2}\right)\tau\right]\right\}, \tag{12}$$

which inherits the explicit dependence on t from the moments of Y. In order to compare the previous expression with real data, following the discussion at the beginning of Sec. III, we take the limit $t_0 \to -\infty$ so that we can replace $\mu_2(t)$ and $\mu_3(t)$ with their asymptotic values, whose general expression, valid for $n < \nu$, is

$$\mu_{n,\mathrm{st}} = K_0^{(n)} = \prod_{k=1}^{n}(-1)^k\,\frac{A_k}{F_k}. \tag{13}$$

Substitution in Eq. (12) reveals that the first term vanishes and the leverage correlation reduces to

$$\mathcal{L}(\tau) = -\rho\, H(\tau)\,\frac{a(2a+c)}{b(a+c)}\exp\left(-\frac{\tau}{\tau^{\mathcal{L}}}\right), \tag{14}$$

where the leverage decay time reads

$$\tau^{\mathcal{L}} = \frac{2}{2|a|-c}.$$

So, the model correctly forecasts the exponential decay of $\mathcal{L}(\tau)$ and its vanishing for negative correlation times.

V. VOLATILITY AUTOCORRELATION

The volatility autocorrelation provides an estimate of how much the volatility at time $t+\tau$ depends on the value it had at time t and it is usually defined as

$$\mathcal{A}(\tau;t) = \frac{\langle dX_t^2\, dX_{t+\tau}^2\rangle - \langle dX_t^2\rangle\langle dX_{t+\tau}^2\rangle}{\sqrt{\mathrm{Var}[dX_t^2]\,\mathrm{Var}[dX_{t+\tau}^2]}}. \tag{15}$$

It is a well known stylized fact [18,33,34] that $\mathcal{A}$ decays with multiple time scales and in particular it shows a long-range memory effect vanishing over a time scale of the order of a few years for stock indexes.

For the model under investigation, the volatility autocorrelation can be computed exactly too. Recalling again the Novikov theorem and the fact that $\delta\, dW_{1,t}/\delta\zeta_{1,t} = 1$, the correlation entering the numerator of (15) becomes

$$\langle dX_t^2\, dX_{t+\tau}^2\rangle = c^2\langle Y_t^2\, Y_{t+\tau}^2\rangle dt^2 + 2\,\rho c^{5/2} H(\tau) \times \langle Y_t^2\, Y_{t+\tau} \exp[\sqrt{c}\,\Delta_t W_2(\tau)]\, dW_{1,t}\rangle dt^2,$$

but, due to the presence of $dW_{1,t}$, the second term results to be of order $O(dt^3)$ and therefore it can be discarded. The exact expression of the autocorrelation function $\langle Y_t^2\, Y_{t+\tau}^2\rangle$ can be obtained as explained in Appendix C, leaving us with

$$\mathcal{A}(\tau;t) = \frac{\exp(a\tau)}{3\mu_4(t) - \mu_2(t)^2}\left\{\frac{2b}{a+c}[\mu_1(t)\mu_2(t) - \mu_3(t)] + \exp[(a+c)\tau]\left[\mu_4(t) + \frac{2b}{a+c}\mu_3(t) - \mu_2(t)\left(\mu_2(t) + \frac{2b}{a+c}\mu_1(t)\right)\right]\right\},$$

where the denominator of Eq. (15) has been approximated with $\mathrm{Var}[dX_t^2] = c^2[3\mu_4(t) - \mu_2(t)^2]\,dt^2$ in view of the stationary limit for Y. After replacing the moments $\mu_n(t)$ with their asymptotic expressions (13) we end with

$$\mathcal{A}(\tau) = \frac{1}{D}\left(N_1 e^{-\tau/\tau_1^{\mathcal{A}}} + N_2 e^{-\tau/\tau_2^{\mathcal{A}}}\right), \tag{16}$$

where the coefficients read

$$D = \frac{(4a^2 - 2ac - 3c^2)(a+c)}{c^2},$$

$$N_1 = -\frac{(2a+3c)(2a+c)}{c},$$

$$N_2 = a,$$

and we also defined the two volatility autocorrelation time scales as

$$\tau_1^{\mathcal{A}} = \frac{1}{|a|} \quad \text{and} \quad \tau_2^{\mathcal{A}} = \frac{1}{2|a|-c}.$$

At this point it is crucial to notice that in deriving Eqs. (14) and (16) we assumed implicitly that the moments of Y_t up to the order $n = 4$ do converge asymptotically. Recalling the expression of the shape parameter ν in (6), we see this assumption imposes

$$\frac{|a|}{c} > \frac{3}{2}, \tag{17}$$

which has to be interpreted as a consistency relation for the model. This constraint imposes the following strict ordering between the time scales of the model:

$$\tau_2^{\mathcal{A}} < \tau_1^{\mathcal{A}} < \tau^{\mathcal{L}}, \quad \text{with} \quad \tau_1^{\mathcal{A}} > \frac{2}{3}\tau^{\mathcal{L}}, \tag{18}$$

where the second inequality for $\tau_1^{\mathcal{A}}$ follows from the convergence of third moment of Y_t which requires $|a|/c > 1$.

The expression obtained for $\mathcal{A}$ fails to capture the persistence of this correlation identified in several analysis reviewed in [35]. The lacking of power law scaling would not be, in principle, a serious drawback as long as one of the two time scales involved in (16) was sufficiently long. However, the ordering (18), which is peculiar to the considered model,

TABLE I. Estimates from return sample averages. We compute the value of the estimators A, B, C, and D for the daily log returns of the S&P500 index during the period 1970–2010, exploiting the means of $|\Delta X|$, ΔX^2, and $|\Delta X|^3$.

Estimators	S&P500 daily returns
A	0.1457 $\text{yr}^{-1/2}$
B	0.0295 yr^{-1}
C	0.0107 $\text{yr}^{-3/2}$
$\|a\|/c$	1.7895

makes these scales too close to each other and the volatility autocorrelation to decay as fast as $\mathcal{L}$, an undesired feature shared with other models such as the Stein-Stein one. The persistence of $\mathcal{A}$ can be accounted for by introducing a nonlinear volatility, as it is for the exponential Ornstein-Uhlenbeck model [18] or coupling a third stochastic equation driving the dynamics of the long run value of Y_t as in [33]. A further possibility to induce a nonexponential time decay would be to consider a nonlinear drift term for the dynamics of Y_t, even though the analytical tractability of the present model will not be preserved.

VI. ESTIMATION OF PARAMETERS

Now we provide a systematic methodology for estimating the model parameters, which are the constants a, b, c entering the dynamics of Y_t, plus the correlation coefficient ρ. We perform the estimation over the Standard & Poor's 500 (S&P500) index daily returns from 1970 to 2010, approximating dX_t with $\Delta X_t = X_{t+\Delta t} - X_t$:

$$dX_t \approx \Delta X_t = \ln\left(\frac{S_{t+\Delta t}}{S_t}\right) - \left\langle \ln\left(\frac{S_{t+\Delta t}}{S_t}\right)\right\rangle,$$

where $\Delta t = 1/250\,\text{yr}$ (one trading day). Taking into account that $dW_{1,t}$ is independent of σ_t and that $|\Delta W_1|$ is distributed accordingly to a folded normal law, the following relations hold for the model (4):

$$A \doteq \frac{\langle|\Delta X|\rangle}{\langle|\Delta W_1|\rangle} = \sqrt{\frac{\pi}{2\Delta t}}\,\langle|\Delta X|\rangle = -\sqrt{c}\,\frac{b}{a},$$
$$B \doteq \frac{\langle\Delta X^2\rangle}{\langle\Delta W_1^2\rangle} = \frac{\langle\Delta X^2\rangle}{\Delta t} = c\,\frac{2b^2}{(2a+c)a},$$
$$C \doteq \frac{\langle|\Delta X|^3\rangle}{\langle|\Delta W_1|^3\rangle} = \sqrt{\frac{\pi}{(2\Delta t)^3}}\,\langle|\Delta X|^3\rangle$$
$$= -\frac{2b^3\,c^{3/2}}{(a+c)\,(2a+c)\,a}.$$

TABLE II. Estimation of the leverage time scale and its limit for $\tau \to 0$, obtained from the fit of the empirical leverage correlation (11) for the daily log- returns of the S&P500 index, with the model predicted expression (14).

Estimators	S&P500 daily returns
$\tau^{\mathcal{L}}$	0.0864 yr
$\mathcal{L}(0^+)$	-30.9515

TABLE III. Model parameters estimated from the daily log-returns of the S&P500 index during 1970–2010 through the relations (19)–(22).

Parameter	Estimate from S&P500
a	-16.0608 yr^{-1}
b	0.8627 yr^{-1}
c	8.9749 yr^{-1}
ρ	-0.5089

The constants A and B can be measured directly from the data, providing us an estimation of the ratio a/c through the relation

$$D \doteq \frac{B}{2(A^2 - B)} = \frac{a}{c}.$$

The value of these quantities extracted from the series of the daily returns of the S&P500 index are reported in Table I. It is crucial to observe that the value obtained for the ratio $|a|/c$ is compatible with the constraint (17), supporting the consistency of our model and the convergence of the volatility autocorrelation. Moreover, the same ratio provides an estimate of $\nu = 4.579$ [see Eq. (6)] implying that the order of the highest converging moment is $n^* = 4$. Consequently, relation (10) indicates the following range for the tail index of $p(x)$:

$$4 < \beta \leqslant 5.$$

The leverage correlation (14) provides a way of obtaining the two further relations needed to fix the four free parameters of the models. Indeed, a two parameters fit of the function $\mathcal{L}(\tau)$ gives estimates for the time scale $\tau^{\mathcal{L}}$ and for the limit $\tau \to 0^+$,

$$\mathcal{L}(0^+) \doteq -\rho\,\frac{a(2a+c)}{b(a+c)},$$

with the results reported in Table II and Fig. 2. In particular, the value obtained for the leverage time scale $\tau^{\mathcal{L}} \approx 21$ days and for its amplitude $\mathcal{L}(0^+)$ are consistent with those quoted in past analysis of different stock indexes such as the Dow Jones

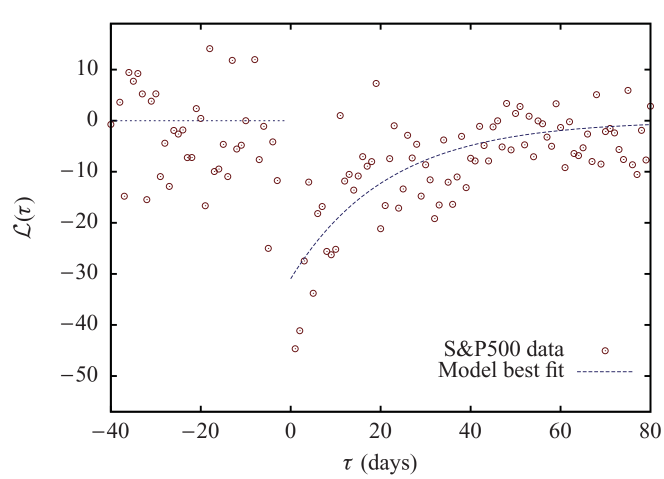

FIG. 2. (Color online) Best fit of the empirical leverage correlation with the model prediction (14) as a function of the two parameters $\tau^{\mathcal{L}}$ and $\mathcal{L}(0^+)$.

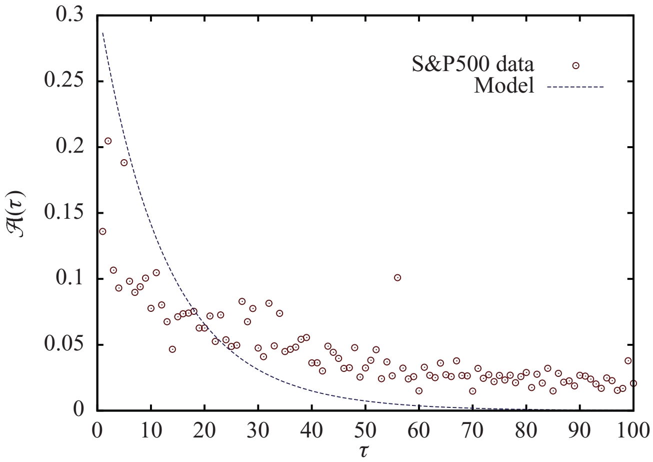

FIG. 3. (Color online) Theoretical prediction for the volatility autocorrelation function of the daily returns of the S&P500 index 1970–2010, Eq. (16).

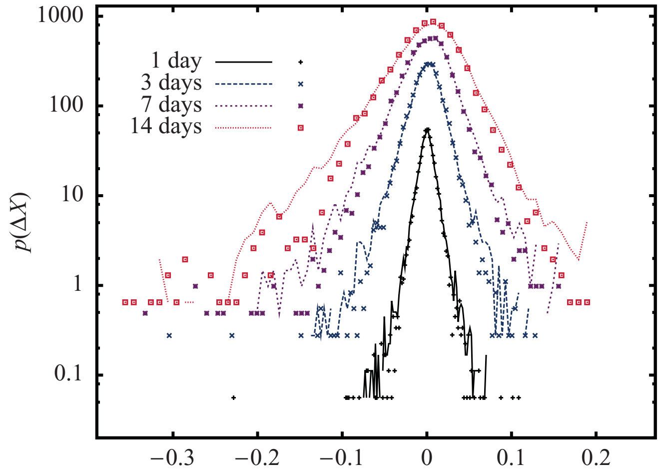

FIG. 5. (Color online) In log-linear scale, return probabilities for the model (lines) vs S&P500 returns (points). Curves have been shifted for sake of readability.

Industrial Average [16,33], and confirm the short range nature of this effect.

At this point all the parameters can be recovered through the following relations:

$$c = -\left[\tau^{\mathcal{L}}\left(D + \frac{1}{2}\right)\right]^{-1}, \tag{19}$$

$$a = c\,D, \tag{20}$$

$$b = -\frac{a+c}{\sqrt{c}}\,\frac{C}{B}, \tag{21}$$

$$\rho = -\frac{b\,(a+c)}{a\,(2a+c)}\,\mathcal{L}(0^+). \tag{22}$$

The final results, reported in Table III, show a negative correlation coefficient in agreement with the known leftward asymmetry of daily return distributions. Moreover, our calibration provides for the relaxation time of the volatility process a finite value $\tau^\sigma \doteq -1/a \approx 15$ days, implying that from a practical point of view the limit $t_0 \to -\infty$ is equivalent to $t_0 \ll -\tau^\sigma$. The fitted values of $\tau^{\mathcal{L}}$ and $\mathcal{L}(0^+)$ provide a good description of real data (as shown in Fig. 2). On the other hand, Fig. 3 shows that the theoretical volatility autocorrelation for the estimated values of the parameters Eq. (16) does not capture the long range persistence of the empirical volatility, as expected from the constraints (18) while it describes correctly the exponential decay for small values of τ.

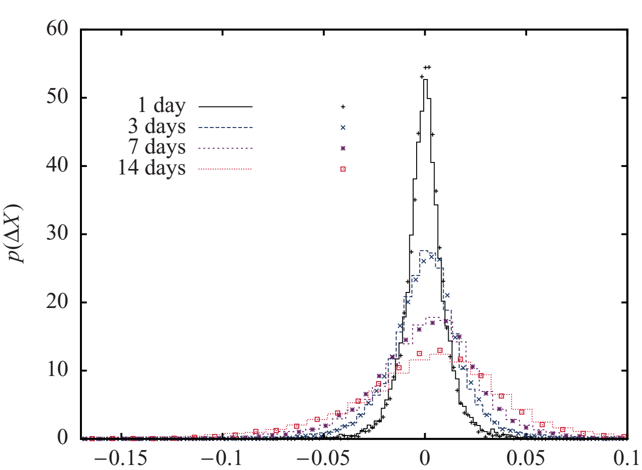

FIG. 4. (Color online) Linear plot showing the comparison between the return PDFs predicted by the model (lines) and the data for the S&P500 index for different time scales.

Finally, it is important to compare the return PDF predicted by the model with the data sample from which the model parameters were estimated. Since we model the return dynamics for increasing t, it is even more important to assess to which extent the diffusion process (4) is able to capture the scaling properties of the empirical distribution over different time horizons. For this aim, with the parameters fixed from the daily S&P500 series, we reconstruct the theoretical PDFs simulating the process at different time scales ($t = 1, 3, 7, 14$ days) and we compare them with the corresponding empirical distributions obtained aggregating the daily returns. This comparison is shown in Figs. 4 and 5. The daily distribution is very well reproduced by the theoretical PDF, which is able to fully capture the leptokurtic nature of the daily data. The plots also confirm that the diffusive dynamics (4), once the parameters have been fixed at the daily scale, follows closely the evolution of the empirical curves for larger t. In particular, it captures the progressive convergence in the central region to a distribution with vanishing skewness and kurtosis.

VII. CONCLUSIONS

In this work we have introduced a class of SVMs where the volatility is driven by the general process with multiplicative noise analyzed in detail in [25]. More specifically, we focused on the set of parameters resulting in an inverse-gamma stationary distribution for the σ_t process. We provided an analytical characterization of the moments of the return distribution, revealing the role played by the power law behavior of the inverse-gamma in the emergence of fat tails. Nevertheless, even though the highest order moments of X diverge for every time lag, the analytical expressions we obtained reveal the vanishing of both the skewness and the kurtosis, in agreement with the normality of returns for long horizons. As far as the estimation procedure is concerned, it is worth noticing that we do not exploit directly the statistical properties of the instantaneous volatility which is a hidden process, but on the contrary we infer the inverse-gamma parameters from well established robust stylized facts holding at the daily scale. Indeed our model correctly predicts zero autocorrelation for the returns and the short-range exponential decay of the leverage. The persistence of the volatility autocorrelation over yearly horizons is not captured, and in this perspective we would like to explore the possibility of coupling a third SDE in the same spirit of [33]. Moreover, we expect that relaxing the time homogeneity of the processes, as done in [25], we may induce time scalings more general than the exponential one. We also expect the proposed dynamics to be a good candidate to describe the price and volatility dynamics even at higher frequencies. This belief is supported by the empirical analysis discussed in the literature [6] concerning the statistical properties of volatility proxies for intraday data. A further perspective would be to explore possible ways to characterize analytically the PDF associated to the process (4) or its characteristic function. This task requires us to solve the Fokker-Planck equation for the PDF or its equivalent version in the Fourier space, analogously to what has been done in [20] for the Heston case. Such a result would also allow for an application of the model in the context of market risk evaluation, possibly exploiting efficient Fourier methodologies such as those proposed in [36,37].

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APPENDIX A: COEFFICIENTS OF $\langle X_t^2\rangle$ AND $\langle X_t^3\rangle$

Here we report the explicit expressions of the coefficients $H_j^{(n)}(t)$ entering the expansion (9) of the moments of X_t for the cases $n=2$ and $n=3$. They were used to plot the analytical curves in Fig. 1:

$$H_0^{(2)}(t) = c\,K_0^{(2)} t,$$
$$H_1^{(2)}(t) = c\,K_1^{(2)}\left[\frac{\exp(F_1 t)-1}{F_1}\right],$$
$$H_2^{(2)}(t) = c\,K_2^{(2)}\left[\frac{\exp(F_2 t)-1}{F_2}\right],$$

$$H_0^{(3)}(t) = 3\,\rho\,c^2\left\{\frac{t}{F_2}\left[A_2\frac{K_0^{(2)}}{F_1} - 2K_0^{(3)}\right] + 2\,K_0^{(3)}\left[\frac{\exp(F_2 t)-1}{F_2^2}\right] + A_2\frac{K_0^{(2)}}{F_2-F_1}\left[\frac{\exp(F_2 t)-1}{F_2^2} - \frac{\exp(F_1 t)-1}{F_1^2}\right]\right\},$$

$$H_1^{(3)}(t) = 3\,\rho\,c^2\left\{\frac{1}{F_2-F_1}\left[A_2\frac{K_1^{(2)}}{F_2-F_1} + 2K_1^{(3)}\right]\times\left[\frac{\exp(F_2 t)-1}{F_2} - \frac{\exp(F_1 t)-1}{F_1}\right] + A_2\frac{K_1^{(2)}}{(F_2-F_1)F_1}\left[\frac{\exp(F_1 t)-1}{F_1} - t\exp(F_1 t)\right]\right\},$$

$$H_2^{(3)}(t) = 3\,\rho\,c^2\left\{-A_2\frac{K_2^{(2)}}{(F_2-F_1)^2}\left[\frac{\exp(F_2 t)-1}{F_2} - \frac{\exp(F_1 t)-1}{F_1}\right] - \frac{1}{F_2}\left[A_2\frac{K_2^{(2)}}{F_2-F_1} + 2K_2^{(3)}\right]\times\left[\frac{\exp(F_2 t)-1}{F_2} - t\exp(F_2 t)\right]\right\},$$

$$H_3^{(3)}(t) = 6\,\rho\,c^2\frac{K_3^{(3)}}{F_3-F_2}\left[\frac{\exp(F_3 t)-1}{F_3} - \frac{\exp(F_2 t)-1}{F_2}\right],$$

where the coefficients $K_j^{(2)}$ and $K_j^{(3)}$, entering the expansion (8) of the moments of Y_t, read

$$K_0^{(2)} = \frac{A_2 A_1}{F_2 F_1},$$
$$K_1^{(2)} = -\frac{A_2}{F_2-F_1}\left[\mu_1(t_0) + \frac{A_1}{F_1}\right],$$
$$K_2^{(2)} = \mu_2(t_0) + \frac{A_2}{F_2-F_1}\left[\mu_1(t_0) + \frac{A_1}{F_2}\right],$$

$$K_0^{(3)} = -\frac{A_3 A_2 A_1}{F_3 F_2 F_1},$$
$$K_1^{(3)} = \frac{A_3 A_2}{(F_3-F_1)(F_2-F_1)}\left[\mu_1(t_0) + \frac{A_1}{F_1}\right],$$
$$K_2^{(3)} = -\frac{A_3}{F_3-F_2}\left\{\mu_2(t_0) + \frac{A_2}{F_2-F_1}\left[\mu_1(t_0) + \frac{A_1}{F_2}\right]\right\},$$

$$K_3^{(3)} = \mu_3(t_0) + \frac{A_3}{F_3-F_2}\times\left\{\mu_2(t_0) + \frac{A_2}{F_3-F_1}\left[\mu_1(t_0) + \frac{A_1}{F_3}\right]\right\}.$$

APPENDIX B: DERIVATION OF EQ. (12)

After expressing $Y_{t+\tau}$ in terms of its integral solution form t to $t+\tau$, the function $f(\tau,t;Y)$ can be rewritten in the form

$$f(\tau,t;Y) = \left\langle Y_t^2\left(Y_t + \int_t^{t+\tau}(aY_s+b)\,ds\right)\exp[\sqrt{c}\Delta_t W_2(\tau)]\right\rangle + \left\langle Y_t^2\left(\sqrt{c}\int_t^{t+\tau} Y_s dW_{2,s}\right)\exp[\sqrt{c}\Delta_t W_2(\tau)]\right\rangle$$

$$= \langle \exp[\sqrt{c}\Delta_t W_2(\tau)]\rangle[\mu_3(t) + b\tau\mu_2(t)] + a\int_t^{t+\tau}\langle Y_t^2 Y_s \exp[\sqrt{c}\Delta_t W_2(\tau)]\rangle ds + \sqrt{c}\int_t^{t+\tau}\langle Y_t^2 Y_s \exp[\sqrt{c}\Delta_t W_2(\tau)]\, dW_{2,s}\rangle.$$

Taking into account that for $t \leqslant s \leqslant t+\tau$ we can always split $\Delta_t W_2(\tau)$ as

$$\Delta_t W_2(\tau) = W_{2,t+\tau} - W_{2,t} = W_{2,t+\tau} - W_{2,s} + W_{2,s} - W_{2,t} = \Delta_s W_2(t+\tau-s) + \Delta_t W_2(s-t),$$

the function $f(\tau,t;Y)$ becomes

$$f(\tau,t;Y) = \langle \exp[\sqrt{c}\Delta_t W_2(\tau)]\rangle[\mu_3(t) + b\tau\mu_2(t)] + a\int_0^{\tau}\langle Y_t^2 Y_{t+\tau'} \exp[\sqrt{c}\Delta_t W_2(\tau')]\rangle \times\langle \exp[\sqrt{c}\Delta_{t+\tau'} W_2(\tau-\tau')]\rangle d\tau' + \sqrt{c}\int_0^{\tau}\langle Y_t^2 Y_{t+\tau'} \exp[\sqrt{c}\Delta_t W_2(\tau')]\rangle \times\langle \exp[\sqrt{c}\Delta_{t+\tau'} W_2(\tau-\tau')]\, dW_{2,t+\tau'}\rangle, \quad \text{(B1)}$$

where we changed the variable of integrations to $\tau' = s - t$. Since the process $\sqrt{c}\,\Delta_{t+\tau'}W_2(\tau-\tau')$ is normally distributed with zero mean and variance $c(\tau-\tau')$, and recalling the expression of the Gaussian characteristic function ϕ^G, we can write

$$\langle \exp[\sqrt{c}\Delta_{t+\tau'} W_2(\tau-\tau')]\rangle = \phi^G(\omega)|_{\omega=-i} = \exp\left[\frac{c}{2}(\tau-\tau')\right].$$

Application of the Novikov theorem also gives

$$\langle \exp[\sqrt{c}\Delta_{t+\tau'} W_2(\tau-\tau')]\, dW_{2,t+\tau'}\rangle = \left\langle \frac{\delta \exp\left(\sqrt{c}\int_{t+\tau'}^{t+\tau}\zeta_{2,s}ds\right)}{\delta\zeta_{W_2}(t+\tau')}\right\rangle d\tau' = \sqrt{c}\,\exp\left[\frac{c}{2}(\tau-\tau')\right],$$

where we expressed the Wiener variation in terms of a Gaussian white noise $\zeta_{2,t}$ as $dW_{2,t} = \zeta_{2,t}\,dt$. Replacing the previous expressions in Eq. (B1) we conclude that $f(\tau,t;Y)$ has to satisfy

$$f(\tau,t;Y) - (a+c)\int_0^{\tau} f(\tau',t;Y)\exp\left[\frac{c}{2}(\tau-\tau')\right]d\tau' = \exp\left(\frac{c}{2}\tau\right)[\mu_3(t) + b\tau\mu_2(t)],$$

which is a Volterra equation of the second kind, whose solution leads to Eq. (12).

APPENDIX C: COMPUTATION OF $\langle Y_t^2 Y_{t+\tau}^2\rangle$

With reference to the model (4), the cross correlation $\langle Y_t^m Y_{t+\tau}^n\rangle$ can be computed exactly. Provided to express $Y_{t+\tau}^n$ as integral solution from t to $t+\tau$

$$Y_{t+\tau}^n = Y_t^n + \int_t^{t+\tau}\left(F_n Y_s^n + A_n Y_s^{n-1}\right)ds + \int_t^{t+\tau}\cdots dW_{2,s}$$

it is straightforward to check that $\langle Y_t^m Y_{t+\tau}^n\rangle$ satisfies the following equation:

$$\frac{d}{d\tau}\left\langle Y_t^m Y_{t+\tau}^n\right\rangle = F_n\left\langle Y_t^m Y_{t+\tau}^n\right\rangle + A_n\left\langle Y_t^m Y_{t+\tau}^{n-1}\right\rangle, \quad \text{(C1)}$$

which is an ODE provided that the correlation $\left\langle Y_t^m Y_{t+\tau}^{n-1}\right\rangle$ has been computed at the lower order $n-1$. In particular, for the case $m = n = 2$ we need the following correlation:

$$\left\langle Y_t^2 Y_{t+\tau}\right\rangle = \exp(a\tau)\mu_3(t) - \frac{b}{a}[1-\exp(a\tau)]\mu_2(t)$$

whose substitution in Eq. (C1) provides the solution

$$\left\langle Y_t^2 Y_{t+\tau}^2\right\rangle = \exp(F_2\tau)\mu_4(t) + \frac{A_2}{a-F_2}[\exp(a\tau) - \exp(F_2\tau)]\mu_3(t) - \frac{b}{a}\left\{\frac{A_2}{F_2}[\exp(F_2\tau) - 1] - \frac{A_2}{a-F_2}[\exp(a\tau) - \exp(F_2\tau)]\right\}\times\mu_2(t).$$

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