PHYSICAL REVIEW E **83**, 041108 (2011)

Diffusion over a fluctuating barrier in underdamped dynamics

Jun-Wen Mao* and Cai-Wan Shen

Department of Physics, Huzhou Teachers College, Huzhou 313000, People's Republic of China

(Received 20 October 2010; revised manuscript received 21 January 2011; published 11 April 2011)

We apply a Langevin model by imposing additive and multiplicative noises to study thermally activated diffusion over a fluctuating barrier in underdamped dynamics. The barrier fluctuation is characterized by Gaussian colored noise with exponential correlation. We present the exact solutions for the first and second moments. Furthermore, we use direct simulations to calculate the asymptotic probability for a Brownian particle passing over the fluctuating barrier. The results indicate that the correlation of the fluctuating barrier is crucial for barrier crossing dynamics.

DOI: 10.1103/PhysRevE.83.041108 PACS number(s): 05.40.−a, 05.10.Gg, 02.50.Ey

I. INTRODUCTION

The problem of thermally activated diffusion over various subsets of barrier potential is ubiquitous in physical contexts, chemical reactions, and biological transport [1,2]. Dynamic processes across a fluctuating barrier potential have attracted much attention for many decades (for a review see Ref. [2]). In many complex environments, e.g., dye lasers with pump noise [3] and chemical reactions in contact with a fluctuating environment [4], the barrier height of the potential fluctuates stochastically over time instead of remaining static. Within the Langevin equation description, the dynamics across a fluctuating barrier is usually modeled as an open system driven simultaneously by additive and multiplicative noises [1,2,5–9]. The internal thermal fluctuation is described by the additive noise and the fluctuation in the barrier height is characterized by multiplicative noise. Unlike the presence of additive noise only, the multiplicative noise with various correlation times may induce nontrivial results such as a noise-induced nonequilibrium phase transition [10,11], anomalous diffusion [12], and stochastic activation in stochastically fluctuating barrier systems [13].

It is noteworthy that much of work mentioned above focuses mostly on the dynamic process in an overdamped limit for the sake of simplicity [2,5–9,11,13]. However, barrier crossing dynamics have shown a significant difference between low- and high-friction limits [14–17]. In a heavy-ion fusion reaction, a compound nucleus can be formed if two colliding nuclei surmount the Coulomb barrier. The fusion probability increases with increasing friction at the low-friction limit, while it decreases with increasing friction in the overdamped case because of fast thermal equilibration and slow diffusion [16,17]. This leads to an evident peak value of fusion probability in the intermediate friction regime that divides two distinct dynamic behaviors. In the present work we study analytically and numerically the problem of diffusion over a fluctuating barrier potential at an underdamped limit. Barrier crossing processes in linear systems with time-dependent potential have been studied theoretically by several authors [18–20]; anomalous diffusion and an interesting probability distribution have been shown in the overdamped region.

In addition to its theoretical extension, our model has a more practical consideration. Recently, experimental data on the synthesis of superheavy nuclei have shown remarkable enhancement of fusion cross sections at energies below and near the barrier potential, which has been explained by the coupled-channel models by introducing a barrier distribution [21–24]. It was found that the barrier height encountered by each fusion event is stochastically distributed as a consequence of couplings of the relative motion to other degrees of freedom such as nuclear shape deformations [23,24]. Due to the complex factors in heavy-ion collisions, such as neck fluctuation during neck formation [25], a time-dependent coupling that may lead to a fluctuating barrier is conceivable. Motivated by this assumption, in this work we consider a Langevin system similar to the one investigated in Ref. [16], except here we examine a time-dependent barrier with random fluctuation illustrated by Gaussian colored noise with exponential correlation. This more general description provides a better understanding of barrier crossing dynamics in heavy-ion fusion reactions.

This paper is organized as follows. In Sec. II we present a Langevin system subjected to additive and multiplicative noises to model the dynamics over a fluctuating barrier. In Sec. III we use direct simulations for various situations to study the stationary probability surmounting the fluctuating barrier. Finally, the conclusion is given in Sec. IV.

II. STOCHASTIC DYNAMICS OVER A FLUCTUATING BARRIER

The dynamics of a Brownian particle in a fluctuating barrier potential is governed by the Langevin equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} - [1+\eta(t)]\Omega^2 x = \xi(t), \tag{1}$$

where γ is the damping parameter. The additive noise $\xi(t)$ comes from the environment in thermal equilibrium, satisfying the fluctuation-dissipation theorem with the statistical properties [16]

$$\langle \xi(t)\rangle = 0, \quad \langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

where $D = \gamma k_B T/m$. The multiplicative noise $\eta(t)$ represents a time-dependent fluctuation in barrier height with

$$\langle \eta(t)\rangle = 0, \quad \langle \eta(t)\eta(t')\rangle = Q\exp(-|t-t'|/\tau), \tag{3}$$

*jwmao@hutc.zj.cn

where τ is the correlation time and Q is the variance of the colored noise. Equation (3) defines an exponentially correlated Gaussian noise, which is typically considered for barrier fluctuation [7,9,14,26,27]. Here Gaussian colored noise $\eta(t)$ is assumed to be uncorrelated with Gaussian white noise $\xi(t)$.

The stochastic variable $\eta(t)$ can be described by the auxiliary stochastic equation [7,9,14,26–28]

$$\frac{d\eta(t)}{dt} = -\frac{1}{\tau}\eta(t) + \sqrt{\frac{2Q}{\tau}}\zeta(t), \tag{4}$$

where $\zeta(t)$ is a standard Gaussian white noise with $\langle\zeta(t)\zeta(t')\rangle = \delta(t-t')$. The probability distribution has the form

$$P[\eta(t)] = \frac{1}{\sqrt{2\pi Q}}\exp\left(-\frac{\eta(t)^2}{2Q}\right), \tag{5}$$

which corresponds to a fluctuating barrier in the Gaussian distribution. The variance Q and the correlation time τ dominate the power spectrum of the colored noise,

$$S_\eta(\omega) = \frac{Q\tau}{\pi(1+\tau^2\omega^2)}. \tag{6}$$

For the case of small τ, a Gaussian white noise with weak noise intensity is obtained. The barrier height fluctuates fast within a finite interval; hence the moving particle experiences an average barrier height. In contrast, the limit $\tau \to \infty$ $[S_\eta(\omega) \to \frac{Q}{\pi}\delta(\omega)]$ characterizes a slow fluctuation in the barrier height of the potential.

A. Exact solutions

To obtain the first moment of x, we rewrite Eq. (1) as two first-order differential equations

$$\dot{x} = v, \quad \dot{v} = -\gamma v + \Omega^2[1+\eta(t)]x + \xi(t). \tag{7}$$

By taking the average of these equations over the ensemble we obtain

$$\dot{\langle x\rangle} = \langle v\rangle, \quad \dot{\langle v\rangle} = -\gamma\langle v\rangle + \Omega^2\langle x\rangle + \Omega^2\langle\eta x\rangle, \tag{8}$$

which contains a new correlator $\langle\eta x\rangle$. To find this correlator, we use the Shapiro-Loginov theorem [29,30], which reads

$$\frac{d\langle\eta g\rangle}{dt} = \left\langle\eta\frac{dg}{dt}\right\rangle - \lambda\langle\eta g\rangle \tag{9}$$

for exponentially correlated noise $\eta(t)$, where $\lambda = 1/\tau$. Here g is any differential function relevant to $\eta(t)$. Thus, by substituting $g = x$ or $g = v$ into Eq. (9), we obtain the linear matrix equation

$$\frac{d\mathbf{F}}{dt} = \mathbb{A}\cdot\mathbf{F}, \tag{10}$$

with

$$\mathbf{F} = \begin{pmatrix}\langle x\rangle\\ \langle v\rangle\\ \langle\eta x\rangle\\ \langle\eta v\rangle\end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix}0 & 1 & 0 & 0\\ \Omega^2 & -\gamma & \Omega^2 & 0\\ 0 & 0 & -\lambda & 1\\ \Omega^2 Q & 0 & \Omega^2 & -(\gamma+\lambda)\end{pmatrix}. \tag{11}$$

Thus a closed system is found for four variables of the first moment and it is solvable for given initial conditions. A similar scheme was used in Refs. [31–33] to complete the system dynamics. To simplify the procedure, here we use the simplest version of the splitting of high-order correlators, i.e., $\langle\eta^2 x\rangle = \langle\eta^2\rangle\langle x\rangle = Q\langle x\rangle$.

The second moments can be obtained by using a similar procedure. Starting from Eq. (7) and making the appropriate transformation, the differential equations for the second moments and cross correlation are written as

$$\begin{aligned}\frac{d\langle x^2\rangle}{dt} &= 2\langle xv\rangle,\\ \frac{d\langle v^2\rangle}{dt} &= -2\gamma\langle v^2\rangle + 2\Omega^2\langle xv\rangle + 2\Omega^2\langle\eta xv\rangle + 2\langle\xi v\rangle,\\ \frac{d\langle xv\rangle}{dt} &= -\gamma\langle xv\rangle + \Omega^2\langle x^2\rangle + \langle v^2\rangle + \Omega^2\langle\eta x^2\rangle + \langle\xi x\rangle.\end{aligned} \tag{12}$$

Notice that there are high-order correlators $\langle\eta xv\rangle$ and $\langle\eta x^2\rangle$ in these equations, which can been found by using the Shapiro-Loginov theorem, just like in the previous case. Similarly, two new correlators $\langle\xi x\rangle$ and $\langle\xi v\rangle$ need to be supplemented. We apply the Shapiro-Loginov theorem again and find $\langle\xi x\rangle = 0$ and $\langle\xi v\rangle = D$ for Gaussian white noise ξ. The details of the derivation are provided in the Appendix. Hence the resulting dynamic system is a six-component linear matrix equation in the form

$$\frac{d\mathbf{G}}{dt} = \mathbb{B}\cdot\mathbf{G} + \mathbf{K}, \tag{13}$$

where

$$\mathbf{G} = \begin{pmatrix}\langle x^2\rangle\\ \langle v^2\rangle\\ \langle xv\rangle\\ \langle\eta x^2\rangle\\ \langle\eta v^2\rangle\\ \langle\eta xv\rangle\end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix}0\\ 2D\\ 0\\ 0\\ 0\\ 0\end{pmatrix}, \tag{14}$$

and $\mathbb{B}$ is a 6×6 drift matrix

$$\mathbb{B} = \begin{pmatrix}0 & 0 & 2 & 0 & 0 & 0\\ 0 & -2\gamma & 2\Omega^2 & 0 & 0 & 2\Omega^2\\ \Omega^2 & 1 & -\gamma & \Omega^2 & 0 & 0\\ 0 & 0 & 0 & -\lambda & 0 & 2\\ 0 & 0 & 2\Omega^2 Q & 0 & -(2\gamma+\lambda) & 2\Omega^2\\ \Omega^2 Q & 0 & 0 & \Omega^2 & 1 & -(\gamma+\lambda)\end{pmatrix}. \tag{15}$$

The formal solutions of the linear matrix equations [Eqs. (10) and (13)] are given by

$$\mathbf{F}(t) = e^{\mathbb{A}t}\mathbf{F}(0), \tag{16}$$

$$\mathbf{G}(t) = e^{\mathbb{B}t}\mathbf{G}(0) - [\mathbf{I} - e^{\mathbb{B}t}]\mathbb{B}^{-1}\cdot\mathbf{K}, \tag{17}$$

where $\mathbf{I}$ is a 6×6 unit matrix.

As an example we discuss the special case of a static barrier potential with no barrier fluctuation ($Q = 0$). In this situation the system is driven by the additive noise only. Equation (10) is reduced to

$$\frac{d}{dt}\begin{pmatrix}\langle x\rangle\\ \langle v\rangle\end{pmatrix}=\begin{pmatrix}0 & 1\\ \Omega^2 & -\gamma\end{pmatrix}\begin{pmatrix}\langle x\rangle\\ \langle v\rangle\end{pmatrix}, \tag{18}$$

which is linear and exactly solvable. The first component of the vector in Eq. (18) is given by

$$\langle x\rangle = a_1 e^{p_1 t} + a_2 e^{p_2 t}, \tag{19}$$

where $p_1 > 0$ and $p_2 < 0$ are eigenvalues of the drift matrix in Eq. (18),

$$p_1 = \frac{1}{2}(\gamma' - \gamma), \quad p_2 = -\frac{1}{2}(\gamma' + \gamma), \tag{20}$$

and

$$a_1 = \frac{-p_2 x_0 + v_0}{\gamma'}, \quad a_2 = \frac{p_1 x_0 - v_0}{\gamma'}. \tag{21}$$

Here $\gamma' = \sqrt{\gamma^2 + 4\Omega^2}$ and x_0 and v_0 are the initial position and velocity of the particle. For time $t \to \infty$, the mean position $\langle x\rangle$ is dominated mostly by the first term of Eq. (19) due to positive p_1.

Similarly, Eq. (13) in a static potential field reads

$$\frac{d}{dt}\begin{pmatrix}\langle x^2\rangle\\ \langle v^2\rangle\\ \langle xv\rangle\end{pmatrix}=\begin{pmatrix}0 & 0 & 2\\ 0 & -2\gamma & 2\Omega^2\\ \Omega^2 & 1 & -\gamma\end{pmatrix}\begin{pmatrix}\langle x^2\rangle\\ \langle v^2\rangle\\ \langle xv\rangle\end{pmatrix}+\begin{pmatrix}0\\ 2D\\ 0\end{pmatrix}. \tag{22}$$

To determine the variance of the motion of the particles we use the covariance matrix [34,35]

$$\sigma = \begin{pmatrix}\sigma_{xx} & \sigma_{xv}\\ \sigma_{vx} & \sigma_{vv}\end{pmatrix} = \begin{pmatrix}\langle x^2\rangle - \langle x\rangle^2 & \langle xv\rangle - \langle x\rangle\langle v\rangle\\ \langle xv\rangle - \langle x\rangle\langle v\rangle & \langle v^2\rangle - \langle v\rangle^2\end{pmatrix}. \tag{23}$$

The linear equation for the elements of the covariance matrix from Eqs. (18) and (22) takes the form

$$\frac{d}{dt}\begin{pmatrix}\sigma_{xx}\\ \sigma_{vv}\\ \sigma_{xv}\end{pmatrix}=\begin{pmatrix}0 & 0 & 2\\ 0 & -2\gamma & 2\Omega^2\\ \Omega^2 & 1 & -\gamma\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{vv}\\ \sigma_{xv}\end{pmatrix}+\begin{pmatrix}0\\ 2D\\ 0\end{pmatrix}. \tag{24}$$

The component of interest in this equation is

$$\sigma_{xx} = \frac{k_B T}{m\Omega^2}(c_1 e^{q_1 t} + c_2 e^{q_2 t} + c_3 e^{q_3 t} - 1), \tag{25}$$

where $q_1 > 0$, $q_2 < 0$, and $q_3 < 0$ are eigenvalues of the drift matrix in Eq. (24), with

$$q_1 = -\gamma + \gamma', \quad q_2 = -\gamma - \gamma', \quad q_3 = -\gamma, \tag{26}$$

and

$$c_1 = \frac{\gamma^2 + \gamma\gamma'}{2\gamma'^2}, \quad c_2 = \frac{\gamma^2 - \gamma\gamma'}{2\gamma'^2}, \quad c_3 = \frac{4\Omega^2}{\gamma'^2}. \tag{27}$$

In Eq. (25) the first term on the right-hand side is the leading one since $q_1 > 0$, which is divergent in the large-time limit. Equations (19) and (25) are in good agreement with the results in Ref. [16]. Other components such as $\langle v\rangle$, σ_{vv}, and σ_{xv} are easily obtained if needed.

In the following we discuss the probability density function (PDF) of the particle position. For the case where the system is subjected to additive noise only, namely, $Q = 0$, the time-evolutionary PDF has a Gaussian form [10,16,20,36]

$$p(x,t;x_0,v_0) = \frac{1}{\sqrt{2\pi\sigma_{xx}(t)}}\exp\left(-\frac{[x(t) - \langle x(t)\rangle]^2}{2\sigma_{xx}(t)}\right), \tag{28}$$

with the initial condition $p(x,t=0) = \delta(x - x_0)\delta(v - v_0)$.

When the barrier fluctuation cannot be neglected, we restrict our consideration to the overdamped case for simplicity. Thus the stochastic system in the large-γ limit reads

$$\gamma\frac{dx}{dt} = [1 + \eta(t)]\Omega^2 x + \xi(t). \tag{29}$$

According to the result given by Refs. [9,37], we obtain the effective Fokker-Planck equation corresponding to Eq. (29) in the form

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\Omega^2}{\gamma}\frac{\partial}{\partial x}xp(x,t) + D_0\frac{\partial^2}{\partial x^2}p(x,t) + D_1\frac{\partial}{\partial x}x\frac{\partial}{\partial x}xp(x,t), \tag{30}$$

where

$$D_0 = \frac{D}{\gamma^2} = \frac{k_B T}{\gamma m}, \quad D_1 = \frac{Q\tau\Omega^4}{\gamma^2}. \tag{31}$$

Here D_0 and D_1 are attributed to additive and multiplicative noises, respectively. At this point it is difficult to find the exact expression for the PDF in Eq. (30), especially a nonstationary solution [10].

B. Numerical simulations

In order to find the nonstationary PDF of a system subjected to additive and multiplicative noises we perform numerical integration of Eq. (1) by Heun's method [38,39]. The exponentially correlated noise $\eta(t)$ is generated by integrating Eq. (4), which is driven by a Gaussian white noise ζ [40]. The time step Δt is set to 0.001 for simulations. The Brownian particles start at the left of the fluctuating barrier potential $x_0 = -2$ and the simulations are performed by averaging over 200 000 trajectories to calculate the dynamical properties. For simplicity we have set the Boltzmann constant $k_B = 1$, the particle mass $m = 1$, and the parameter $\Omega = 1$. The friction coefficient $\gamma = 0.1$ in the simulations, unless stated otherwise.

The numerical results of the nonstationary PDF are shown in Fig. 1. A non-Gaussian distribution is evident even in this linear Langevin system, but it is driven by additive and multiplicative noises simultaneously [9,41]. The non-Gaussian PDF indicates that it cannot be determined by the first and second moments only.

We calculate the time-evolutionary ratio of the average position and the variance, $r = -\langle x\rangle/\sqrt{\sigma_{xx}}$. For the Gaussian distribution the ratio approaches a constant value at large times, which indicates a stable spreading with respect to the center of the Gaussian distribution. As expected, this non-Gaussian distribution results in an unstable ratio that decreases monotonically, as shown in Fig. 2. For the case of a large correlation time and low-friction coefficient, the

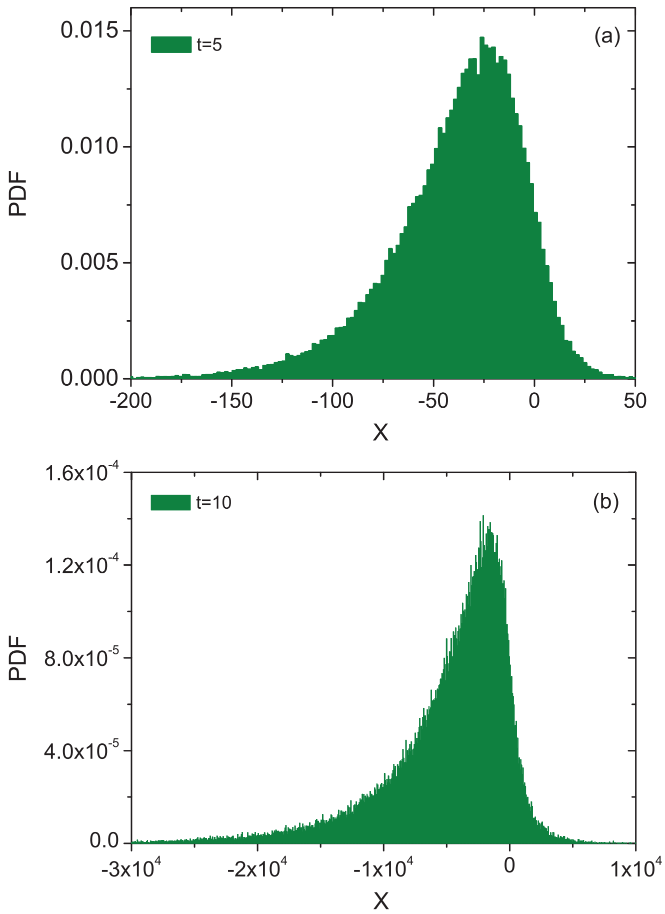

FIG. 1. (Color online) Normalized PDF in the presence of additive and multiplicative noises. The initial position $x_0 = -2$ and initial velocity $v_0 = 1.5$. $B = \frac{1}{2}m\Omega^2 x_0^2$ is the barrier height measured from the initial position. Here $T/B = 0.5$, $\gamma = 0.1$, $Q = 0.1$, and $\tau = 1$. (a) $t = 5$. (b) $t = 10$.

ratio allows a fast decay, implying a wide spreading of the distribution.

III. PROBABILITY OF PASSING OVER A BARRIER

A. Static barrier

The barrier crossing problem has been considered as a simplified model that describes the fusion mechanism of the synthesis of heavy elements in heavy-ion reactions [16,36,42]. For the static barrier potential with $Q = 0$, Eq. (1) results in a Gaussian PDF [Eq. (28)] for the particle position. The asymptotic probability for a particle passing over the static barrier is given, as in Refs. [16,17,20,43–45], by

$$P(t \to \infty) = \int_0^\infty p(x,t;x_0,v_0)\,dx = \frac{1}{2}\mathrm{erfc}\left(-\frac{1}{\sqrt{T}\sqrt{1-y^2}}(\sqrt{B} - y\sqrt{K})\right), \quad (32)$$

where $y = p_1/\Omega$ is a dimensionless parameter, with p_1 the positive eigenvalue in Eq. (20). Here $B = \frac{1}{2}m\Omega^2 x_0^2$ is the barrier height measured from the initial position and $K = \frac{1}{2}mv_0^2$ is the initial kinetic energy.

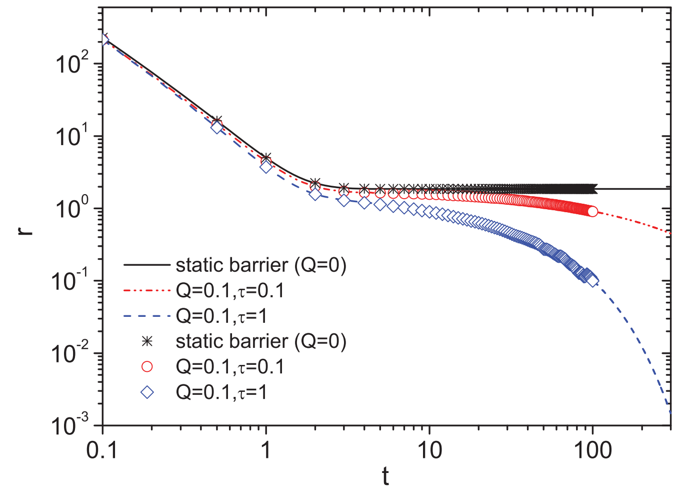

FIG. 2. (Color online) A log-log plot of the time-dependent ratio $r = -\langle x\rangle/\sqrt{\sigma_{xx}}$ in the system with a static barrier $Q = 0$ and fluctuating barriers $Q = 0.1$, $\tau = 0.1$ and $Q = 0.1$, $\tau = 1$, respectively. The lines are the analytical results of Eqs. (16) and (17) and the scatters denote numerical integration results. Other parameters are the same as in Fig. 1.

The critical kinetic energy K_c, which is defined as the energy necessary for half of the particles to pass over the barrier, has the form [16,42]

$$\frac{K_c}{B} = \frac{1}{y^2} = \left[\frac{\tilde{\gamma}}{2} + \sqrt{\left(\frac{\tilde{\gamma}}{2}\right)^2 + 1}\right]^2. \quad (33)$$

Notice that the ratio of the critical kinetic energy to the barrier height is determined just by the dimensionless friction coefficient $\tilde{\gamma} = \gamma/\Omega$, regardless of the environment temperature T.

We are interested in the effect of temperature on the overpassing probability. The asymptotic probability [Eq. (32)] as a function of the environment temperature T is shown in Fig. 3. In the weak-friction-coefficient regime there are three distinct behaviors. For low initial kinetic energy $K < K_c$, a monotonically increasing probability with increasing temperature is observed close to the asymptotic value $P = 0.5$ in the high-temperature limit. For $K > K_c$, the overpassing probability decreases as the temperature increases, which is not expected intuitively. In fact, all particles with initial kinetic energy larger than the barrier height will pass over the barrier in a nondissipative system of $\tilde{\gamma} = 0$. For the system with small $\tilde{\gamma}$, most of the particles can surmount the barrier due to low-energy dissipation and thus a large overpassing probability $P > 0.5$ is exhibited. When the environment temperature T increases, on average some of the particles that have passed over the barrier potential will return to the starting side, resulting in a reduced overpassing probability, namely, negative dP/dT. When $K = K_c$, the asymptotic probability takes the uniform value $P = 0.5$ regardless of the environment temperature. Thus the energy regimes can be divided into three parts, as shown in Fig. 4: $dP/dT < 0$ for $K > K_c$, $dP/dT > 0$ for $K < K_c$, and $dP/dT = 0$ for $K = K_c$.

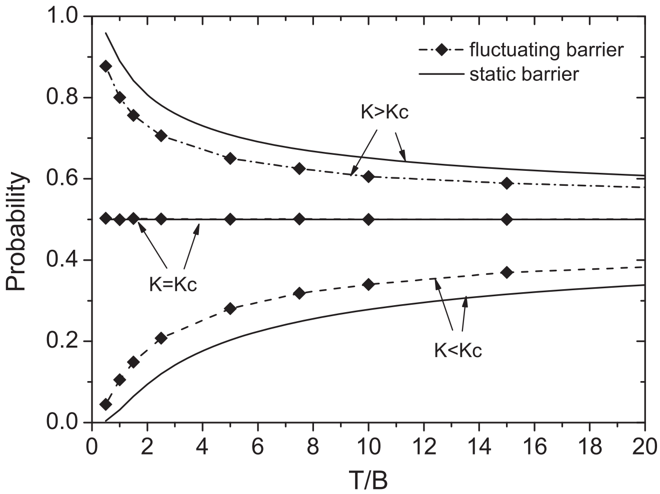

FIG. 3. Temperature dependence of the probability [Eq. (32)] for the weak-friction case of $\gamma = 0.1$. Here the critical kinetic energy $K_c/B = 1.1051$. The initial kinetic energies from top to bottom correspond to $K/B = 1.5625, 1.1051$, and 0.5625, respectively. The barrier height $B = 2$. For a fluctuating barrier, $Q = 0.1$ and $\tau = 0.1$.

B. Fluctuating barrier

Now we consider the probability of particles passing over a fluctuating barrier [their motion is described by Eqs. (1)–(3)].The fluctuating barrier has a Gaussian distribution dominated by two parameters: the variance Q and the correlation time τ. For $Q = 0$ the barrier remains static, corresponding to the situation of no barrier fluctuation. A large Q, such as $Q \sim 1$, might lead to a temporary harmonic-oscillator potential. The correlation time $\tau \to 0$ corresponds to a Gaussian white noise; therefore a moving particle experiences a fast fluctuation in the barrier height. In contrast, a slow fluctuation of barrier height is shown for $\tau \to \infty$. In the latter case, each particle starting at an initial position experiences an almost fixed barrier when moving along its trajectory, which is stochastically distributed in the form

$$P(\eta) = \frac{1}{\sqrt{2\pi Q}} \exp\left(-\frac{\eta^2}{2Q}\right). \tag{34}$$

Here η is a time-independent stochastic value. Initially, one might take it for granted that Eq. (34) is intuitively the same as Eq. (5). In fact, Eq. (5) is true for a wide range of τ, leading to a time-dependent barrier, while Eq. (34) is appropriate only for $\tau \to \infty$, implying that a time-independent barrier is encountered by each particle.

We now turn to numerical simulations to study the overpassing probability. We have investigated the time-evolutionary probability extensively and have demonstrated a stationary probability after a period of time for either small Q or small τ, as shown in Fig. 5(a). In addition, for large Q and large τ, the probability evolves to the asymptotic value of $P = 0.5$ after a very long time, as plotted in Fig.5(b).

The asymptotic probability as a function of Q for small τ in the low-friction regime is shown in Fig. 6. If the initial kinetic energy is below or near the barrier potential, $K < B$, the overpassing probability is less than 0.5 and increases slowly with increasing variance Q. In contrast, for $K > B$ the probability is larger than 0.5, implying that an overwhelming majority of particles have surmounted the fluctuating barrier, as expected. Furthermore, larger Q, and hence a large barrier fluctuation, will not help more particles pass over the fluctuating barrier. On the contrary, some of the particles return to the starting point if Q increases; hence a

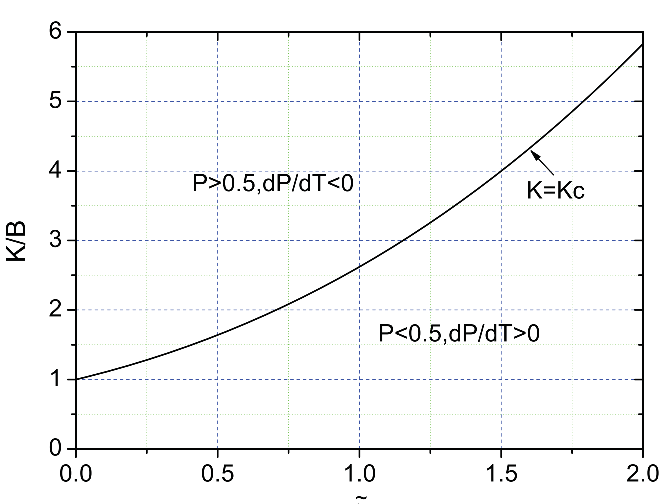

FIG. 4. (Color online) Energy regimes with completely different characteristics for the temperature dependence. The solid line is the result of Eq. (33), acting as the boundary of the two regimes. The barrier height $B = 2$.

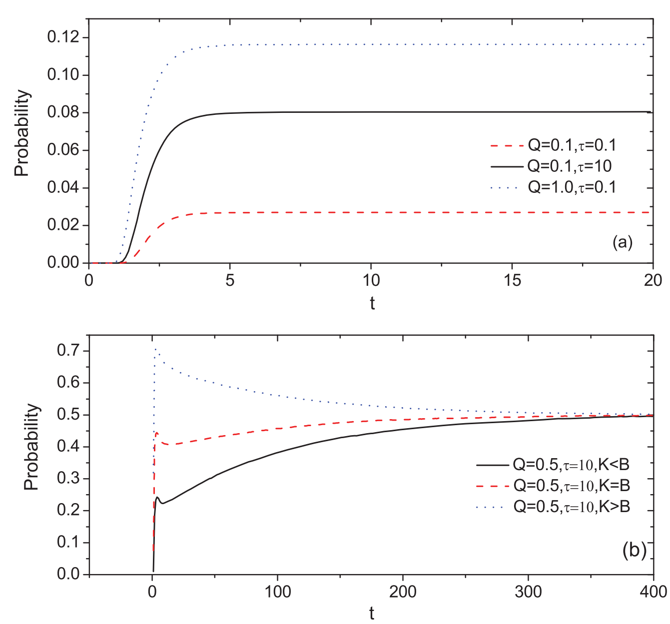

FIG. 5. (Color online) Probability of passing over a fluctuating barrier as a function of time. The temperature $T/B = 0.5$ and friction $\gamma = 0.1$ in the simulations. (a) Stationary probability obtained after a time for either small Q or small τ. From bottom to top the three lines indicate $Q = 0.1$ and $\tau = 0.1$ (small Q and small τ), $Q = 0.1$ and $\tau = 10$ (small Q), and $Q = 1.0$ and $\tau = 0.1$ (small τ), respectively. The initial kinetic energy $K/B = 0.5$. (b) Nonstationary probability for large Q and large τ. Notice that the time scale is much greater than that in (a).

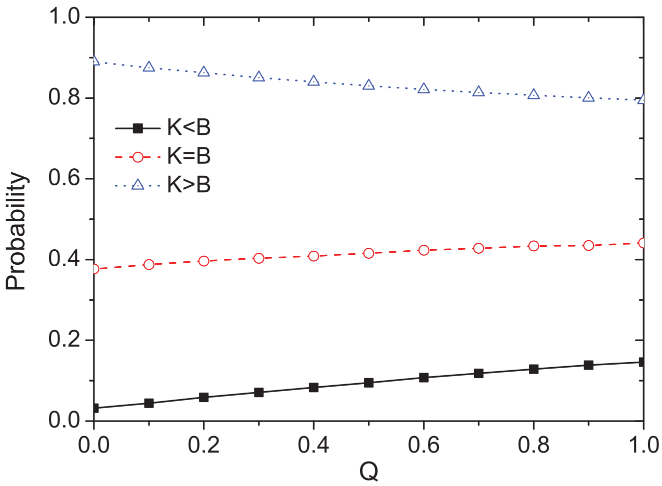

FIG. 6. (Color online) Asymptotic probability of passing over a fluctuating barrier as a function of the variance Q for three cases. A small correlation time $\tau = 0.1$ is taken to ensure a stationary probability within the simulation time. Other parameters are the same as in Fig. 5.

decreasing probability with increasing Q is observed for large K and low γ.

In the following we restrict our consideration to the case of $Q \ll 1$ to ensure a barrier potential, thus avoiding a harmonic oscillator, even temporarily. Extensive numerical simulations show completely distinct behaviors in the three energy regimes with respect to the case of a static barrier: the enhanced probability in the energy area $K < K_c$, a suppressed probability for $K > K_c$, and an invariable probability for $K = K_c$, as shown in Fig. 3. It is interesting that the energy regimes in Fig. 4 are still in effect even in the presence of a fluctuating barrier. As a result we come to the conclusion that

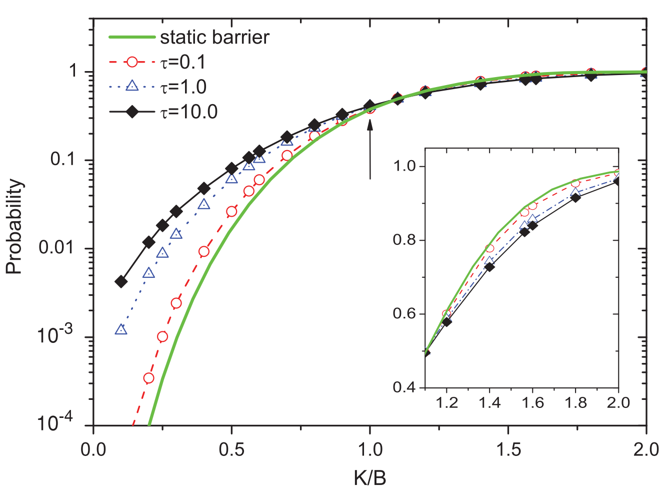

FIG. 7. (Color online) Semilogarithmic plot of the overpassing probability as a function of the initial kinetic energy. The temperature $T/B = 0.5$. The solid line is the result of Eq. (32), which corresponds to the static barrier potential ($Q = 0$), and the scatters from bottom to top correspond to the fluctuating barrier with variance $Q = 0.1$ and correlation times $\tau = 0.1, 1.0$, and 10.0, respectively. The arrow indicates the average barrier height. Inset: Probability at higher energies on a linear scale.

neither the temperature of the environment nor the fluctuating barrier will affect the critical kinetic energy.

Finally, it is instructive to study the effect of correlation time τ on asymptotic probability. A wide range of correlation times has been used in our simulations to obtain the characteristics. The overpassing probability as a function of initial kinetic energy is shown in Fig. 7. Large τ leads to great enhancement of the probability at energies well below the barrier. At energies near the barrier, probability increases noticeably with increasing τ. In contrast, a slight decrease in probability occurs at high energies, even for the case of large τ (see the inset of Fig. 7). The results are qualitatively consistent with those of previous work on the enhanced fusion cross sections for heavy-ion synthesis by introducing a barrier distribution, especially at low, subbarrier energies [21–24,46]; while a decreased probability at high energies is reported in Ref. [47]. Indeed, the fusion cross section is dominated by the fusion probability [42,48] and a realistic fusion reaction includes a more complex potential and specific parameters. Our results have demonstrated that both enhancement and suppression of the fusion probability are relevant to not only the barrier distribution, but the correlation time of the fluctuating barrier as well.

IV. CONCLUSION

In conclusion, we have proposed a Langevin model subjected to additive and multiplicative noises to describe thermally activated diffusion over a fluctuating barrier. The analytical predictions of the first and second moments and relevant dynamical characteristics have been obtained by using two linear matrix equations. Numerical analysis of the Langevin model provides a nonstationary PDF in a non-Gaussian form. Our results can also be applied to stochastically modulated harmonic oscillators (e.g., a Kubo oscillator) only if the inverted parabolic barrier is replaced by a harmonic-oscillator potential.

We also showed interesting dynamics in the underdamped region where the exchange of energy with the environment is limited due to low friction. For the case of a static barrier with $Q = 0$, there are three regimes that depend on the initial kinetic energy with respect to the critical energy [Eq. (33)]. For $K < K_c$, the asymptotic probability P of passing over a static barrier is less than 0.5 and increases with increasing temperature. In contrast, the asymptotic probability is greater than 0.5 and decreases with increasing temperature for $K > K_c$. This indicates that, contrary to expectation, higher temperatures do not always help the particles surmount the barrier. At energy $K = K_c$, the probability remains unchanged at $P = 0.5$ for various temperatures of the environment. Furthermore, the simulations showed that this holds true even for a fluctuating barrier. Consequently, the implication is that the critical kinetic energy K_c has nothing to do with the temperature of the environment or the stochastic modulation of the barrier.

We have explored the influence of barrier fluctuation (in relevant scales of correlation time) on the probability of passing over fluctuating barriers. For a large correlation time τ, the overpassing probability is enhanced remarkably at subbarrier energies, while a slight decrease in probability occurs at high energies, as compared to the case of the static barrier. A similar enhancement of fusion cross sections has been found in previous work on experimental studies in heavy-ion fusion reactions and was explained by a theoretical analysis of the coupled-channel effect using a barrier distribution. It should be noted that our model presents a qualitative prediction of the dynamics in the fluctuating barrier with various scales of correlation. Our results indicate that the correlation time of the fluctuating barrier plays an important role in barrier crossing dynamics. We hope this model will provide a better understanding of the barrier crossing problem in heavy-ion fusion reactions, particularly at energies below and near the Coulomb barrier.

ACKNOWLEDGMENTS

J.-W.M. thanks Y.Q. Li for helpful discussions. J.-W.M. is grateful for financial support provided by the Natural Science Foundation of China under Grant No. 10947012. C.-W.S. acknowledges support provided by the Natural Science Foundation of China under Grants No. 10979024 and No. 10905021, the Key Project of Science and Technology Research of the Education Ministry of China under Grant No. 209053, and the Natural Science Foundation of Zhejiang province, China under Grant No. Y6090210. We are also grateful to the C3S2 computing center for help with the calculation.

APPENDIX

For the stochastic variables x and v described by the two-component Langevin equation [Eq. (7)], the correlators $\langle \xi x \rangle$ and $\langle \xi v \rangle$ cannot be found directly by using the Shapiro-Loginov theorem for Gaussian white noise ξ. In the Stratonovich representation, which supposed a δ-correlated process as an approximation to a colored process with zero mean and finite correlation time, $\langle \xi(t)\xi(t') \rangle = (D/\tau_G)\exp(-|t-t'|/\tau_G)$. On the very small time scale τ_G, Gaussian white noise ξ with noise intensity D is obtained [10,27,28,49]. This is considered an appropriate interpretation for most physical situations. With such an assumption, application of the Shapiro-Loginov theorem yields two differential equations

$$\frac{d\langle \xi x \rangle}{dt} = -(1/\tau_G)\langle \xi x \rangle + \langle \xi v \rangle,$$
$$\frac{d\langle \xi v \rangle}{dt} = \Omega^2\langle \xi x \rangle - [\gamma + (1/\tau_G)]\langle \xi v \rangle + D/\tau_G, \quad \text{(A1)}$$

where the approximation $\langle \xi \eta x \rangle = \langle \xi \eta \rangle \langle x \rangle = 0$ has been used. Equation (A1) is exactly solvable, but in a very complicated form. Here we give the concise results $\langle \xi x \rangle \to 0$ and $\langle \xi v \rangle \to D$ in the limit $\tau_G \to 0$.

[1] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990), and references cited therein.
[2] P. Reimann and P. Hänggi, in *Stochastic Dynamics*, edited by L. Schimansky-Geier and T. Pöschel, Lecture Notes in Physics Vol. 484 (Springer, Berlin, 1997), p. 127.
[3] K. Kaminishi, R. Roy, R. Short, and L. Mandel, Phys. Rev. A **24**, 370 (1981).
[4] R. D. Astumian and M. Bier, Phys. Rev. Lett. **72**, 1766 (1994).
[5] P. Hänggi, Chem. Phys. **180**, 157 (1994).
[6] A. J. R. Madureira, P. Hänggi, V. Buonomano, and W. A. Rodrigues, Phys. Rev. E **51**, 3849 (1995).
[7] J. Iwaniszewski, Phys. Rev. E **54**, 3173 (1996).
[8] M. Marchi, F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. E **54**, 3479 (1996).
[9] H. Hasegawa, Physica A **387**, 2697 (2008).
[10] G. Hu, *Stochastic Force and Nonlinear Systems* (Shanghai Scientific and Technological Education Press, Shanghai, 1994).
[11] C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).
[12] K. Mallick and P. Marcq, Phys. Rev. E **66**, 041113 (2002).
[13] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992).
[14] T. D. Shepherd and R. Hernandez, J. Chem. Phys. **115**, 2430 (2001).
[15] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).
[16] Y. Abe, D. Boilley, B. G. Giraud, and T. Wada, Phys. Rev. E **61**, 1125 (2000).
[17] B. Yilmaz, S. Ayik, Y. Abe, and D. Boilley, Phys. Rev. E **77**, 011121 (2008).
[18] F. Lillo and R. N. Mantegna, Phys. Rev. E **61**, R4675 (2000).
[19] K. S. Fa, Chem. Phys. **287**, 1 (2003).
[20] J.-W. Mao, J.-X. Chen, W.-H. Huang, B.-Q. Li, and W.-K. Ge, Phys. Rev. E **81**, 031123 (2010).
[21] A. Stefanini, in *FUSION08: New Aspects of Heavy Ion Collisions Near the Coulomb Barrier, Chicago, 2008*, edited by K. Ernst Rehm, Birger B. Back, Henning Esbensen, and C. J. Lister, AIP Conf. Proc. No. 1098 (AIP, New York, 2009), p. 3.
[22] W. Reisdorf, F. P. Hessberger, K. D. Hildenbrand, S. Hofmann, G. Monzenberg, K.-H. Schmidt, J. H. R. Schneider, W. F. W. Schneider, K. Summerer, and G. Wirth, Nucl. Phys. A **438**, 212 (1985).
[23] K. Hagino, N. Rowley, and A. Kruppa, Comput. Phys. Commun. **123**, 143 (1999).
[24] N. Wang, Z. Li, and W. Scheid, J. Phys. G **34**, 1935 (2007).
[25] J.-D. Bao and D. Boilley, Nucl. Phys. A **707**, 47 (2002).
[26] J. M. Moix and R. Hernandez, J. Chem. Phys. **122**, 114111 (2005).
[27] P. Hänggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995), and references cited therein.
[28] J. Luczka, Chaos **15**, 026107 (2005).
[29] V. E. Shapiro and V. M. Loginov, Physica A **91**, 563 (1978).
[30] V. M. Loginov, Acta Phys. Pol. B **27**, 693 (1996).
[31] J. R. Chaudhuri, P. Chaudhury, and S. Chattopadhyay, J. Chem. Phys. **130**, 234109 (2009).
[32] M. Gitterman, Phys. Rev. E **69**, 041101 (2004).
[33] M. Gitterman and V. I. Klyatskin, Phys. Rev. E **81**, 051139 (2010).
[34] C. Zerbe, P. Jung, and P. Hänggi, Phys. Rev. E **49**, 3626 (1994).

[35] Y. Deng, J. Bechhoefer, and N. R. Forde, J. Opt. A **9**, S256 (2007).
[36] D. Boilley, Y. Abe, and J.-D. Bao, Eur. Phys. J. A **18**, 627 (2003).
[37] G. Liang, L. Cao, and D. Wu, Physica A **335**, 371 (2004).
[38] R. Toral, in *Computational Field Theory and Pattern Formation*, edited by P. L. Garrido and J. Marro, Lecture Notes in Physics Vol. 448 (Springer, Berlin, 1995), p. 3.
[39] K. Burrage, I. Lenane, and G. Lythe, SIAM J. Sci. Comput. **29**, 245 (2007).
[40] R. F. Fox, I. R. Gatland, R. Roy, and G. Vemuri, Phys. Rev. A **38**, 5938 (1988).
[41] H. Wio and K. Lindenberg, in *Modern Challenges in Statistical Mechanics: Patterns, Noise, and the Interplay of Nonlinearity and Complexity*, edited by V. M. Kenkre and K. Lindenberg, AIP Conf. Proc. No. 658 (AIP, New York, 2003), p. 1.
[42] Y. Abe, C. Shen, G. Kosenko, D. Boilley, and B. Giraud, Int. J. Mod. Phys. E **17**, 2214 (2008).
[43] J.-D. Bao, Phys. Rev. E **69**, 016124 (2004).
[44] D. Boilley and Y. Lallouet, J. Stat. Phys. **125**, 473 (2006).
[45] A. Quint *et al.*, Z. Phys. A **346**, 119 (1993).
[46] Z.-Q. Feng and G.-M. Jin, Phys. Rev. C **80**, 037601 (2009).
[47] S. Misicu and H. Esbensen, Phys. Rev. C **75**, 034606 (2007).
[48] S. Ayik, B. Yilmaz, and D. Lacroix, Phys. Rev. C **81**, 034605 (2010).
[49] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).