Statistics of reciprocal distances for random walks of three particles in one dimension

Riccardo Campari[1,2,*] and Davide Cassi[2,†]
[1]*Università San Raffaele Roma, via Fantoli 16/15, I-20138 Milano, Italy*
[2]*Dipartimento di Fisica, Università di Parma, Viale Usberti 7/A, I-43100 Parma, Italy*
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We investigate the collective properties of three particles performing both independent and interacting random walks on an infinite line by studying the problem on the corresponding *distance graph*. In the large times limit, we obtain the asymptotically exact behavior of several probability functions regarding maximum and minimum mutual distances among particles. Finally, we suggest possible applications of our results.

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I. INTRODUCTION

The collective properties of several random walkers have been studied in some detail since the 1980s; although a thorough account of the results achieved in the last 30 years is beyond the scope of this article, a short summary of the main directions of research is in order. In a short review by Fisher [1] several results are sketched, for both noninteracting and interacting walkers; other authors focused on more specific topics: vicious walkers, who cannot pass one another [2–6], osculating walkers, who can sit on the same site but will never go along the same link [7,8], n-friendly walkers, who can go along the same path for no more than n links [3,4,9], as well as other types of models [10–12].

In this paper we are going to present several results on some collective properties of three independent random walkers on an infinite line. In particular, we have calculated the probability that minimum and maximum distances among the walkers are in a given range, and we have also derived the asymptotic behavior of the probability that the three particles first meet at time t, for large times.

We have performed the calculation in the graph of the interparticle distances, which displays several advantages: It reduces the problem to a two-dimensional lattice, which is a quite classical trick when dealing with multiple random walkers, it has a simple and intuitive geometric form, it discards the useless details concerning the absolute position of the particles while retaining all the meaningful interparticle relations, and it allows for a straightforward introduction of interparticle attraction or repulsion.

For large times, the asymptotic behavior of the probability distributions of maximum and minimum mutual distances d, which are strictly nonlinear functions, displays to first order a diffusion-like scaling as d^2/t, when d is large. At higher orders, this scaling breaks down, and for intermediate distances a richer behavior is revealed.

We have also studied two different models for vicious interacting walkers, which we have been able to map onto the model for independent walkers; the aforementioned results thus hold in these cases as well.

The paper is organized as follows: First, we define the system under study and introduce the interparticle distance graph; next we compute the exact $t \to \infty$ asymptotic form of the probability $P(\vec{x},t)$ of finding the walkers at a point $\vec{x}$ in the distance graph, and then we infer the behavior of the first-return probabilities. In the following sections we first compute the probability that the minimum and maximum distances between pairs of adjacent particles are smaller than some d, then focus on the probability that the walkers are no closer than d_{inf} and no farther than d_{sup}. We then study two models for vicious random walkers, and finally we analyze our results.

II. THREE PARTICLES ON A LINE

A simple discrete random walk on a one-dimensional lattice is a process in which, at discrete time steps $t = 1,2,\ldots,$ a particle located at position $i(t) \in \mathcal{Z}$ moves to an adjacent vertex on the lattice with equal probability:

$$p_{i\to(i+1)}(t) = p_{i\to(i-1)}(t) = \frac{1}{2} \quad \forall t.$$

Let $P^{\text{lin}}_{ij}(t)$ be the probability that a walker starting from i is at position j after t steps.

We now take three independent walkers on the same lattice, leaving from i at time 0: The probability $P^{(3)}_{1D}(t)$ for all of them to meet at a time t is

$$P^{(3)}_{1D}(t) = \sum_j \left[P^{\text{lin}}_{ij}(t)\right]^3.$$

A good choice of coordinates to tackle this problem is the following: Let x be the distance between the leftmost walker and the central one, and let y be the distance between the rightmost walker and the central one. A straightforward analysis of the transition probabilities gives the following results:

(i) If the position at time $(t-1)$ is (0,0):

$$p\left((0,0) \to (0,0)\right) = \frac{1}{4},$$
$$p\left((0,0) \to (0,2)\right) = p\left((0,0) \to (2,0)\right) = \frac{3}{8};$$

(ii) If the position at time $(t-1)$ is $(x,0)$ or $(0,y)$:

$$\begin{aligned} p\left((x,0) \to (x,0)\right) &= p\left((x,0) \to (x-2,2)\right) \\ &= p\left((x,0) \to (x,2)\right) = \frac{1}{4}, \end{aligned}$$
$$p\left((x,0) \to (x+2,0)\right) = p\left((x,0) \to (x-2,0)\right) = \frac{1}{8},$$

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*riccardo.campari@fis.unipr.it
†davide.cassi@fis.unipr.it

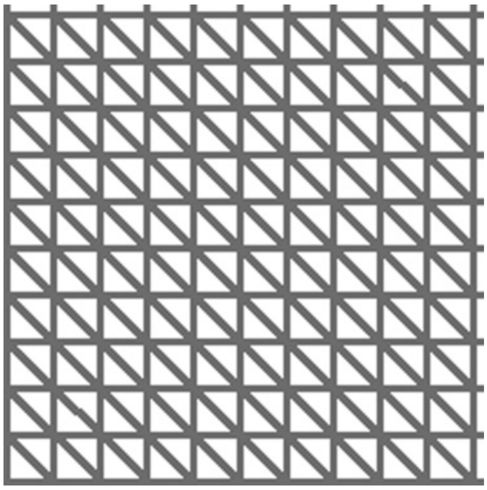

FIG. 1. The distance graph for the random walk of three particles on a line is a two-dimensional lattice with diagonals.

and

$$p((0,y) \to (0,y)) = p((0,y) \to (2,y-2)) = p((0,y) \to (2,y)) = \frac{1}{4},$$

$$p((0,y) \to (0,y+2)) = p((0,y) \to (0,y-2)) = \frac{1}{8};$$

(iii) If the position at time $(t-1)$ is (x,y):

$$p((x,y) \to (x,y)) = \frac{1}{4},$$

$$p((x,y) \to (x+2,y)) = p((x,y) \to (x-2,y)) = \frac{1}{8},$$

$$p((x,y) \to (x,y+2)) = p((x,y) \to (x,y-2)) = \frac{1}{8},$$

$$p((x,y) \to (x+2,y-2)) = p((x,y) \to (x-2,y+2)) = \frac{1}{8}.$$

The end result is that $P^{(3)}_{1D}(t)$ corresponds to the probability of return to the origin, for a simple random walk with waiting probability $\frac{1}{4}$, on a two-dimensional structure with diagonals, as depicted in Fig. 1, or equivalently on a slice of a triangular lattice, as in Fig. 2. On both structures each step corresponds to a change of two for the reciprocal distances of the particles.

Since the graph we have now obtained is inhomogeneous, a natural way to tackle it is by transforming it into a full triangular lattice (Fig. 3). This feat is accomplished by noting that, at (0,0), the probability of departing from the origin—which is 6/8—corresponds to a choice of one among the six axes; on the axes of Fig. 2, on the other hand, the probability 1/4 of departing from the current axis corresponds to the choice between two adjacent slices in the full triangular lattice. A cogent interpretation of the full triangular lattice is obtained by noting that each of the six slices corresponds to the possible orderings of the walkers on a one-dimensional lattice, and that each crossing of an axis implies a change in this order (see Fig. 3).

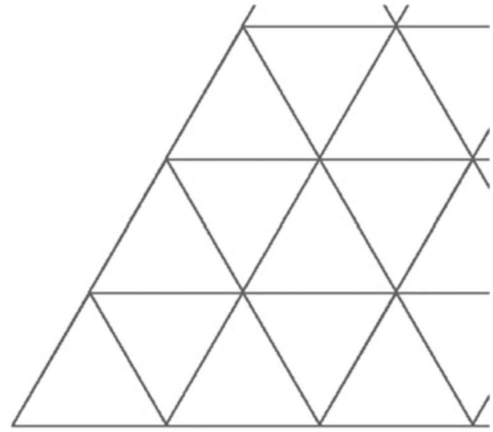

FIG. 2. The two-dimensional lattice with diagonals is equivalent to a triangular lattice.

The graphs in Figs. 1, 2, and 3 can all be defined as *distance graphs* of the multiple random walk we are considering. It is just for computational simplicity that we will stick to the latter henceforth.

This reduction to a two-dimensional problem is a quite classical trick, but it should be stressed that it is not guaranteed to succeed for multiple random walks in inhomogeneous graphs, as it can prove impossible to construct a distance graph; in such cases, interesting phenomena regarding the probability of meeting can arise.

The full triangular lattice is a homogeneous graph, and in every point the transition probabilities are $\frac{1}{8}$ for each of the six first neighbors, with probability $\frac{1}{4}$ of not moving at all.

The return probabilities for this lattice can be computed via a straightforward Fourier transform:

$$P(\vec{x},t) = \frac{9}{16\pi^2}\int d^2\vec{K}\, e^{i\frac{3}{2}\sum_j K_j n_j}\left\{\frac{1}{4}\left[1+\cos\left(\frac{3}{2}K_1\right) + \cos\left(\frac{3}{2}K_2\right)+\cos\left(\frac{3}{2}K_3\right)\right]\right\}^t,$$

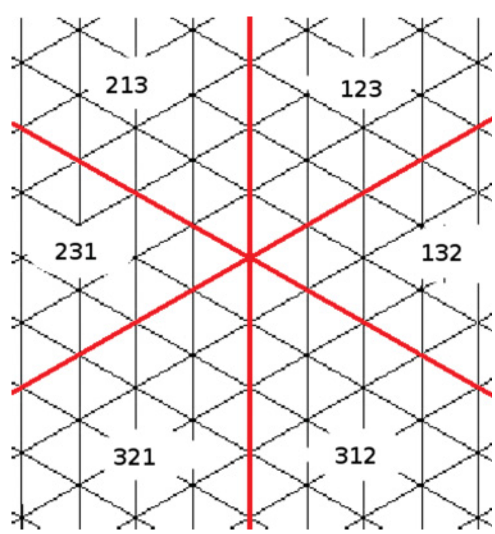

FIG. 3. (Color online) The full triangular lattice. Each slice corresponds to an ordering of the three particles, as indicated in the picture; crossing an axis corresponds to swapping two particles.

where $k_1+k_2+k_3=0$, $\vec{x}=\sum_j n_j\vec{e}_j$ is a site on the full triangular lattice, and $\vec{e}_j$ is one of the three first-neighbor vectors

$$\vec{e}_1=(1,0),$$
$$\vec{e}_2=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right).$$

The asymptotic behavior of the function can be obtained exactly as $t\to\infty$ by the steepest descent method: The maximum value of the expression

$$\frac{1}{4}\left[1+\cos\left(\frac{3}{2}K_1\right)+\cos\left(\frac{3}{2}K_2\right)+\cos\left(\frac{3}{2}K_3\right)\right]$$

is obtained for $K_1=K_2=0$. The integral is then well approximated by

$$\frac{9}{16\pi^2}\int d^2\vec{K}e^{i\frac{3}{2}\sum_j K_j n_j}e^{-\frac{9}{16}t[K_1^2+K_2^2+K_1K_2]}.$$

The Gaussian integral is straightforwardly solved by the following change of variables:

$$l_1=K_1+K_2,$$
$$l_2=K_1-K_2,$$

which yields

$$\begin{aligned}P(\vec{x},t)&\sim\frac{9}{32\pi^2}\int d^2\vec{l}\exp\left\{i\left[\frac{l_1+l_2}{2}(n_1-n_3)\right.\right.\\&\quad\left.\left.+\frac{l_1-l_2}{2}(n_2-n_3)\right]-\frac{9}{64}t\left(3l_1^2+l_2^2\right)\right\}\\&=\frac{2}{\sqrt{3}\pi}\frac{1}{t}\exp\left(-\frac{4}{3}\frac{\vec{x}^2}{t}\right),\end{aligned}\tag{1}$$

where $\vec{x}^2=n_1^2+n_2^2+n_1n_2$ and is valid for every point on the lattice as $t\to\infty$.

Taking into account the fact that each step in the distance graph corresponds to a distance 2 in the original lattice, the following holds:

$$P_{\text{line}}(x,y,t)=\begin{cases}6P\left(\frac{x}{2}\vec{e}_1+\frac{y}{2}\vec{e}_2,t\right) & \{x,y\neq0\}\\ 3P\left(\frac{x}{2}\vec{e}_1+\frac{y}{2}\vec{e}_2,t\right) & \{x=0,y\neq0\}\cup\{x\neq0,y=0\}\\ P\left(\vec{0},t\right) & \{x=0,y=0\}\end{cases}.\tag{2}$$

III. NORMALIZATION OF THE ASYMPTOTIC PROBABILITY

The probability in Eq. (1) is asymptotically exact as $t\to\infty$, and the normalization provided is the correct one for the exact initial formula. It proves useful, however, to compute in an approximate way the normalization factor, as a way to test both the saddle point method and the technique we will use later to compute various probability functions. Without further ado, the result is once again Eq. (1), as we now proceed to prove.

We look for N in the following equation:

$$1=\sum_{\vec{x}}P(\vec{x},t)=\frac{N}{t}\sum_{\vec{x}}\exp\left(-\frac{4}{3}\frac{\vec{x}^2}{t}\right).$$

We now restrict our attention to one of the six slices, which is described by the base vectors $\vec{e}_1$ and $\vec{e}_2$. The sum over the lattice can be written as follows:

$$\sum_{\vec{x}}=6\sum_{\text{slice}}-6\sum_{\text{axis}}-5P(\vec{0},t),$$

where the second and third terms are necessary as the axes and the origin are considered multiple times in the first term. As a first step, we can evaluate the weight of these terms:

$$\sum_{\text{axis}}P(\vec{x},t)=\frac{N}{t}\sum_{i=0}^{\infty}e^{-\frac{4}{3}\frac{\vec{x}^2}{t}}$$

satisfies

$$\sum_{\text{axis}}P(\vec{x},t)>\frac{N}{t}\int_0^{\infty}dx\ e^{-\frac{4}{3}\frac{x^2}{t}}=\frac{N}{t}\frac{1}{2}\sqrt{\frac{3\pi t}{4}}=\frac{N}{4}\sqrt{\frac{3\pi}{t}}$$

and

$$\sum_{\text{axis}}P(\vec{x},t)<\frac{N}{t}\left(1+\int_1^{\infty}dx\ e^{-\frac{4}{3}\frac{(x-1)^2}{t}}\right)=\left(\frac{N}{t}+\frac{N}{4}\sqrt{\frac{3\pi}{t}}\right)$$

so that

$$\sum_{\text{axis}}P(\vec{x},t)=\frac{N}{4}\sqrt{\frac{3\pi}{t}}+O(t^{-1})=O(t^{-\frac{1}{2}})$$

and

$$6\sum_{\text{axis}}P(\vec{x},t)+5P\left(\vec{0},t\right)=O(t^{-\frac{1}{2}}).$$

The next step consists of computing the sum over each slice of the exponential. This sum is best performed using the variables

$$\begin{aligned}l&=n_1+n_2\\m&=n_1-n_2\end{aligned}.$$

In this base the exponential reads

$$\exp\left[-\frac{4\left(\frac{3}{4}l^2+\frac{1}{4}m^2\right)}{3t}\right]=\exp\left(-\frac{l^2}{t}-\frac{m^2}{3t}\right).$$

The sum over the points of the slice, $\sum_{n_1}\sum_{n_2}$, is converted into the following sum:

$$\sum_{l=0,1,\ldots}\quad\sum_{m=-l,-l+2,\ldots,l-2,l}^{l};$$

the sum can now be split into two terms, for even and odd l,

$$\begin{aligned}&\textstyle\sum_{\vec{x}}P(\vec{x},t)=S+T\\&S=\textstyle\sum_{l=0,2,\ldots}e^{-\frac{l^2}{t}}\sum_{m=-l,-l+2,\ldots,l}e^{-\frac{m^2}{3t}}\\&T=\textstyle\sum_{l=1,3,\ldots}e^{-\frac{l^2}{t}}\sum_{m=-l,-l+2,\ldots,l}e^{-\frac{m^2}{3t}}.\end{aligned}$$

Using as a sum indices $q=\frac{l}{2}$ and $v=\frac{m}{2}$, S becomes

$$S=2\sum_{q=0,1,\ldots}e^{-\frac{4q^2}{t}}\sum_{v=0,1,\ldots,q}e^{-\frac{4v^2}{3t}}-\sum_{q=0,1,\ldots}e^{-\frac{4q^2}{t}};\tag{3}$$

as we have already seen, the second term is $O(t^{\frac{1}{2}})$, so we will drop it and focus on the double sum, which has now to be bound as we did earlier.

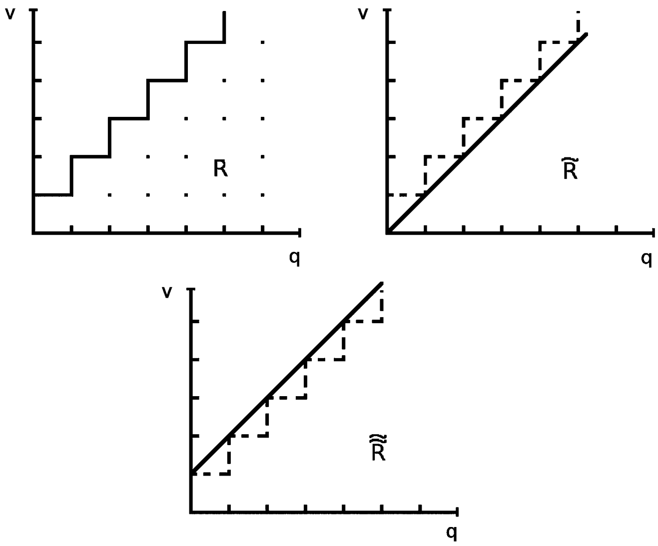

FIG. 4. The sum in Eq. (3), over the set of points in region R, can be bounded from below by an integral over $\tilde{R}$, and from above by an integral over $\tilde{\tilde{R}}$.

It is easily seen that

$$\sum_{q=0,1,\ldots} e^{-\frac{4q^2}{t}} \sum_{v=0,1,\ldots,q} e^{-\frac{4q^2}{3t}} > \int_{\tilde{R}} dq\, dv\, \exp\left(-\frac{4q^2}{t} - \frac{4v^2}{3t}\right)$$

$$= \frac{\sqrt{3}t}{4} \int_0^\infty R\, dR \int_0^{\frac{\pi}{6}} d\phi\, e^{-R^2} = \frac{\pi}{16\sqrt{3}} t,$$

where the regions R and $\tilde{R}$ are as in Fig. 4.

As to the other bound

$$\sum_{q=0,1,\ldots} e^{-\frac{4q^2}{t}} \sum_{v=0,1,\ldots,q} e^{-\frac{4v^2}{3t}} = \sum_q e^{-\frac{4q^2}{t}} + \sum_{q=1}^{\infty} \sum_{v=1}^{\infty} \exp\left(-\frac{4q^2}{t} - \frac{4v^2}{3t}\right)$$

$$< [1 + O(\sqrt{t})] + \sum_{q=0}^{\infty} \sum_{v=0}^{q} \exp\left[-\frac{4(q+1)^2}{t} - \frac{4(v+1)^2}{3t}\right]$$

$$< \int_{\tilde{\tilde{R}}} dq\, dv\, \exp\left[-\frac{32(q+1)^2}{27t} - \frac{32(v+1)^2}{27t}\right] + O(t^{\frac{1}{2}})$$

$$= \frac{\sqrt{3}t}{4} \int_0^\infty dr\, R \int_0^{\frac{\pi}{6}} d\phi\, e^{-R^2} + O(t^{\frac{1}{2}}) = \frac{\pi}{16\sqrt{3}} t + O(t^{\frac{1}{2}}),$$

which together with the previous result implies

$$S = 2\frac{\pi}{16\sqrt{3}} t + O(t^{\frac{1}{2}}) = \frac{\pi}{8\sqrt{3}} t + O(t^{\frac{1}{2}}).$$

The computation of T is perfectly analogous, with some complications in the division of the integration regions into subregions. In the end we get the result that, as above,

$$T = \frac{\pi}{8\sqrt{3}} t + O(t^{\frac{1}{2}}),$$

and

$$1 = 6N\left[\frac{\pi}{8\sqrt{3}} t + \frac{\pi}{8\sqrt{3}} t + O(t^{\frac{1}{2}})\right],$$

$$N = \frac{2}{\sqrt{3}\pi} + O(t^{-\frac{1}{2}}).$$

The normalized form of the probability is then proved to be as in Eq. (1).

IV. FIRST-RETURN PROBABILITY

It is also easy to give an estimate for the asymptotic probability $F(\vec{0},t)$ of first return to the origin: The generating functions satisfy

$$\tilde{F}(\vec{0},\lambda) = 1 - \frac{1}{\tilde{P}(\vec{0},\lambda)}.$$

From Eq. (1),

$$\tilde{P}(\vec{0},\lambda) = \sum_t \lambda^t P(\vec{0},t) \sim \frac{2}{\sqrt{3}\pi} \ln(1-\lambda),$$

as $\lambda \sim 1$, so

$$\tilde{F}(\vec{0},\lambda) \sim 1 - \frac{\sqrt{3}\pi}{2} \frac{1}{\ln(1-\lambda)}.$$

Standard asymptotic series analysis yields

$$F(\vec{0},t) \sim \frac{1}{t\,(\ln t)^2}$$

as $t \to +\infty$.

V. EUCLIDEAN DISTANCE FROM THE ORIGIN

While not physically meaningful for the three-particles problem, the probability that a random walker on the triangular lattice is inside a circle of radius ρ in the two-dimensional plane will be useful later. By definition

$$Q_\rho(t) = \sum_{\{\vec{x}:\,|\vec{x}|<\rho^2\}} P(\vec{x},t).$$

The calculation is a straightforward generalization of the steps taken earlier: The regions $\tilde{R}$ and $\tilde{\tilde{R}}$ are now of finite radii, respectively, ρ and $\rho+\sqrt{2}$. As to the details, for S the following holds:

$$\frac{\sqrt{3}\pi t}{2} S > \frac{3\sqrt{3}}{2} t \frac{\pi}{3} \int_0^{\frac{2\rho}{\sqrt{3t}}} dR\, \mathrm{Re}^{-R^2} + O(t^{\frac{1}{2}})$$
$$= \frac{\sqrt{3}\pi t}{4}\left(1 - e^{-\frac{4\rho^2}{3t}}\right) + O(t^{\frac{1}{2}}),$$

$$\frac{\sqrt{3}\pi t}{2} S < \frac{3\sqrt{3}}{2} t \frac{\pi}{3} \int_0^{\frac{2(\rho+\sqrt{2})}{\sqrt{3t}}} dR\, \mathrm{Re}^{-R^2} + O(t^{\frac{1}{2}})$$
$$= \frac{\sqrt{3}\pi t}{4}\left[1 - e^{-\frac{4(\rho+\sqrt{2})^2}{3t}}\right] + O(t^{\frac{1}{2}}).$$

Let now

$$E(\rho) = \frac{\sqrt{3}\pi t}{4} e^{-\frac{4\rho^2}{3t}}\left(1 - e^{-\frac{8}{3t}(\sqrt{2}\,\rho+1)}\right).$$

As we will be mostly interested in the case $1 \ll \rho \ll t$, we can write

$$E(\rho) = \frac{2\sqrt{2}\pi\rho}{\sqrt{3}} e^{-\frac{4\rho^2}{3t}}$$

and

$$\frac{\sqrt{3}\pi t}{2} S = \frac{\sqrt{3}\pi t}{4}\left\{1 - e^{-\frac{4\rho^2}{3t}}\left[1 + O\left(\frac{\rho}{t}\right)\right]\right\}.$$

The same procedure holds for T. The contribution of the points on the edges of each slice is $O(t^{-\frac{1}{2}})$, and it can be dropped as long as $\rho \gg 1$; one then obtains

$$Q_\rho(t) = 1 - e^{-\frac{4\rho^2}{3t}}\,[1 + O(\rho/t)]. \tag{4}$$

VI. MINIMUM DISTANCE BETWEEN PARTICLES

We have previously defined x and y as the distance between the central particle and, respectively, the leftmost particle and the rightmost one. What are the chances that, at some time t, the minimum between x and y, $d_{\min} = \min\{x,y\}$, is lower than some given value $2d$? This type of quantity can be of interest, e.g., when investigating chemical kinetics, as reactions happen at a limited range.

In the complete triangular lattice, $d_{\min}$ is the distance to the nearest of the three axes, so the probability that $d_{\min}(t) < d$ is

$$P(d_{\min} < 2d,t) = \sum_{\vec{x}\in R_d} P(\vec{x},t),$$

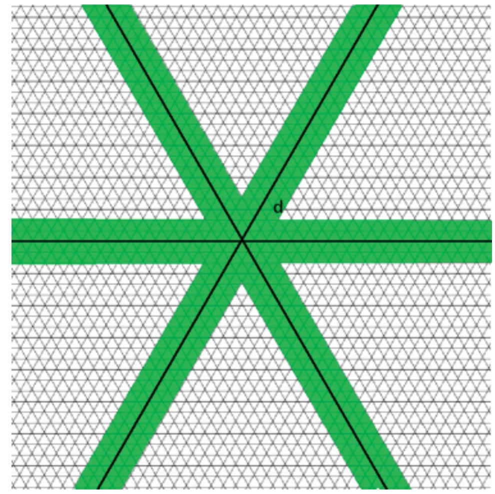

FIG. 5. (Color online) The shaded area is R_d, the region of summation for $P(d_{\min} < d,t)$.

where R_d comprises the lattice points in a region similar to the one shaded in Fig. 5.

The region R_d can be divided into two main subregions: a circle of radius $\sqrt{3}d$ and six identical infinite beams, each collinear to one of the three main axes. The weight of the circle is given by $Q_{\sqrt{3}d}(t)$, while the contribution of the infinite beams is to be computed now.

We make now two approximations: to start, we divide one beam into two pieces, one comprising all the points at distance greater than $\sqrt{3}d$, the other one the remaining points. The contribution of the latter is $\approx e^{-\frac{4d^2}{t}} O(t^{-1})$, and in hindsight we can safely ignore it. The former piece gives the main contribution, to both the beam and the $P(d_{\min} < 2d,t)$: we approximate it in a rather crude way, which can be justified for $1 \ll d \ll t^{1/2}$, of considering the distance fixed and equal to R^2 for all the lattice points orthogonal to a given point on the axis.

A procedure analogous to the one employed earlier yields, for $1 \ll d \ll t^{-1/2}$,

$$\begin{aligned}
&P(d_{\min} < 2d,t)\\
&\quad = 1 - e^{-4d^2/t} + \frac{6d}{\sqrt{\pi t}}\left[1 - \mathrm{Erf}\left(\frac{2d}{\sqrt{t}}\right)\right][1 + O(d^{-1})]\\
&\quad = 1 - e^{-4y^2} + \frac{6y}{\sqrt{\pi}}[1 - \mathrm{Erf}(2y)][1 + O(d^{-1})]\\
&\quad = \frac{6d}{\sqrt{\pi t}}[1 + O(d^{-1})]\\
&\qquad - \frac{4(6-\pi)d^2}{\pi t}[1 + O(d^{-1})] + O\left(\frac{d^2}{t}\right)^{3/2}\\
&\quad = \frac{6y}{\sqrt{\pi}}[1 + O(d^{-1})]\\
&\qquad - \frac{4(6-\pi)y^2}{\pi}[1 + O(d^{-1})] + O(y^3),
\end{aligned} \tag{5}$$

where we have set $y = \sqrt{d^2/t}$; it is worthwhile to explicitly note that the leading terms of the probability are a function of d^2/t, which thus describes the time evolution of a fixed probability surface. This regular diffusion-like behavior, unlike in the two-particle case, is not trivial: even though the probability of meeting at a given point is led, for long times, by a Gaussian propagator, the minimum distance is a strictly nonlinear function of the positions of the particles, so that *a priori* it is hazardous to guess this behavior. Furthermore, the diffusion-like behavior is not guaranteed to hold at higher orders because of the $O(d^{-1})$ error that one introduces when approximating the sums with integrals.

VII. MAXIMUM DISTANCE BETWEEN PARTICLES

Another quantity of interest is the probability that at any given moment the maximum distance between two adjacent particles is lower than $2d$: the set of points satisfying $d_{\mathrm{Max}} < 2d$ corresponds to the shaded area R in Fig. 6:

$$P\left(d_{\mathrm{Max}} < 2d, t\right) = \sum_{\vec{x} \in R} P\left(\vec{x}, t\right).$$

Such a quantity can be of interest, e.g., when investigating the robustness of a dynamic network of wireless devices, whose nodes can move randomly in space [13].

Instead of computing sums, we compute integrals, as we did in the previous sections: By doing so we introduce errors which are $O(d^{-1})$, so that we can safely ignore them as long as $1 \ll d \ll t$; we now divide R into two regions, the circle of radius d, C_d, and $R \setminus C_d$, and write

$$P\left(d_{\mathrm{Max}} < 2d, t\right) = Q_d(t) + \sum_{\vec{x} \in R \setminus C_d} P\left(\vec{x}, t\right).$$

The second piece can be written by substituting

$$l = \frac{n_1 + n_2}{2}, \qquad m = \frac{n_1 - n_2}{2}$$

and approximated with an integral over the slices as follows:

$$\sum_{\vec{x} \in R \setminus C_d} P(\vec{x}, t) = \left(\frac{1}{2\pi}\right) 12 \int_{2d/\sqrt{3t}}^{2d/\sqrt{t}} d\bar{q}\, \frac{\pi}{3} \left(\frac{\frac{2d}{\sqrt{t}} - \bar{q}}{\sqrt{3} - 1}\right) e^{-\bar{q}^2}[1 + O(d^{-1})].$$

The result can be stated in an analytic, if somewhat unclear, formula as follows:

$$P\left(d_{\mathrm{Max}} < 2d, t\right) = 1 + \left\{-\frac{\sqrt{3}}{\sqrt{3} - 1} e^{-\frac{4d^2}{3t}} + \frac{1}{\sqrt{3} - 1} e^{-\frac{4d^2}{t}} + \frac{2\sqrt{\pi}}{\sqrt{3} - 1} \frac{d}{\sqrt{t}} \left[\mathrm{Erf}\left(\frac{2d}{\sqrt{t}}\right) - \mathrm{Erf}\left(\frac{2d}{\sqrt{3t}}\right)\right]\right\} [1 + O(d^{-1})].$$

A straightforward computation leads, in the same limits as before, $1 \ll d \ll t^{1/2}$, to

$$P\left(d_{\mathrm{Max}} < 2d\right) = \frac{4}{\sqrt{3}} \frac{d^2}{t}[1 + O(d^{-1})] - \frac{8}{27}[3 + 4\sqrt{3} + O(d^{-1})] \left(\frac{d^2}{t}\right)^2 + \frac{32}{405}[12 + 13\sqrt{3} + O(d^{-1})] \left(\frac{d^2}{t}\right)^3 + O\left(\frac{d^2}{t}\right)^4.$$

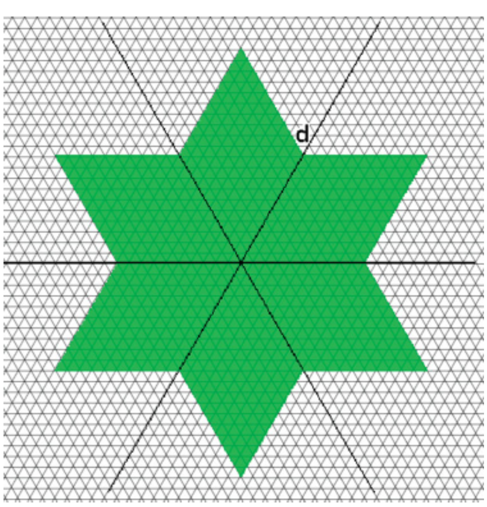

FIG. 6. (Color online) The shaded area is R, the region of summation for $P\left(d_{\mathrm{Max}} < 2d, t\right)$.

VIII. PARTICLES WITH BOUNDED MAXIMUM AND MINIMUM DISTANCE

To complete the picture, we next compute the probability that the pairs of adjacent particles are not closer than $2d_{\mathrm{inf}}$, and not farther apart than $2d_{\mathrm{sup}}$:

$$P(d_{\min} > 2d_{\mathrm{inf}}, d_{\mathrm{Max}} < 2d_{\mathrm{sup}}, t) = \sum_{R_m} P(\vec{x}, t),$$

where R_m is the shaded region in Fig. 7. This quantity is of interest in reactions involving a catalyst, e.g., in a cell, as it is known [14] that an optimal distance exists at which the reaction is favored.

As in the previous section, we approximate the sum with an integral and obtain

$$P(d_{\min} > 2d_{\inf}, d_{\mathrm{Max}} < 2d_{\sup}, t) = \frac{12}{\sqrt{3}\pi}\left[\int_{2d_{\inf}/\sqrt{t}}^{(d_{\inf}+d_{\sup})/\sqrt{t}} dq\left(q - \frac{2d_{\inf}}{\sqrt{t}}\right)e^{-q^2} + \int_{(d_{\inf}+d_{\sup})/\sqrt{t}}^{2d_{\sup}/\sqrt{t}}\left(\frac{2d_{\sup}}{\sqrt{t}} - q\right)e^{-q^2}\right]\left(1 + O\left[d_{\inf}^{-1}\right)\right]$$
$$= 6\left\{\frac{e^{-\frac{4d_{\inf}^2}{t}} + e^{-\frac{4d_{\sup}^2}{t}} - 2e^{-\frac{(d_{\inf}+d_{\sup})^2}{t}}}{\sqrt{3}\pi} + \frac{2\left[\mathrm{Erf}\left[\frac{2d_{\inf}}{\sqrt{t}}\right]d_{\inf} + \mathrm{Erf}\left[\frac{2d_{\sup}}{\sqrt{t}}\right]d_{\sup} - \mathrm{Erf}\left[\frac{d_{\inf}+d_{\sup}}{\sqrt{t}}\right](d_{\inf}+d_{\sup})\right]}{\sqrt{3\pi t}}\right\}\left[1 + O\left(d_{\inf}^{-1}\right)\right]. \quad (6)$$

This exact result is quite cumbersome, but one can extract the leading terms in the expansion under the conditions $1 \ll d_{\inf} < d_{\sup} \ll t^{1/2}$:

$$P(d_{\min} > 2d_{\inf}, d_{\mathrm{Max}} < 2d_{\sup}, t) = \frac{12\left(d_{\inf}^2 - 2d_{\inf}d_{\sup} + d_{\sup}^2\right)}{\sqrt{3}\pi t}\left[1 + O(d_{\inf}^{-1})\right] - 6\frac{7d_{\inf}^4 - 4d_{\inf}^3 d_{\sup} - 6d_{\inf}^2 d_{\sup}^2 - 4d_{\inf}d_{\sup}^3 + 7d_{\sup}^4}{3\sqrt{3}\pi t^2} \times \left[1 + O(d_{\inf}^{-1})\right].$$

As soon as we set $d_{\sup} = kd_{\inf}$, the asymptotic diffusion-like behavior becomes apparent:

$$P(d_{\min} > 2d_{\inf}, d_{\mathrm{Max}} < 2d_{\sup}, t) = \frac{12}{\sqrt{3}\pi t}(k-1)^2 d_{\inf}^2\left[1 + O(d_{\inf}^{-1})\right].$$

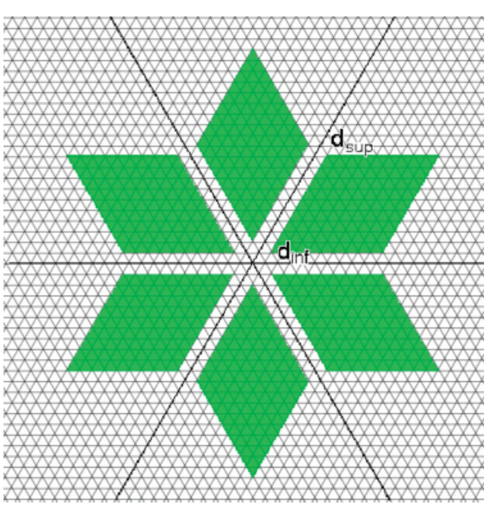

FIG. 7. (Color online) The region of summation for $P(d_{\min} > 2d_{\inf}, d_{\mathrm{Max}} < 2d_{\sup}, t)$ is the shaded area.

IX. INTERACTING MULTIPLE RANDOM WALKS

It is indeed possible to extend the previous results to two different types of interacting multiple random walks.

First, consider a random walk, on an infinite line, for three particles that can stay on the same vertex but never change their ordering. Let x be the distance between the leftmost particle and the central one, and let y be the distance between the central particle and the rightmost one. The motion of the particles can be easily represented on the slice in Fig. 2. As long as x and y are both greater than zero (everywhere outside the axes and the origin), the most straightforward choice for the transition rates is the same as in the case of noninteracting particles. When $x = 0$ or $y = 0$, instead, the probability distribution for the outcoming configurations are arbitrary, depending on what physical sense we give to the encounters among particles.

For the probability of transition on the axes (excluding the origin), we decide that when two particles try to cross one another they bounce back. This immediately entails the same probability of transition as in the noninteracting case, and we can once again describe the problem in a graph such as the one in Fig. 3. The difference is that each triangular slice is a mirror copy of the adjacent ones, while in the first case each slice carries a specific ordering of the particles. The probability $P^{\mathrm{nc}}(\vec{x},t)$ of finding the three noncrossing particles at given distances $(2x,2y)$ is then obtained by summing the probability for noninteracting particles over all the corresponding points:

$$P^{\mathrm{nc}}(\vec{x},t) = \begin{cases} 6P(\vec{x},t) & x,y > 0, \\ 3P(\vec{x},t) & x = 0, y > 0, \\ 3P(\vec{x},t) & x > 0, y = 0, \\ P(\vec{x},t) & x = y = 0. \end{cases} \quad (7)$$

The other quantities, which are already summed over equivalent points, are the same as in the noninteracting case.

A slightly more complex case is when the three walkers are solid and can neither sit on the same vertex nor cross each other. In this case the motion can be described on the interior of the slice in Fig. 2, i.e., on sites of the form

$$\vec{x} = \frac{x}{2}\vec{e}_1 + \frac{y}{2}\vec{e}_2,$$

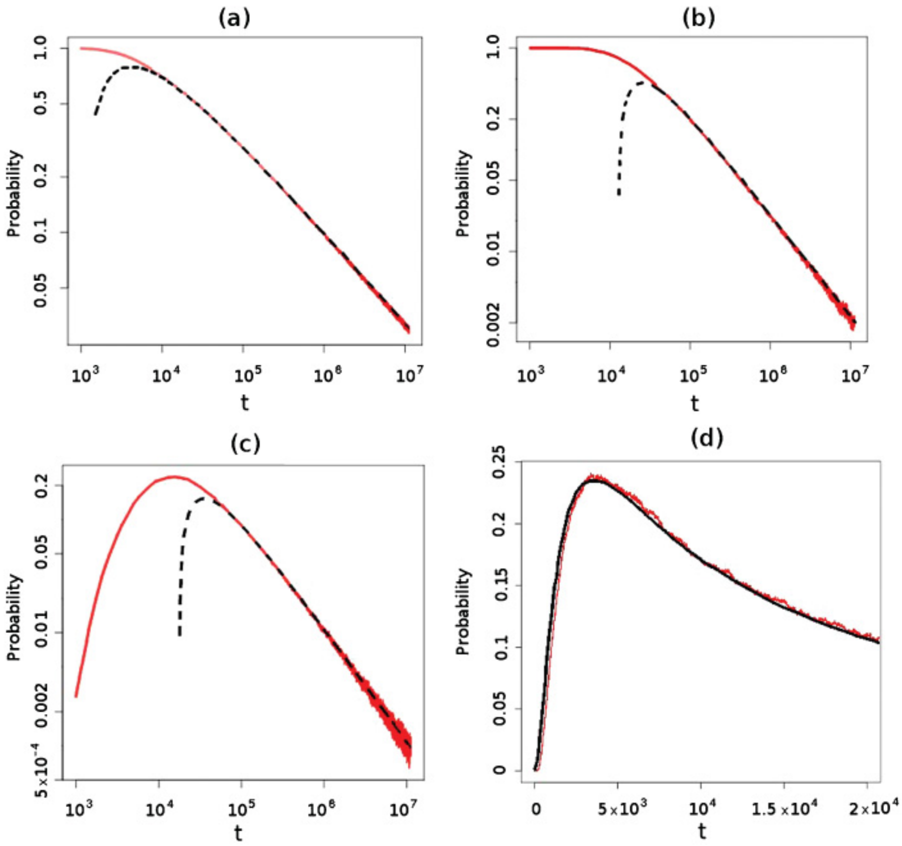

FIG. 8. (Color online) In panels (a) to (c), the black dashed lines show the asymptotic behavior, and solid red lines are the results of simulations for $d_{\mathrm{inf}} = 30$, $d_{\mathrm{sup}} = 100$. In panels (d) the solid regular black line shows the asymptotic behavior, while the jagged red line is the results of simulations with the same parameters. (a) $P(d_{\mathrm{min}} < 2d_{\mathrm{inf}},t)$ is plotted together with the first two terms of Eq. (5). (b) $P(d_{\mathrm{Max}} < 2d_{\mathrm{sup}},t)$ and (c) $P(d_{\mathrm{inf}} < 2d_{\mathrm{min}},d_{\mathrm{Max}} < 2d_{\mathrm{sup}},t)$ are also plotted together with the second-order behavior. (d) The expression for $P(d_{\mathrm{inf}} < 2d_{\mathrm{min}},d_{\mathrm{Max}} < 2d_{\mathrm{sup}},t)$ from Eq. (6) is plotted at intermediate times together with simulation results. Other choices for d_{inf} and d_{sup} give similar results.

with $x,y \geqslant 2$, which correspond to distances (x,y). Whenever $x,y > 2$, the probability of transition is the same as in the noninteracting case. When two particles are adjacent ($x = 2$ or $y = 2$, but not both), our choice about bouncing yields the following probabilities of transition:

$$p\,((2,y) \to (2,y)) = \frac{1}{4},$$

$$p\,((2,y) \to (2,y-2)) = p\,((2,y) \to (2,y+2)) = \frac{1}{8},$$

$$p\,((2,y) \to (4,y-2)) = p\,((2,y) \to (4,y)) = \frac{1}{4}.$$

When the particles are all adjacent ($x = y = 2$), the probabilities are as follows:

$$p\,((2,2) \to (2,2)) = \frac{1}{4},$$

$$p\,((2,2) \to (2,4)) = p\,((2,2) \to (4,2)) = \frac{3}{8}.$$

If we now send (x,y) to $(x-2,y-2)$, we can immediately recognize that all these probability are the same as in the case of noncrossing particles. Thus for solid random walkers the probability function $P^{\mathrm{srw}}(\vec{x},t)$ satisfies

$$P^{\mathrm{srw}}(\vec{x},t) = P^{\mathrm{nc}}(\vec{x} - \vec{e}_1 - \vec{e}_2),t), \tag{8}$$

and analogously for the other quantities we have computed.

While the choice we have made for the transition probability of adjacent particles can be justified, from an intuitive point of view, by calling up independence of the Brownian motion of different particles and momentum conservation in collisions, it should be stressed that it is nevertheless an arbitrary position, and different choices will yield different results.

X. DISCUSSION

The present paper explores an area that up to now has been just grazed by research; we have been able to compute the leading terms of several probability functions, all related to collective properties of the random walks of three particles, namely the minimum and maximum distances among particles. These quantities are not of mere academic interest, as they have possible applications in diverse fields, from the study of dynamic wireless networks [13] to reaction kinetics involving catalysts [14].

The asymptotic behaviors for the three main quantities computed have been verified by means of simulations, as exemplified in Fig. 8: after a transient, the asymptotic behavior sets in for times of the order $t \gtrsim d^2$, as expected.

For large times t and large distances d, our results highlight that such probability distributions display a diffusion-like scaling as d^2/t, although they regard nonlinear functions of mutual distances. This regularity, which is hinted at by the Gaussian shape of the probability of meeting but is nevertheless nontrivial because of the nonlinearity of the minimum and maximum distances, breaks down for intermediate distances, where a richer behavior is revealed as the integral approximation of the sum over lattice sites no longer holds.

The procedure we followed also allows for the computation of other interesting quantities, related to collective properties of the random walks, which we neglected; furthermore it lets one straightforwardly introduce interparticle interactions as biases in the distance graph. In particular, we have examined two different models for vicious interacting walkers, and we have shown their relation to the noninteracting case.

The extension to four and more walkers of our results involves random walks in multiple dimensions on structures analogous to the triangular lattice in Fig. 3, and will be the object of further research.

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