Electronic and mechanical realizations of one-way coupling in one and two dimensions

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One-way or unidirectional coupling is a striking example of how topological considerations—the parity of an array of multistable elements combined with periodic boundary conditions—can qualitatively influence dynamics. Here we introduce a simple electronic model of one-way coupling in one and two dimensions and experimentally compare it to an improved mechanical model and an ideal mathematical model. In two dimensions, computation and experiment reveal richer one-way coupling phenomenology: In media where two-way coupling would dissipate all excitations, one-way coupling enables solitonlike waves to propagate in different directions with different speeds.

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I. INTRODUCTION

One-way or unidirectional coupling has the notable effect of facilitating the propagation of solitary waves or solitons in media that would normally be dissipative. It is a recent paradigm for the effects of topology on dynamics [1–7]. This type of coupling, described in detail below, was first introduced by In *et al.* to improve the performance of fluxgate magnetometers [1]. Recently, Lindner *et al.* realized a simple mechanical model of generic one-way coupling that facilitates the study of arrays of many elements [8].

Here we describe an even simpler physical model of one-way coupling. Our apparatus uses common electronic components instead of mechanical elements and readily generalizes to two and higher dimensions. We experimentally compare the behavior of the electronic model in one dimension with an improved version of the mechanical model and demonstrate their qualitatively similar dynamics.

The compact electronic array permits the experimental exploration of two-dimensional one-way coupled arrays. We compare the behavior of the two-dimensional arrays with predictions from theory and simulation, where soliton-antisoliton pairs spatially separate domains of degenerate ground states. For different initial conditions, solitons propagate at different speeds and directions corresponding to topologically distinct modes. Common to our ideal mathematical model and our mechanical and electronic apparatuses is the logical idea of a "reverser," a kind of coupling among array elements in which the state of one element is reversely proportional to the state of the previous element.

II. IDEAL REVERSER ARRAY

Consider an array of bistable oscillators described by

$$\phi'_x = \phi_x - \phi_x^3 - \kappa\phi_{x-1}, \tag{1}$$

for $x = 1,2,\ldots,N$, with periodic boundary conditions $\phi_0 = \phi_N$, where the primes indicate differentiation with respect to time t and κ is the coupling strength. The final term $-\kappa\phi_{x-1}$ in Eq. (1) is the *reverser*, a torque reversely proportional to the previous oscillator's deflection.

For two dimensions, generalize this to

$$\phi'_{x,y} = \phi_{x,y} - \phi_{x,y}^3 - \kappa(\phi_{x,y-1} + \phi_{x-1,y}), \tag{2}$$

with periodic boundary conditions. Because each oscillator is influenced only from below and left (on a standard Cartesian plane), we define the coupling direction to be from bottom-left to top-right, or $\hat{v}_{11} = (\hat{x} + \hat{y})/\sqrt{2}$.

The phenomenology of the one-dimensional array is well known [5]. From random initial conditions, moving discontinuities referred to as solitons separate regions of opposite degenerate ground states in which oscillators alternate between their two stable equilibria. These are soliton-antisoliton pairs in the sense that a trailing soliton reverses the oscillator equilibria left by a leading soliton, and they annihilate in pairs upon close approach. In even arrays, annihilations result in a global ground state of quiescent oscillators. However, in odd arrays, one soliton is always left over, propagating endlessly in a frustrated attempt to reach an impossible global equilibrium. Soliton speed increases with coupling, but also depends on noise and disorder (temporal and spatial inhomogeneity).

Here we extend these results to higher dimensions using computer simulations. We numerically integrate Eq. (2) using a fourth-order Runge-Kutta algorithm implemented in C/C++ with typical time steps of $dt = 0.01$. We check this with backward differentiation formula (BDF) and Adams integrators implemented using MATHEMATICA.

In two dimensions, periodic boundary conditions enforce a toroidal topology. Moving discontinuities still separate regions of degenerate ground states, but are now spatially extended from points to lines. For small coupling κ, individual oscillators are stuck in their own equilibria and nothing propagates.

For intermediate coupling, topologically distinct modes move at discretely different speeds and directions $\vec{v}_{mn}$, where m:n represents a soliton that wraps m times in the x direction for n times in the y direction, as in Fig. 1. Symmetric modes have $m = n$ and asymmetric modes have $m \neq n$. The maximum speed v_{11} is only realized for solitons that propagate in the coupling direction $\hat{v}_{11}$.

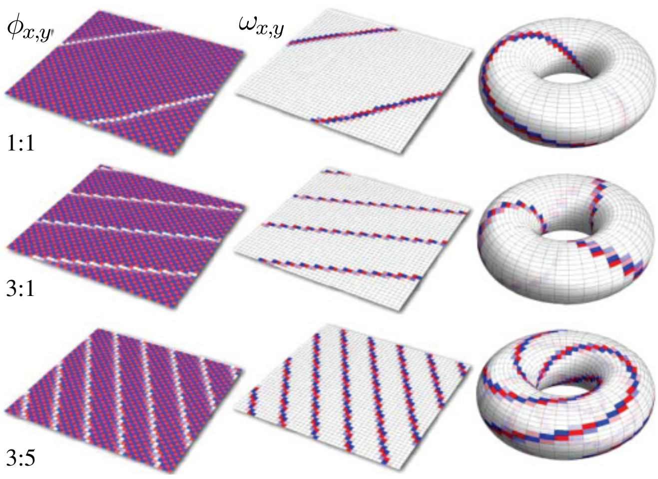

FIG. 1. (Color online) Simulated angles $\phi_{x,y}$ (left) and angular velocities $\omega_{x,y}$ (right) for two-dimensional m:n modes in a 41 × 41 array for coupling $\kappa = 0.74$; toroidal views emphasize doubly periodic boundary conditions. Such solitons are robust with respect to noise and disorder. Contrasting shades indicate positive or negative values, white represents zero.

For large coupling, random initial conditions generate additional very slow modes at large propagation angles $\alpha > 45^\circ$ from the symmetric 1 : 1 mode that are mirror images of the asymmetric modes. For example, the wave front of a −1 : 3 mode has the negative slope of the wave front of a 1 : 3 mode, as shown in Fig. 2(top).

About half of randomly selected initial conditions result in static −1 : 1 dislocations orthogonal to the fast 1 : 1 wave front, but all others induce solitons. Almost all these solitons are in the fast 1 : 1 mode but about 1 in 10^2 are in the slower ±1 : 3 and ±3 : 1 modes at angles $\alpha \approx 27^\circ$ and $\alpha \approx 63^\circ$ to either side of the coupling direction. For sufficiently large arrays, about 1 in 10^4 solitons is in the 3 : 3 mode, as summarized by Fig. 2(bottom).

As coupling κ increases, the time to hop between stable equilibria decreases. Consequently, soliton speed v_{mn} increases monotonically with coupling. As the propagation angle α from the coupling direction increases, torque on individual oscillators decreases and is directed increasingly along the wave front (instead of orthogonal to it). Consequently, soliton speed v_{mn} decreases with propagation angle α. All modes are robust with respect to noise and disorder.

III. MECHANICAL REVERSER ARRAY

A recent paper [8] describes a mechanical apparatus built at The College of Wooster that realizes one-way coupling and uses it to investigate the annihilation of soliton-antisoliton pairs. Here, we improve this device and use the improved version as a benchmark for our electronic apparatus described below in Sec. IV.

Like the Wooster apparatus, our device employs seesawlike bistable oscillators consisting of inverted pendulums balanced by restoring springs. If one seesaw rotates clockwise, a mechanical reverser, described by Fig. 3, rotates the next seesaw counterclockwise. The reversers direct falling water to force adjacent seesaws into opposite equilibria, thereby

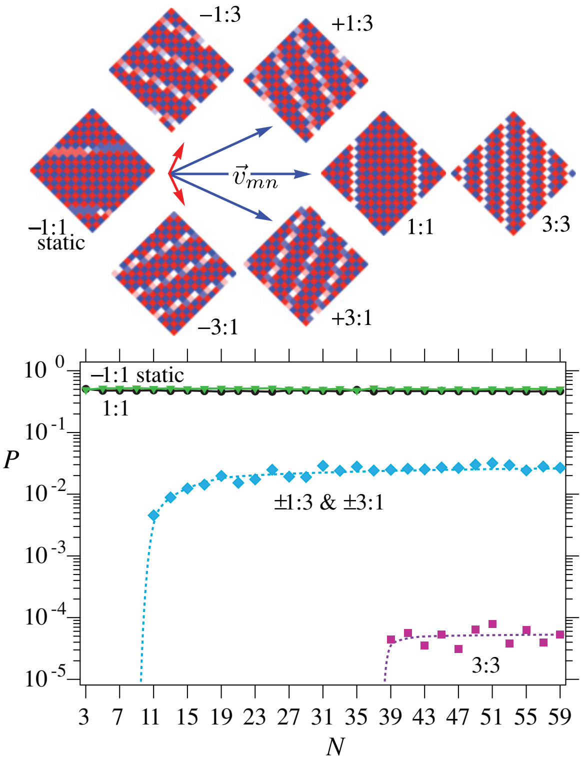

FIG. 2. (Color online) (top) Simplest modes of a 15 × 15 array and their propagation velocity vectors $\vec{v}_{mn}$ for coupling $\kappa = 1$. (bottom) Numerically estimated steady state probabilities P for $N \times N$ arrays starting from random initial conditions indicate relative basin of attraction sizes for such modes.

exploiting the fact that the downward force of the water jet is independent of the transverse force that directs it. Unlike the Wooster device, where water weight supplements jet pressure, our device uses jet pressure alone to torque each oscillator, thereby more closely mimicking the $-\kappa\phi_{x-1}$ reverser terms in the idealized one-dimensional array of Sec. II [9].

To test the array, we record the time to annihilation T as a function of the initial soliton separation ΔN for an $N = 16$

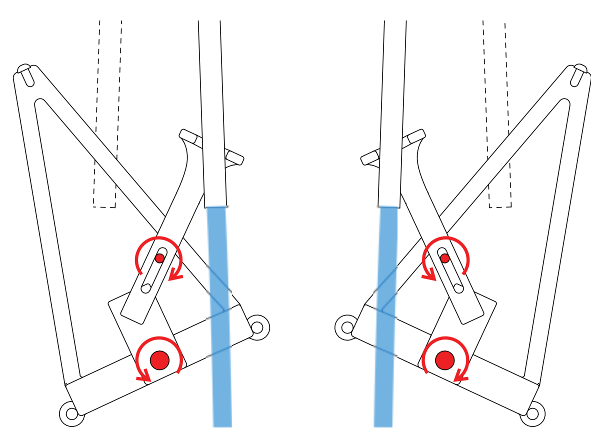

FIG. 3. (Color online) Mechanical instantiation of the Eq. (1) reverser harnesses falling water. When the previous water jet (dashed) deflects a seesaw counterclockwise, the linked arm deflects the next water jet clockwise (left), and vice versa (right). Shaded disks represent fixed axles parallel to the array. Video online [9].

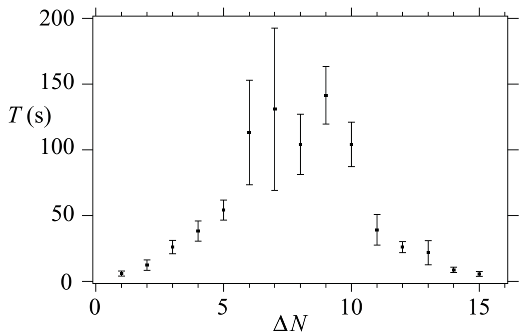

FIG. 4. Experimental distribution of 150 mechanical array annihilation times T in seconds for solitons initially separated by ΔN oscillators in an array of length $N = 16$.

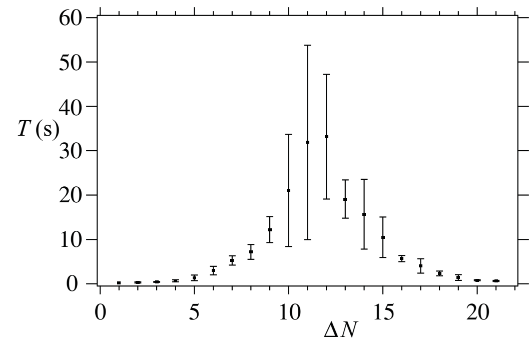

FIG. 6. Experimental distribution of 305 electronic array annihilation times T in seconds for solitons initially separated by ΔN oscillators in an array of length $N = 24$.

mechanical array, as reported in Fig. 4. As expected, the larger the initial separations of the solitons, the longer they survive before annihilation. While periodic boundary conditions make ΔN and $N - \Delta N$ equivalent, asymmetries in the plot reflect slight rotational asymmetries in the oscillators. Annihilation time studies are difficult computationally because of the very long transient times, but the results agree well with previous work [8].

IV. ELECTRONIC REVERSER ARRAY

Generalizing our mechanical reverser array to study one-way coupling dynamics in two dimensions is nontrivial. Instead, we fabricate an electronic circuit that mimics the mathematical reverser terms in the idealized arrays of Sec. II and the mechanical reverser of Sec. III.

The key component is a complementary metal oxide silicon (CMOS) inverter, which contains both an n-type and a p-type field effect transistor. High input voltage causes current to flow through the n-channel producing low output voltage, while low input voltage causes current to flow through the p-channel producing high output voltage, as in Fig. 5. The high and low voltages correspond to the bistable states of the oscillators in the ideal array.

Connecting an odd number of such inverters (or NOT gates) in series with periodic boundary conditions forms a *ring oscillator*. Such oscillators are well known in electrical engineering [10]. For example, they often provide the frequency standard for phase lock loop control systems, with frequencies typically in the megahertz range. Recently, ring oscillators have been connected by diodes to create a chaotic circuit [11].

At the nodes that connect the output of one inverter to the input of the next inverter, we insert a capacitor in series with bidirectional bicolored light-emitting diodes (LEDs) to ground. A constant 16-V voltage across the inverters powers the coupling, a 470-μF capacitance slows the solitons to 1-s time scales, and the LEDs visualize the currents. We use the open source analog electronic circuit simulator SPICE [12] to vet our designs. (For our larger arrays, we generate the SPICE code algorithmically using MATHEMATICA.)

The phenomenology of the electronic arrays resembles that of the ideal and mechanical arrays. For example, in an array of even length, briefly connecting two nodes, either manually or using a simple timer circuit, creates a soliton-antisoliton pair. We record the time to annihilation T as a function of initial separation ΔN, as reported in Fig. 6. As in the mechanical array, the larger the initial separation of the solitons, the longer they survive before annihilation.

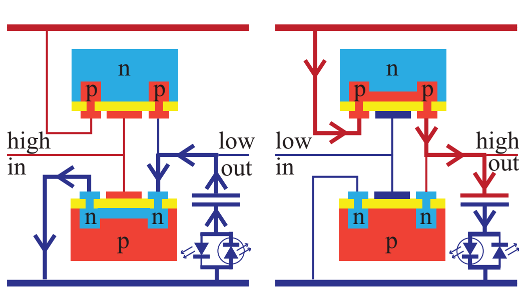

FIG. 5. (Color online) Electronic instantiation of the Eq. (1) reverser employs a CMOS inverter. High input voltage causes current to flow through the n channel discharging the capacitor and producing low output voltage (left), and vice versa (right). Bidirectional LEDs visualize the currents.

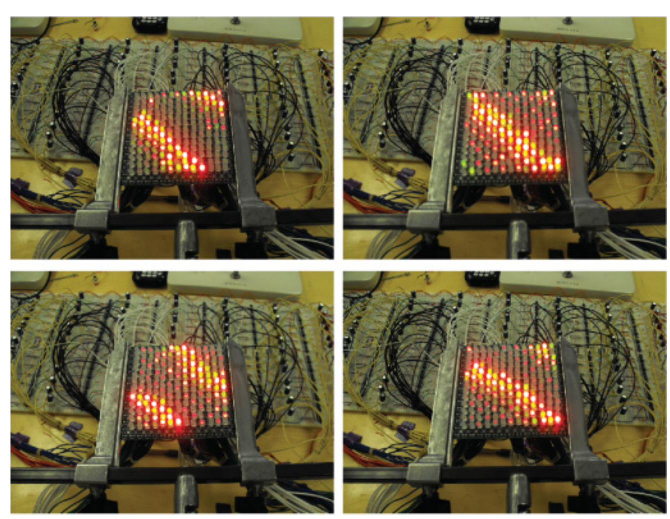

FIG. 7. (Color online) Inexpensive electronic 13×13 array of CMOS inverters and capacitors on prototype boards (background) facilitates exploration of one-way coupling in two dimensions. Sequence of a soliton in a 1 : 1 mode visualized by a square grid of LEDs on a printed circuit board (foreground).

In two dimensions, our electronic analog of one-way coupling features four CMOS inverters per node in a square array with periodic boundary conditions. Note that the Eq. (2) coupling term is proportional to the *average* of the below and left oscillators. Consequently, we connect each node to CMOS inverter outputs from the left and below and to CMOS inverter inputs to the right and above [13]. As before, a capacitor and an LED in series connect each node to ground. Figure 7 shows a 13×13 node electronic array with $2 \times 13 \times 13$ CMOS inverters built on prototype boards as it sustains a soliton in a $1:1$ mode.

To initialize the array, $2 \times 13 + 1 = 27$ solid state relays controlled by a single mechanical switch isolate one 13×1 row. Manually closing the switch closes all the relays, reconnects the row, and initiates either a $1:1$ soliton or a $-1:1$ static discontinuity with approximately equal probability, in good agreement with the Fig. 2 basin of attraction sizes.

V. CONCLUSIONS

One-way coupling is more than a mathematical curiosity. We have realized mechanical and electronic arrays that are radically different instantiations of the same one-way physics. The key feature is coupling that reverses adjacent states. We have also demonstrated that one-way coupling induces rich additional behavior in two dimensions.

In the future, we hope to use electronic analogs of one-way coupling in two and higher dimensions to systematically investigate the dynamical effects of noise and disorder. Field programmable gate arrays (FPGAs) might facilitate realization of larger electronic analogs. Preliminary computer simulations already suggest additional phenomena in three dimensions.

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