Controlling birhythmicity in a self-sustained oscillator by time-delayed feedback

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Time-delayed feedback is a practical method for controlling various nonlinear dynamical systems. We consider its influence on the dynamics of a multicycle van der Pol oscillator that is birhythmic in nature. It has been shown that depending on the strength of delay the bifurcation space can be divided into two subspaces for which the dynamical response of the system is generically distinct. We observe an interesting collapse and revival of birhythmicity with the variation of the delay time. Depending on the parameter space the system also exhibits a transition between birhythmicity and monorhythmic behavior. Our analysis of amplitude equation corroborates with the results obtained by numerical simulation of the dynamics.

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I. INTRODUCTION

Limit cycle oscillations are ubiquitous in natural phenomena and are universally encountered in a variety of nonlinear dissipative dynamical systems. Examples are abundant, with periods ranging from cardiac rhythms of seconds, glycolysis over minutes, and circadian oscillations over 24 hours, while epidemiological oscillations extend even over years. A limit cycle oscillator, by nature, is self-sustained and possesses a mechanism to damp itself when grows too large and a source of energy to pump when it becomes too small. Although natural dynamical systems are mostly characterized by only one stable limit cycle, the coexistence of two stable limit cycles or attractors separated by an unstable one is not unknown. This is referred to as birhythmicity characterized by two different amplitudes and frequencies depending upon the initial condition. Such birhythmic behavior can be observed in biological systems, as, for example, in a glycolytic oscillator [1,2], enzymatic reactions, and some biochemical reactions [3–6]. The focus of this study is to explore how to control the nonlinear dynamics of birhythmicity. The prototypical example chosen for this purpose is a variant of well-known van der Pol oscillator proposed by Kaiser [7–9] to model enzyme reactions in some biosystems.

To put the discussion in an appropriate perspective we begin with a note that a multicycle nonlinear oscillator is characterized by its multistability. Appropriate control of multistability by destroying some of the coexisting attractors while preserving the others offers an opportunity to control dynamical systems [10,11]. A method for controlling bistability by annihilation of an individual attractor was first suggested by Pisarchik and Goswami [12]. Extension of the work to control multistability [13] also followed. This has been proved to be useful for applications in optical physics. For example, in lasers, the multistability plays a key role in limiting its performance characteristics. Recently, the control of multistability in nonautonomous systems with coexisting attractors has been realized in semiconductors and fiber lasers with modulated pump parameters [14,15]. A convenient method of control is based on use of appropriate time delay on the dynamical system since time delay is important in many natural systems due to the finite speed of propagation of signals, finite processing times in synapses, and specific reaction time in elementary steps of a chemical reaction. An early attempt in this direction was made by Ott *et al.* who employed input signals constructed from the difference between the current and past states [16]. Another simple, efficient and noninvasive scheme of time delayed feedback control is due to Pyragas [17,18] since the method does not require detailed information of the system to be controlled. Pyragas control was used in Refs. [19,20]. The method of time-delayed feedback and its variants have also been used to control chaos and stabilize unstable oscillations or steady states in spatially homogeneous as well as inhomogeneous systems [21–25]. The control of bistability by changing both the feedback strength and delay time was applied in a semiconductor laser [26]. The object of this paper is to understand the dynamic response of a birhythmic system toward a delayed feedback. We are particularly interested in the time-delayed feedback of the form $F(t) = K[s(t-\tau) - s(t)]$ on a multicycle van der Pol oscillator, where $s(t)$ is the signal coming from the system, K is the feedback strength, and τ is the delay time. We show that the amplitude equation for this oscillator assumes the normal form of a codimension-two saddle-node bifurcation. The delay time acts as an imperfection parameter that can be tuned to control the bifurcation scenarios. Depending upon the strength of delay it is possible to divide the bifurcation parameter space corresponding to differential response of the system toward delayed feedback. This may result in the transition from birhythmicity to monorhythmic behavior and collapse of birhythmicity on a steady state and its subsequent revival.

The paper is organized as follows: In the next section we explore the influence of time-delayed feedback on the multicycle van der Pol oscillator. The amplitude equations are derived, and the existence and stability of the dynamical attractors have been analyzed. Our theoretical analysis is verified by numerical simulation of the dynamical equation. The paper is concluded in Sec. III.

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II. DYNAMICS OF A SELF-SUSTAINED BIRHYTHMIC OSCILLATOR UNDER DELAYED FEEDBACK

A. The multicycle van der Pol oscillator

We begin with a variant of a classical van der Pol oscillator with a damping term that is a nonlinear function of polynomial of higher order described by the following equation:

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = 0. \tag{2.1}$$

Here the overdot denotes the derivative with respect to time and μ, α, β are positive parameters that can modulate the nonlinearity of the system. This simple equation is a specific form of Lienard's equation given by $\ddot{x} + f(x)\dot{x} + g(x) = 0$, which has been interpreted [27] mechanically as an equation of a particle with unit mass subjected to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$. Equation (2.1) was also considered earlier [7–9] to simulate certain specific processes in biophysical systems. When employed to model biochemical systems [28], namely, the enzymatic-substrate reactions, x in Eq. (2.1) is considered to be proportional to the population of enzyme molecules in the excited polar state. The parameters α and β measure the degree of tendency of the system to a ferroelectric instability, and μ is a parameter that effectively refers to strength of nonlinear damping. We refer to Ref. [28] for more details. The oscillator described by Eq. (2.1) exhibits self-sustained oscillations possessing more than one limit cycle, which is the condition for occurrence of birhythmicity.

Following Kadji *et al.* [28], we briefly summarize the nature of dynamical attractors of the multicycle van der Pol oscillator described by Eq. (2.1). The periodic solutions of Eq. (2.1) can be approximated by

$$x(t) = A\cos\omega t. \tag{2.2}$$

The approximate analytic estimates of amplitude A and frequency ω can be readily obtained [28]. It has been found that the amplitude A is independent of parameter μ, which enters only in the frequency ω. The amplitude equation is given by

$$\frac{5\beta}{64}A^6 - \frac{\alpha}{8}A^4 + \frac{1}{4}A^2 - 1 = 0, \tag{2.3}$$

which appears generic for codimension-two saddle-node bifurcation. Depending on the value of the parameters α and β the oscillator possesses one or three limit cycles. Equation (2.3) can give rise to one or three positive real roots that correspond, respectively, to one stable limit cycle or three limit cycle solutions (of which two are stable and one is unstable). The unstable limit cycle represents the separatrix between the basins of attractions of the two stable limit cycles. Figure 1 describes the bifurcation lines that enclose the region (white) of birhythmicity in the two-parameter domain (α,β). The two bifurcation lines meet at a cusp point along with the generation of saddle-node bifurcations for the outer or larger amplitude limit cycle and the inner or smaller amplitude limit cycle [29–31]. In the single-limit cycle zone below the lower bifurcation line, it is found that the limit cycle is of smaller amplitude compared to the single-limit cycle with larger amplitude in the region above the upper bifurcation line. The analytic prediction thus corresponds to the stable limit cycles and their corresponding basins of attractionm which can be deduced from a direct numerical integration of Eq. (2.1) using the fourth-order Runge-Kutta algorithm.

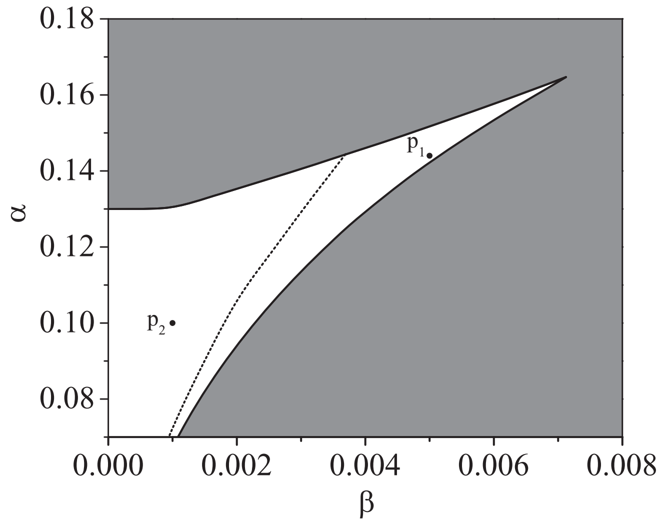

FIG. 1. Parameter domain for the existence of a single limit cycle (area denoted by gray region) and the three limit cycles (area denoted by the white region without the dotted line in absence of delayed feedback). The dotted line denotes the bifurcation line showing the two parameter zones corresponding to the points p_1 (0.005,0.144) and p_2 (0.001,0.1) for two distinct dynamical scenarios in presence of delayed feedback corresponding to Figs. 4 and 5, respectively.

B. Control of birhythmicity using delayed feedback: Amplitude equation

A time delayed feedback is known to be an efficient tool [17–22,28–32] for continuous-time control of dynamical systems. The dynamical behavior of various physical and biological systems under the influence of delayed feedback can be modeled by including terms with delayed arguments in the equations of motion. Application of delayed feedback results in the following equation:

$$\ddot{x} + x = \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + K(\dot{x}_\tau - \dot{x}). \tag{2.4}$$

Here $\dot{x}_\tau$ denotes the delayed variable $\dot{x}(t-\tau)$, K is the strength of the delayed feedback, and τ is the delay time ($\tau > 0$). The time delay τ is considered to be our control parameter. The approximate amplitude and frequency of the system described by Eq. (2.4) can be found by using harmonic balance method. For this purpose let us assume the approximate solution of the above equation as

$$x(t) = A\cos\Omega t, \tag{2.5}$$

where A is the amplitude and Ω is the frequency of the oscillator with delayed feedback. We then arrive at

$$\begin{aligned}[-A\Omega^2 + A - KA\Omega\sin\Omega\tau]\cos\Omega t\\ = -\mu A\Omega\left[1 - \frac{1}{4}A^2 + \frac{\alpha}{8}A^4 - \frac{5\beta}{64}A^6\right]\sin\Omega t\\ - KA\Omega(\cos\Omega\tau - 1)\sin\Omega t\\ + \text{higher harmonic terms.}\end{aligned} \tag{2.6}$$

Ignoring the higher harmonic terms (the neglect of the higher harmonics may be justified [33] by regarding them as additional input in the form of forcing terms to linear equation $\ddot{x} + x = 0$. The periodic response of the system to a forcing term characterizing the nth harmonic is proportional to $\cos(n\Omega t)/(n^2\Omega^2 - 1)$, which rapidly diminishes with increase of n; we derive the following equations for frequency and amplitude:

$$\Omega^2 - 1 + K\Omega\sin\Omega\tau = 0 \tag{2.7}$$

and

$$\mu\left(1 - \frac{A^2}{4} + \frac{\alpha A^4}{8} - \frac{5\beta A^6}{64}\right) + K(\cos\Omega\tau - 1) = 0. \tag{2.8}$$

In the absence of delayed feedback, i.e., $K = 0$ or $\tau = 0$, the amplitude equation [Eq. (2.8)] is reduced to Eq. (2.3), and furthermore $\Omega = 1$ corresponds to the frequency in the harmonic limit. The three roots refer to the amplitudes of three limit cycles. The focal theme of the present section is to study the effect of delayed feedback on the existence and stability of the limit cycles.

This stability test is better checked using the energy balance method, which, we shall see, is another way of getting a hint of amplitude of the limit cycles. For this we return to (2.4) and note that for $\mu = 0$ and $K = 0$ the harmonic solution is given by

$$x(t) = A\cos(t + \phi), \tag{2.9}$$

where ϕ is the initial phase. For convenience, we may settle at $\phi = 0$ and $A > 0$. The solution describes a circular orbit in the phase plane with a period $T \simeq 2\pi$. In the harmonic limit we see that Ω for the aforesaid feedback is close to 1, and hence for $\mu \neq 0$ and $\mu \to 0$, in the presence of delayed feedback one may still approximate

$$x(t) \simeq A\cos t, \quad \dot{x}(t) \simeq -A\sin t, \quad \text{and} \quad T \simeq 2\pi, \tag{2.10}$$

where $\Omega \simeq 1$. The terms on the right-hand side of Eq. (2.4) give an estimate for the change in the energy ΔE over the period $0 \leqslant t \leqslant T$ if one treats $[\mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + K(\dot{x}_\tau - \dot{x})]$ as an external force so that we write

$$\begin{aligned}\Delta E &= E(T) - E(0)\\ &= \int_0^T [\mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + K(\dot{x}_\tau - \dot{x})]\dot{x}\,dt.\end{aligned} \tag{2.11}$$

It is obvious that for a limit cycle periodic solution ΔE should be zero. Hence using the expressions (2.10) in (2.11) and integrating Eq. (2.11) we arrive at the following expression:

$$\begin{aligned}\Delta E = \mu\frac{A^2}{2}\left(1 - \frac{A^2}{4} + \frac{\alpha A^4}{8} - \frac{5\beta A^6}{64}\right)2\pi\\ + K(\cos\tau - 1)\frac{A^2}{2}2\pi = 0.\end{aligned} \tag{2.12}$$

This yields

$$f(A^2) = \mu\left[1 - \frac{A^2}{4} + \frac{\alpha A^4}{8} - \frac{5\beta A^6}{64}\right] + K(\cos\tau - 1) = 0, \tag{2.13}$$

which is identical to the amplitude Eq. (2.8). In absence of delayed feedback the amplitude equation described by Eq. (2.13) reduces to the previously obtained expression (2.3). Equation (2.13) suggests that time delay τ can be tuned appropriately to control the saddle-node bifurcation condition. We have chosen the parameter region where three limit cycles coexist, that is, the region of birhythmicity. Based on the analytic estimate from the amplitude equation, Eq. (2.13), the number of positive roots that decides the presence and absence of limit cycles can be obtained. Furthermore the stability of the limit cycles can be checked by taking the derivative of amplitude equation with respect to A. Stability implies

$$\left[\frac{d}{dA}\Delta E(A)\right]_{\text{Limit cycle}} < 0. \tag{2.14}$$

It is thus apparent that the delayed feedback as an external force on the system can efficiently govern the dynamics of limit cycle oscillations.

To illustrate the scheme we now determine the roots of amplitude equation by finding the zeros of the function $f(A^2)$ for the parameter values $\alpha = 0.144$ and $\beta = 0.005$, $\mu = 0.1$ and with a weak feedback strength $K = 0.1$. This corresponds to the point p_1 of Fig. 1 in the bifurcation diagram. The number of roots signifies the presence or absence of the number of limit cycles of the system. The plots $f(A^2)$ versus A^2 exhibit typically bistable, monostable, and steady-state characteristics of oscillation with the variation of time delay τ as shown in Fig. 2(a). For $\tau = 0.0$ we observe three roots, of which two are stable corresponding to two limit cycles or birhythmicity, the other being the unstable one. With an increase of τ the curve is gradually pushed downward so that it crosses the zero line only once, showing a single root indicating one stable limit cycle or monorhythmic behavior. Further increase of τ beyond a critical time delay ($\tau = 0.527\pi$) precludes the possibility of any real root implying the loss of self-sustained oscillatory character. The oscillation is again recovered with the further increase of τ with the appearance of first monorhythmic followed by birhythmic behavior. The time delay changes the dynamics of the system in a periodic fashion. This can be better understood by plotting the function $f(A^2)$ versus τ for some fixed values of A^2. It is shown in Fig. 2(b).

To what extent is the dynamics as discussed above parameter specific? To answer this question we now examine further another dynamical scenario for the parameter values $\alpha = 0.1$ and $\beta = 0.001$ as depicted in Fig. 3. This corresponds to the point p_2 shown in the bifurcation diagram (Fig. 1). For $\tau = 0.0$, i.e., for the system without delayed feedback we observe as usual the bistable behavior of $f(A^2)$ versus A^2 curve, implying birhythmicity and an unstable limit cycle as shown in Fig. 3(a). With gradual increase of τ the inner cycle becomes smaller in amplitude and ultimately vanishes for a critical τ, and the system becomes monorhythmic in nature. Again with further increase of τ the inner cycle reappears, and birhythmicity is recovered. Since the $f(A^2)$ versus A^2

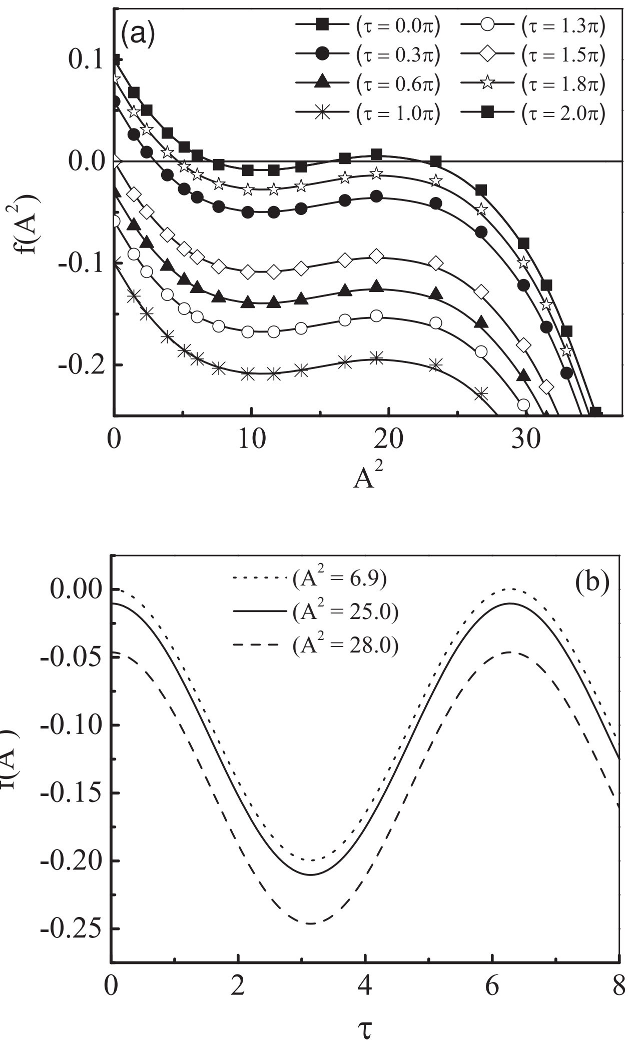

FIG. 2. (a) Plot of $f(A^2)$ versus square of amplitude (A^2) for the values of parameters $\mu = 0.1$, $K = 0.1$, $\alpha = 0.144$, and $\beta = 0.005$ at different time delay (τ) ranging from 0 to 2π. $f(A^2)$ versus A^2 curve moves downward for τ values for 0.0π (three roots), 0.3π (one root), 0.6π (no root), or 1.0π (no root) and then moves upward 1.3π (no root), 1.5π (no root), 1.8π (one root), or 2.0π (three roots). (b) Plot of $f(A^2)$ versus τ for some fixed values of A^2 with other parameters remaining the same.

curves for several τ values from 0 to 2π appearing in Fig. 3(a) are too closely spaced we have zoomed the square block of the left-lower corner of the figure and shown (enlarged) in Fig. 3(b). It is found from Figs. 3(a) and 3(b) that delay time can hardly affect the outer stable and the unstable limit cycles. We have further demonstrated the periodic change of the dynamics of the system in Fig. 3(c) in which $f(A^2)$ is plotted as a function τ for three fixed values of A^2. Thus it follows that depending on the parameter space the delayed feedback can be suitably tuned to control the nature of the attractors. While investigating the nature of stability of the limit cycles from Eq. (2.14) we found that for the parameter zone with $\alpha = 0.144$, $\beta = 0.005$ (corresponding to the point p_1 of Fig. 1) stable limit cycles exist for $A_1 = 2.63$ and $A_3 = 4.83$, while a limit cycle having amplitude $A_2 = 3.96$ is unstable. The unstable limit cycle separates the basin of two stable attractors.

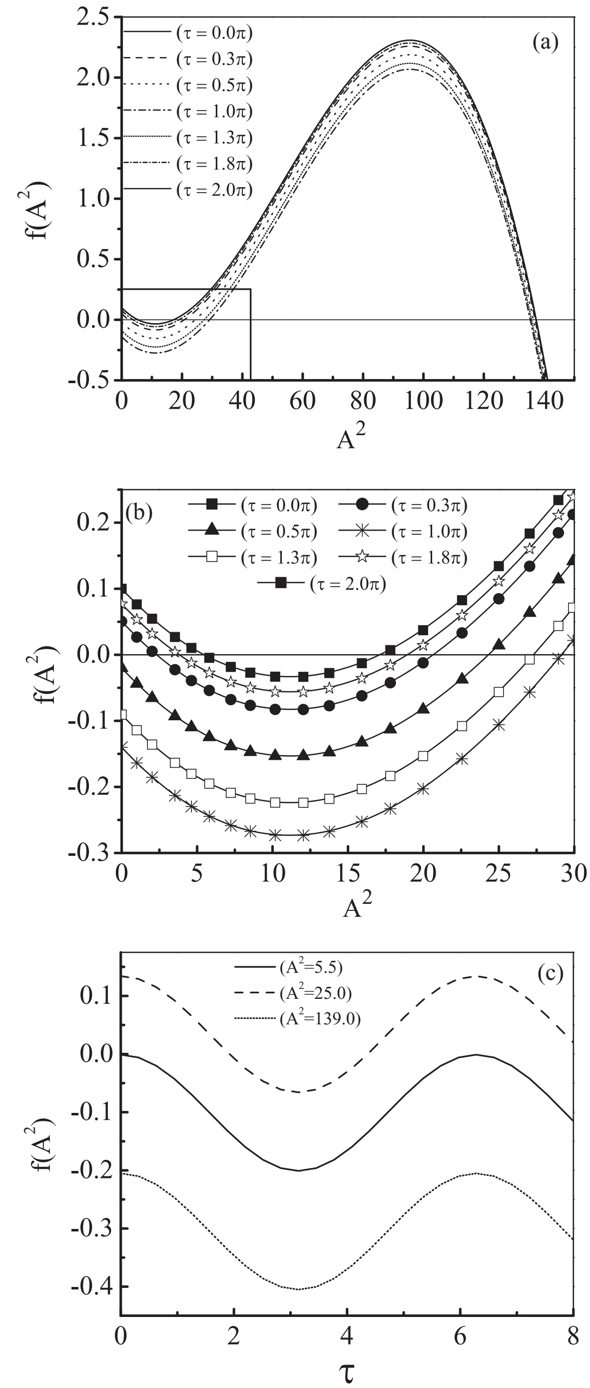

FIG. 3. (a) Plot of $f(A^2)$ versus square of amplitude (A^2) for the values of parameters $\mu = 0.1$, $K = 0.1$, $\alpha = 0.1$, and $\beta = 0.001$ at different time delay (τ) ranging from 0 to 2π. (b) The square box of the left-lower corner of Fig. 3(a) is shown (enlarged) in Fig. 3(b). (c) Plot of $f(A^2)$ versus τ for some fixed values of A^2 for the parameter set mentioned above.

For the parameter set $\alpha = 0.1$, $\beta = 0.001$ (corresponding to the point p_2 of Fig. 1) the outer stable limit cycle that remains unaffected by delay has an amplitude $A = 11.72$.

In Table I we provide a summary of dynamical features for the two distinct parameter zones (divided by the dotted line in Fig. 1), both lying inside the birhythmic zone.

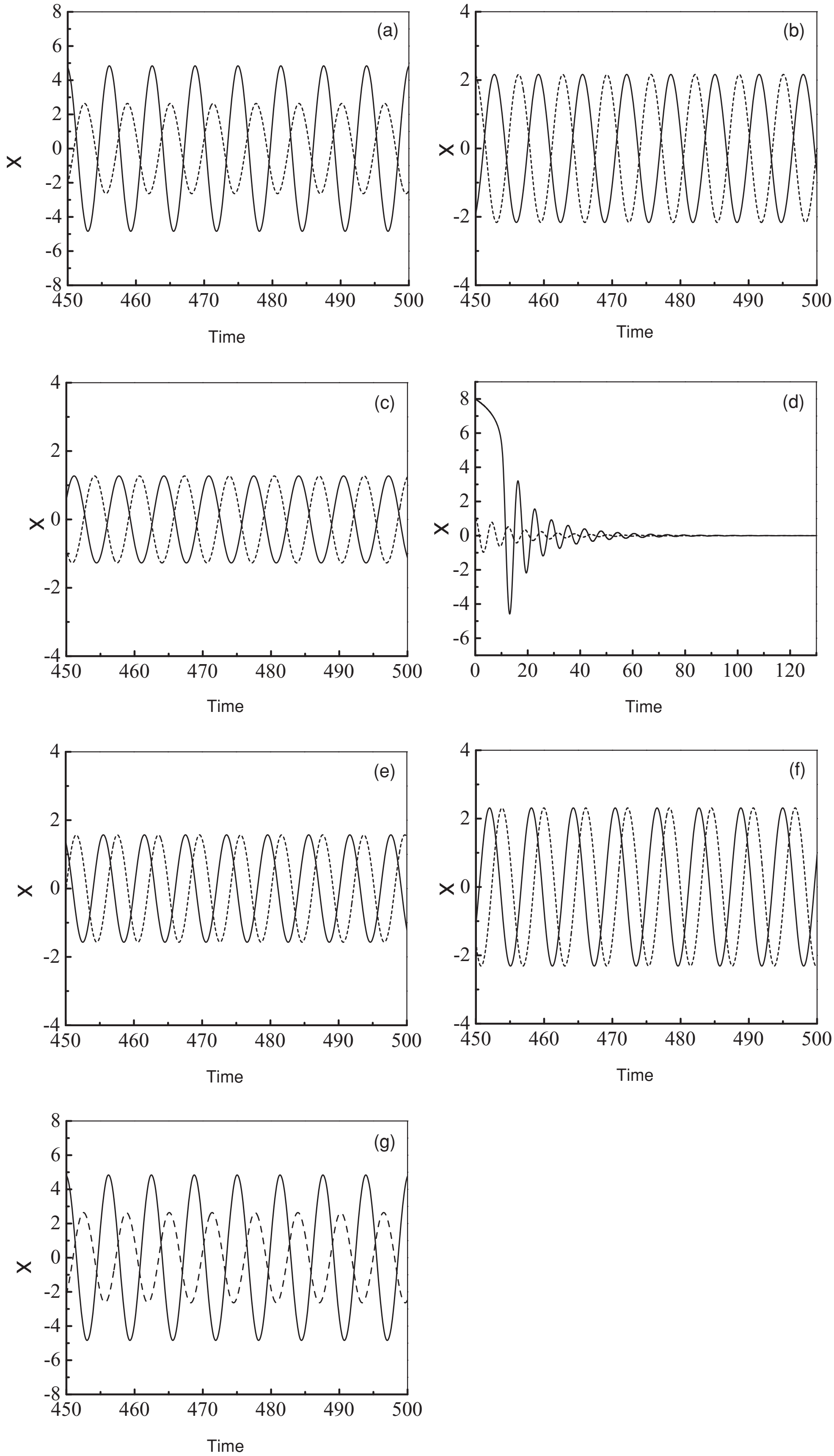

FIG. 4. Control of attractors by delayed feedback. Numerical simulation: Time evolution profile for the parameter values $\alpha = 0.144$, $\beta = 0.005$, $\mu = 0.1$, and $K = 0.1$ for several values of time delay τ with two initial conditions. Dotted line is for the initial values $x_0 = 1.0, \dot{x}_0 = 0.0$, and the solid line for $x_0 = 8.0$, $\dot{x}_0 = 0.0$. (a) $\tau = 0.0\pi$, (b) $\tau = 0.2\pi$, (c) $\tau = 0.4\pi$, (d) $\tau = 1.0\pi$, (e) $\tau = 1.6\pi$, (f) $\tau = 1.8\pi$, and (g) $\tau = 2.0\pi$.

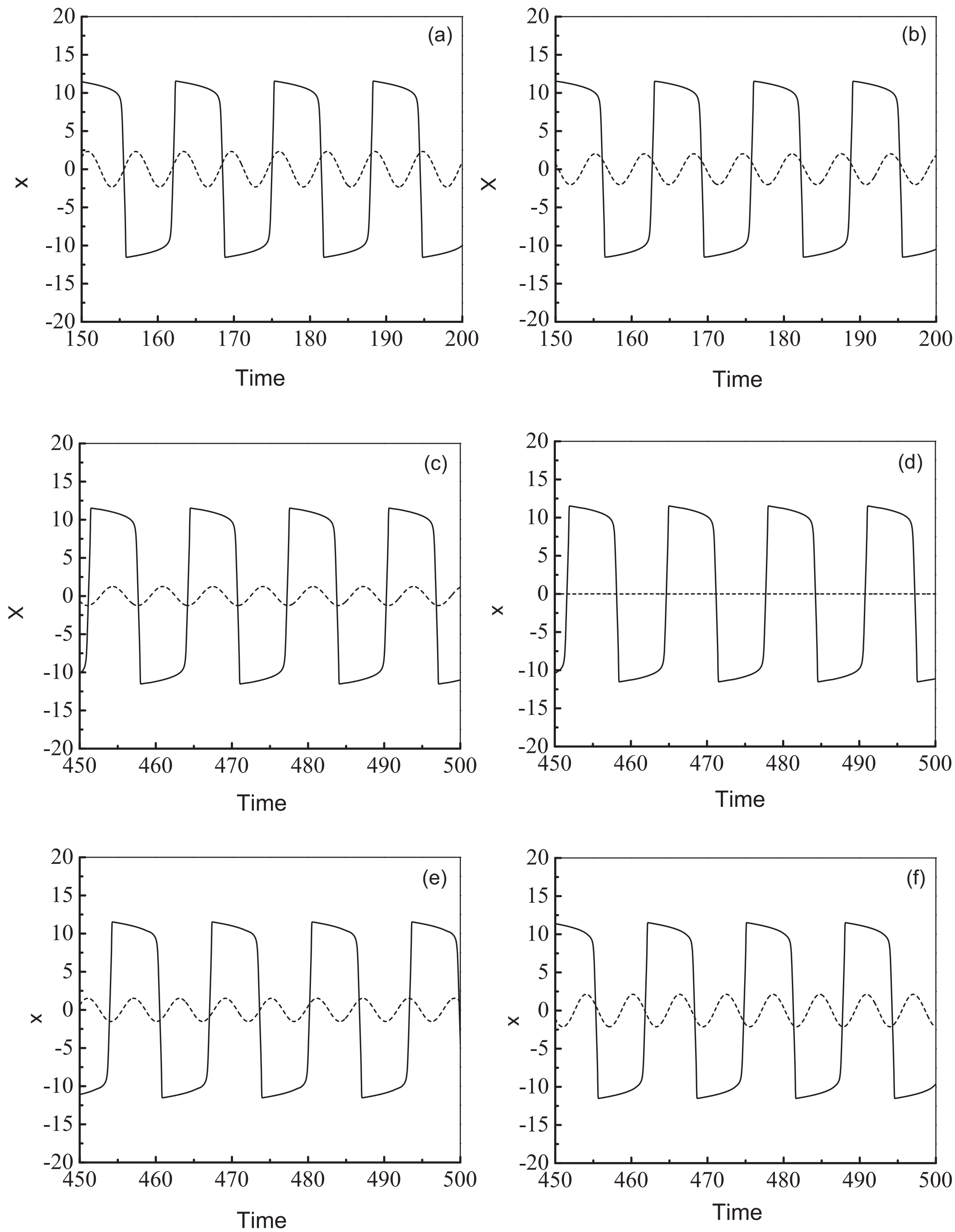

FIG. 5. Control of attractors by delayed feedback: Numerical simulation: Time evolution profile for the parameter values $\alpha = 0.1$, $\beta = 0.001$, $\mu = 0.1$, and $K = 0.1$ for several values of time delay τ with two initial conditions: dotted line is for the initial values $x_0 = 1.0, \dot{x}_0 = 0.0$, and the solid line for $x_0 = 8.0$, $\dot{x}_0 = 0.0$. (a) $\tau = 0.0\pi$, (b) $\tau = 0.2\pi$, (c) $\tau = 0.4\pi$, (d) $\tau = 0.598\pi$, (e) $\tau = 1.6\pi$, and (f) $\tau = 1.8\pi$.

C. Numerical simulations

In order to compare the analytic predictions on the control of dynamical attractors with numerical simulations we have carried out direct numerical integration of Eq. (2.4) for two sets of parameter space ($\alpha = 0.144, \beta = 0.005$ and $\alpha = 0.1, \beta = 0.001$), using weak strength of feedback at $K = 0.1$ and for $\mu = 0.1$. We use the fourth-order Runge-Kutta method for numerical integration with time step $dt = 0.01$. The results are demonstrated in Figs. 4 and 5. The numerical observations of Fig. 4 clearly indicate that when there is no delayed feedback, i.e., $\tau = 0$, two stable limit cycles coexist, resulting in birhythmicity [Fig. 4(a)]. With increase of delay time τ the birhythmic nature of oscillation tends to vanish generating monorhythmicity [Figs. 4(b) and 4(c)]. Beyond a critical value

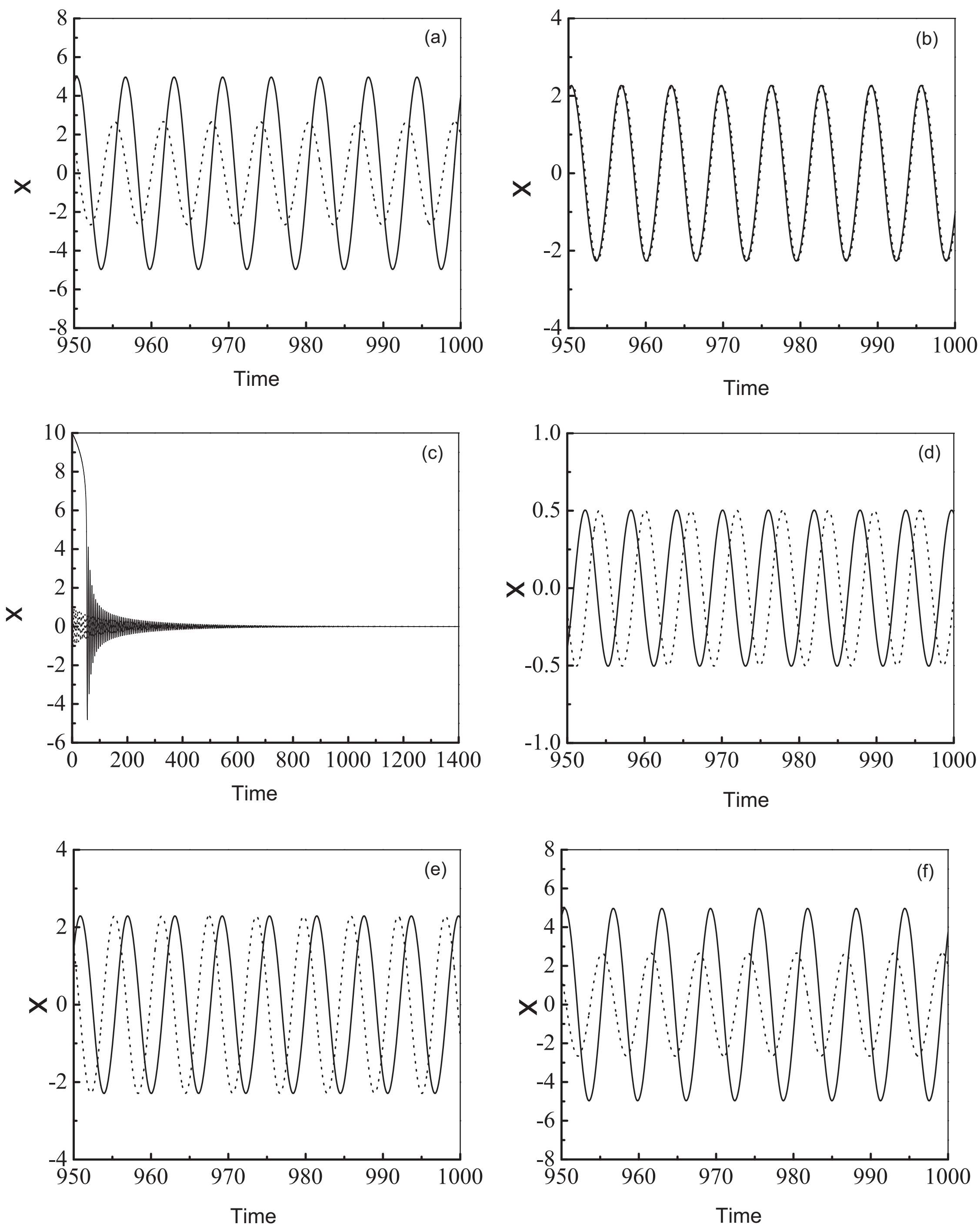

FIG. 6. Control of attractors by delayed feedback. Numerical simulation: Time evolution profile for the parameter values $\alpha = 0.145$, $\beta = 0.005$, $\mu = 0.1$, and $K = 0.12$ for several values of time delay τ with two initial conditions: dotted line is for the initial values $x_0 = 1.0, \dot{x}_0 = 0.0$, and the solid line for $x_0 = 10.0, \dot{x}_0 = 0.0$. (a) $\tau = 0.0\pi$, (b) $\tau = 0.16\pi$, (c) $\tau = 0.5\pi$, (d) $\tau = 1.48\pi$, (e) $\tau = 1.8\pi$, and (f) $\tau = 2.0\pi$.

of delay time ($\tau = 0.527\pi$) the self-sustained oscillation is totally lost, and the system resides in the steady state [Fig. 4(d) for $\tau = \pi$]. The monorhythmic and birhythmic oscillations, however, reappear as shown in Figs. 4(e), 4(g) and 4(f) with further increase in delay time.

The numerical simulation on Eq. (2.4) is further extended to the other parameter regime defined by $\alpha = 0.1$ and $\beta = 0.001$ for the same set of parameter values of K and μ. The results on variation of delay time are shown in Fig. 5. It is clear that the outer stable limit cycle is never destroyed, and it remains robust against the variation of time delay; but the amplitude of the smaller inner cycle starts gradually diminishing before finally setting down on a stationary fixed point. However, further increase of τ results in a revival of the limit cycle, signifying regeneration of birhythmicity.

TABLE I. A comparison between theoretical analysis and numerical simulation of the limit cycles in the presence of time delay in two different sets of parameter regions for $\mu = 0.1$, $K = 0.1$.

Parameter (α,β)	Delay (τ)	No. of roots (analytically)	Stability (analytically)	No. of limit cycles (Numerically)
	0.0	3	2 stable, 1 unstable	2 stable
	0.2π	1	Sable	1 stable
	0.4π	1	Stable	1 stable
	0.527π	No roots	No roots	No limit cycles
(0.144,0.005)	π	No roots	No roots	No limit cycles
	1.6π	1	Stable	1 stable
	1.8π	1	Stable	1 stable
	2.0π	3	2 stable, 1 unstable	2 stable
	0.0	3	2 stable, 1 unstable	2 stable
	0.2π	3	2 stable, 1 unstable	2 stable
	0.4π	3	2 stable, 1 unstable	2 stable
(0.1,0.001)	0.598π	1	Stable	1 stable
	1.6π	3	2 stable, 1 unstable	2 stable
	1.8π	3	2 stable, 1 unstable	2 stable
	2.0π	3	2 stable, 1 unstable	2 stable

We now emphasize a conspicuous difference between the amplitude equation without delay [Eq. (2.3)] and the amplitude equation with delayed feedback [Eq. (2.13)]. The former is independent of μ, which makes its presence felt in Eq. (2.13) in the presence of K(and τ). It is therefore worthwhile understanding to what extent the dynamics depends on the choice of K and μ. A simple consideration shows that a change in the strength of feedback, K, gives rise to a small shift of the dividing (dotted) line in Fig. 1 (for the sake of clarity in the figure we have not shown explicitly this shifted dividing line). That this small shift does not change the generic nature of dynamical response has been ascertained by carrying out a further numerical simulation of Eq. (2.4) for the parameter values $\mu = 0.1$ and $K = 0.12$ corresponding to a point $\alpha = 0.145$, $\beta = 0.005$. The results are shown in Fig. 6. Qualitatively the same behavior as observed in Fig. 4 is noted.

Summarizing the above analysis we may note that the white region (without the gray region) in the codimension-two bifurcation diagram (Fig. 1), which characterizes the birhythmicity domain in the absence of any delayed feedback breaks into two regions denoted by the dotted line when the feedback is set to act on the dynamics. These two regions as shown in Fig. 1 corresponding to points p_1 and p_2 are typically distinct with respect to the response of the dynamical system toward delayed feedback. In region where p_1 lies, the outer stable limit cycle with larger amplitude is suppressed by delayed feedback, and instead an inner cycle with smaller amplitude is obtained. A further increase in delay time τ gives rise to a collapse of oscillation on the steady state and its subsequent revival with larger amplitude of the stable outer limit cycle. In the region where p_2 resides, on the other hand, the delayed feedback has little influence on the outer stable limit cycle; with increase of delay time τ the inner stable limit cycle gradually becomes smaller in amplitude and collapses on the stationary state after a critical value of τ, signifying the suppression of birhythmicity. For further increase of delay τ the regeneration of mono and birhythmicity occurs.

III. CONCLUSION

Using a multicycle van der Pol oscillator, we have shown that the rhythmic properties of the oscillator can be efficiently controlled by a time-delayed feedback. Our analysis reveals that the amplitude equation assumes a generic normal form for the saddle-node bifurcation, which can be controlled by an appropriate time delay. In the harmonic approximation the time-delayed feedback appears as a phase-shifted feedback. We have shown how this feedback does modify the bifurcation scenario in a generic way; specifically the bifurcations can be tuned by time delay in such a way that one may realize in the bifurcation diagram a dividing line that separates the space into two subspaces depending on the dynamic response of the system toward the feedback. This results in interesting dynamical features, such as transition between a birhythmic state and a steady state and a crossover from birhythmic to monorhythmic behavior. Although the present study of dynamical control of attractors by delayed feedback has been carried out on a simple specific model that admits of birhythmicity, we believe that some of the conclusions drawn are expected to be useful for other multicycle oscillations in general. Depending on the delay and its strength it is possible to figure out different zones in the bifurcation diagram for which the dynamical scenarios are generically distinct. The strength of delay is likely to play a significant role in modulating the position of the dividing line(s) between the zones.

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[1] O. Decroly and A. Goldbeter, Proc. Natl. Acad. Sci. USA **79**, 6917 (1982).
[2] S. Kar and D. S. Ray, Europhys. Lett. **67**, 137 (2004).
[3] M. Morita, K. Iwamoto, and M. Seno, Phys. Rev. A **40**, 6592 (1989).
[4] J. C. Leloup and A. Goldbeter, J. Theor. Biol. **198**, 445 (1999).
[5] I. M. De la Fuente, BioSystems **50**, 83 (1999).
[6] M. Stich, M. Ipsen, and A. S. Mikhailov, Physica D **171**, 19 (2002); Phys. Rev. Lett. **86**, 4406 (2001).
[7] F. Kaiser, *Coherent Excitations in Biological Systems: Specific Effects in Externally Driven Self-sustained Oscillating Biophysical Systems* (Springer-Verlag, Berlin, 1983).
[8] F. Kaiser and C. Eichwald, Int. J. Bifurcation Chaos Appl. Sci. Eng. **1**, 485 (1991).
[9] C. Eichwald and F. Kaiser, Int. J. Bifurcation Chaos Appl. Sci. Eng. **1**, 711 (1991).
[10] C. R. Fuller, S. J. Eliot, and P. A. Nelson, *Active Control of Vibration* (Academic, London, 1997).
[11] A. H. D. Cheng, C. Y. Yang, K. Hackl, and M. J. Chajes, Int. J. Nonlinear Mech. **28**, 549 (1993).
[12] A. N. Pisarchik and B. K. Goswami, Phys. Rev. Lett. **84**, 1423 (2000).
[13] A. N. Pisarchik, Phys. Rev. E **64**, 046203 (2001).
[14] A. N. Pisarchik and B. F. Kuntsevich, IEEE J. Quantum Electron. **38**, 1594 (2002).
[15] A. N. Pisarchik, Yu. O. Barmenkov, and A. V. Kiryanov, Phys. Rev. E **68**, 066211 (2003).
[16] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[17] A. G. Balanov, N. B. Janson, and E. Scholl, Phys. Rev. E **71**, 016222 (2005).
[18] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[19] B. E. Martinez-Zerega, A. N. Pisarchik, and L. S. Tsimring, Phys. Lett. A **318**, 102 (2003).
[20] B. E. Martinez-Zerega and A. N. Pisarchik, Phys. Lett. A **340**, 212 (2005).
[21] K. Pyragas, Phys. Lett. A **206**, 323 (1995).
[22] M. E. Bleich and J. E. S. Socolar, Phys. Lett. A **210**, 87 (1996).
[23] S. Sen, P. Ghosh, S. S. Riaz, and D. S. Ray, Phys. Rev. E **80**, 046212 (2009).
[24] P. Ghosh, S. Sen, and D. S. Ray, Phys. Rev. E **81**, 026205 (2010).
[25] S. Sen, P. Ghosh, and D. S. Ray, Phys. Rev. E **81**, 056207 (2010).
[26] S. Rajesh and V. M. Nandakumaran, Physica D **213**, 113 (2006).
[27] S. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, Reading, MA, 1994).
[28] H. G. Enjieu Kadji, J. B. Chabi Orou, R. Yamapi, and P. Woafo, Chaos Solitons Fractals **32**, 862 (2007).
[29] V. S. Anishchenko, V. Astakhov, A. Neiman, T. Vadivasova, and L. Schimansky-Geier, *Nonlinear Dynamics of Chaotic and Stochastic Systems: Tutorial and Modern Developments* (Springer, New York, 2007).
[30] R. Yamapi, G. Filatrella, and M. A. Aziz-Alaoui, Chaos **20**, 013114 (2010).
[31] R. Yamapi, B. R. Nana Nbendjo, and H. G. Enjieu Kadji, Int. J. Bifurcation Chaos **17**, 1343 (2007).
[32] N. B. Janson, A. G. Balanov, and E. Scholl, Phys. Rev. Lett. **93**, 010601 (2004).
[33] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Oxford University Press, New York, 1999).