Mathematics and morphogenesis of cities: A geometrical approach

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Cities are living organisms. They are out of equilibrium, open systems that never stop developing and sometimes die. The local geography can be compared to a shell constraining its development. In brief, a city's current layout is a step in a running morphogenesis process. Thus cities display a huge diversity of shapes and none of the traditional models, from random graphs, complex networks theory, or stochastic geometry, takes into account the geometrical, functional, and dynamical aspects of a city in the same framework. We present here a global mathematical model dedicated to cities that permits describing, manipulating, and explaining cities' overall shape and layout of their street systems. This street-based framework conciliates the topological and geometrical sides of the problem. From the static analysis of several French towns (topology of first and second order, anisotropy, streets scaling) we make the hypothesis that the development of a city follows a logic of division or extension of space. We propose a dynamical model that mimics this logic and that, from simple general rules and a few parameters, succeeds in generating a large diversity of cities and in reproducing the general features the static analysis has pointed out.

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I. INTRODUCTION

The city is a living structure: It is an open system, always in motion. A city is born, develops, heals over injuries (war damages, etc.), and sometimes dies in part or totally. Its development responds to internal and external constraints in such a way that local geography acts as a shell that sculpts it. As general living systems, cities exhibit a huge range of diversity both on their overall shape (that can be circular, sprawling, linear, or even fractal) and on the appearance of their street systems (regular, organic, treelike). Such a diversity can also be observed at the level of a single city that has not developed homogeneously.

We seek out to show that behind this diversity stands a single principle: A city develops within a logic of division or extension of space. The intra- and inner diversity of shapes can be seen as a variation in a coherent global phenomenon. Our approach is street based: We consider an infinitesimal piece of street as the elementary component of a city and wager it contains important information. The dual approach is to consider the buildup area (buildings, parks, etc.) as the unit of formation. Several points of view have been used in the past to model cities: cellular automata, multiagent systems, fractals, stochastic geometry, L-systems, and graph theory leading to complex systems' theory.

Cellular automata and multiagent systems have widely and successfully been used to simulate the dynamics of populations and of land use [1]. The fractal description of cities [2] gathers these simulations into a theory and points out the advantages of a fractal-shaped city from the point of view of the buildup area. Nonetheless, the basis of these models is either a discrete field or the map of a given city. They explain the global differentiation of space when the street network is known or ignored.

Since the famous "Bridges of Konigsberg" problem by Euler, one is tempted to describe a city as a graph with streets as edges and their intersections as vertices. This provides a relational representation of the city [3]. One of the difficulties then is the particular embedding of these graphs that make random graphs unable to stick to a city's map representation.

Stochastic geometry gets around this problem by considering stationary tessellations (Poisson Voronoï, Poisson Delaunay, or Poisson Line Tessellation, and their superpositions and iterations) that are geometrical objects embedded in a compact subset of the space and deduces geometrical random graphs from their induced topology [4]. L-systems with procedural programming make a map evolve from local coherence rules and input data that incorporate global constraints [5,6]. The stochastic geometry approach gets good results at analyzing optimization problems on street networks, and L-systems are successfully used in graphics, but they do not explain the underlying phenomena at work to determine the appearance of a city.

For a few years a complex network-based study has been adopted to describe cities [7,8]. But the main conclusion is that a city behaves neither as a classical scale-free network nor a small-world network, essentially because of its spatial embedding [9]. Cities then clearly need a dedicated mathematical framework, taking into account both the topology and, what is new, the geometry of a street network. The scope of this article is a street-based approach to cities that allows analyzing, manipulating, and explaining cities' morphogenesis. In Sec. II we define a mathematical formalism to handle with cities both in a relational and a geometrical way, the very substance of the geometry being that street segments are lined up into sets called streets, which we model via a hypergraph additional structure.

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Section III presents several description measurements (topology of first and second order, anisotropy, and streets scaling) to obtain quantitative comparison elements between cities, and exhibit some features of the global mechanic of cities system (organic shape, log scaling of length, and small topological radius). These features call for the modeling of the city as a process of division and extension of space. We use ten French towns and their centers without having to restrict them to a square window to illustrate this.

Eventually we present a morphogenetic model of the city (Sec. IV) and its simulation (Sec. V) that implements the idea of space division or extension. This model reproduces the general features pointed out in Sec. III, and the variation of a few normalized parameters allows recovering a large range of diversity. The model is expressed in terms that are as relevant for mathematicians or physicists as they are for town planners or social scientists. No sampling of space is required for the simulation.

II. CITIES' SPACE

Since the famous resolution of the "Bridges of Konigsberg" problem by Euler [3], one is tempted to look at cities with a formal and relational point of view: A city is a graph whose edges are streets and vertices are their intersections.

Nevertheless, this approach does not take into account the physical constraints (of being a functional object on the plan) that are exerted on cities. Furthermore, it enhances a fundamental disjunction between intersections—that would be objects of interest—and street segments that would simply bind them. Under those street segments or edges lies a characteristic geometry. Each point of this geometry should be seen as an object in relation with other similar objects. This paragraph aims at introducing a terminological frame that permits manipulating cities as "continuous graphs" embedded in a two-dimensional Euclidian space. The vocabulary used in this article is freely adapted from general graph theory [10] to respond to our specific needs.

When importing a map (via a .MIF file), the raw data is coded into a list of polylines. In fact, these polylines come from the sampling of curved streets that are difficult to represent in a computer. We consider it is possible to transform a *geometrical graph* into a *straight graph* arbitrary close to it. The important paradigm of degree 2 points comes up. To *rectify* the map we have added degree 2 vertices, but we want to consider a version of the same object. In short, a segment with its two extremities and the same segment with its extremities plus its midpoint should be seen as the same entity. We define a measure on a geometrical graph that allows to see that graph as a "continuum" and to consider each point of its geometry in the same time. The data rather represents *street segments* that actual *streets*. The particularity of a city's geometry is that its street segments are coherently arranged into disjoint geometrical sets: the streets. We try to record the notion of streets with a *hypergraph* additional structure. This provides a multiscale representation of the city.

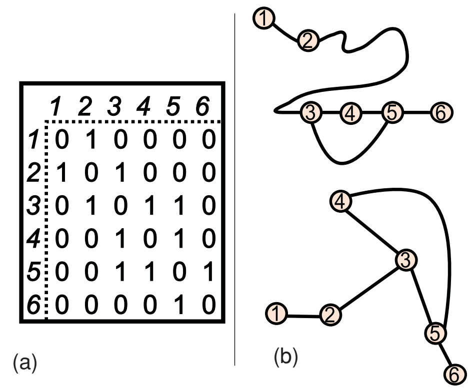

FIG. 1. (Color online) The representation of a graph by its symmetric adjacency matrix (a). This graph is planar: It admits at least two geometrical graphs as drawings (b).

A. Graphs and planar graphs

Let S be a set, V (for vertices) a finite subset of S, E (for edges) a symmetric part of $V \times V$, and then $G = (V,E)$ is said to be a (undirected) graph.

A drawing of G is an injective function from V to $\mathbb{R}^2$ and from E to the set of continuous paths such that the image of an edge has for limits the images of the vertices it binds and does not pass through images of vertices it does not bind. An edge crossing is the intersection of the images of two edges outside the image of V.

If there exists a drawing without any edge crossing, the graph is said to be planar (Fig. 1). The first characteristic of city graphs is their planarity.

B. Geometrical graphs

A geometrical graph can be seen as a particular drawing of a planar graph. Let the available space $\mathbb{A}$ be a connected and compact subset of $\mathbb{R}^2$, V a finite subset of $\mathbb{A}$, and E a set of almost everywhere derivable paths included in $\mathbb{A}$ from one element of V to another that does not intersect outside of V. Then $G = (V,E)$ is an element of the space of geometrical graphs $\mathcal{G}_g(\mathbb{A})$. If E is restricted to straight segments [$G \in \mathcal{G}_s(\mathbb{A})$], G is a straight graph. To a geometrical graph G, one associates π_G, the subset of $\mathbb{A}$ defined by

$$\pi_G = \{x \in \mathbb{A},\ \exists e \in E,\ x \in e\}. \tag{1}$$

π_G is compact so we can provide $\mathcal{G}_g(\mathbb{A})$ with an Hausdorff distance:

$$d_H(g_1||g_2) = \max_{x\in\pi_{g_1}} \min_{y\in\pi_{g_2}} ||x-y||. \tag{2}$$

A drawing $G' = (V',E')$ is a rectification of the geometrical graph $G = (V,E)$ if $V \subset V' \subset \pi_G$ and if each element of E' is a segment. G' is not necessarily a planar straight graph because edges can possibly intersect outside of vertices (Fig. 2). The idea is that one should be able to add to a geometrical graph as many vertices of degree 2 as he wishes and still consider the same mathematical object. In this article we will admit that Pr1 and Pr2 hold for a large enough class of geometrical

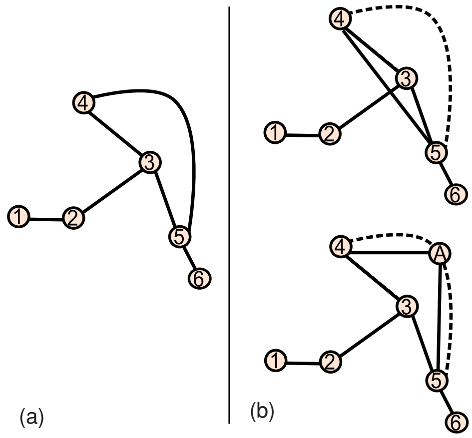

FIG. 2. (Color online) A geometrical graph (a) and two of its rectifications (b). The upper one is not straight because two edges intersect outside of the vertices set.

graphs:

Pr1: Every geometrical graph admits a planar rectification.

Pr2: Every geometrical graph is the limit of a sequence of straight graphs.

To get (Pr1), one has to sample an original graph with enough additional vertices of degree 2 and a small enough edges length. (Pr2) states that it is possible to approximate any geometrical graph by a straight graph arbitrary close to it in the sense of d_H. This is of practical importance as it allows to work only with straight planar graphs, while a city's original maps can have curved edges.

C. Measure

As a compact part of $\mathbb{A}$, π_G is a Polish space (complete and separable) on which one can define a Borelian measure μ_G. For instance, $\int_G d\mu_G$ is the total length of edges in G or in (III C) we define the angular density of a graph $\Psi(\cdot)$ with $\mu_G(\cdot)$. This measure respects

$$\int f(g) d\mu_G = \sum_{e \in E} \left| \int_e f(x) d\mu_1(x) \right|, \tag{3}$$

where μ_i is the i-dimensional Borelian measure, f is a positive function, and $\int_e$ is the integral along the path e. If f is a continuous function defined on $\mathbb{R}^2$, we have

$$\int f(g) d\mu_G = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int f(x,y)(\pi_G \oplus B_\epsilon) d\mu_2(x,y), \tag{4}$$

where $\oplus$ is the Minkowski addition and B_ϵ is the closed Euclidian ball of radius ϵ: $x \oplus B_\epsilon = \{y \in \mathbb{R}^2 \; ||x - y|| \leqslant \epsilon\}$. So μ_G is a measure between μ_1 and μ_2 that allows to make quantitative measurements on the whole geometrical graph.

D. Hypergraph structure

If H is an equivalence relationship on E then $[(V,E),H]$ is said to be a hypergraph.

Let (V,E) be a graph and R be a reflexive relationship on E^2. Then the relationship $\hat{R}$, defined by

$$e_1 \hat{\mathrm{R}} e_2 \text{ iif } \exists\, \alpha_1 = e_1, \alpha_2, \ldots, \alpha_n = e_2 \in E |, \quad \alpha_1 R \alpha_2, \alpha_2 R \alpha_3, \ldots, \alpha_{n-1} R \alpha_n, \tag{5}$$

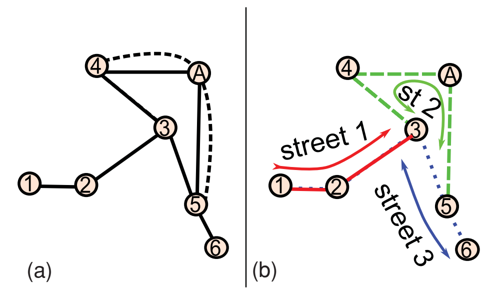

FIG. 3. (Color online) A straight graph (a) and its hypergraph structure (b) deduced from $R_{\pi/20}$. Viewed as a city's map, this graph contains seven streets segments but three streets.

is an equivalence relationship. From this, one can consider R_θ:

$$e_1 R_\theta e_2 \text{ iif} (e_1 \star_2 e_2) \vee [(e_1 \star e_2) \wedge (|\measuredangle(e_1,e_2) - \pi| \leqslant \theta)], \tag{6}$$

where $e_1 \star e_2$ means that e_1 and e_2 intersect, and $e_1 \star_2 e_2$ means that e_1 and e_2 intersect in a vertex of degree 2. $\measuredangle(e_1,e_2)$ stands for the angle between e_1 and e_2 oriented with the same origin.

For instance, think of considering the map of a city and the relationship where "these two edges are pieces of the same street." This R_θ allows recovering the notion of "streets" even if the input data do not contain such labels. The algorithm labeling street segments with a street number does not depend on its starting point and is fast to run. The price to pay is that some special cases are where forks of two segments make a very small angle with a third one will be considered as a single street. This additional structure is essential as it gives a way to analyze the overall structures of planar graphs, and, in particular, of cities. (See Fig. 3.)

E. City graphs

We define pragmatically the set of city graphs $\mathcal{G}_C$ as the subset of $\mathcal{G}_S$ that represents an existing city or a city that could have existed. We restrict this definition to $\mathcal{G}_s$ because many of the cities' streets are straight, and even if it is not the case, there exists a straight approximation that is as accurate as we want.

In the following, we will write $C = (V,E)$ to be a city and canonically provide C with a hypergraph structure from a relationship R_θ. For the sake of simplicity we keep the same notation to designate the hypergraph : $C = [(V,E),H]$. Its Borelian measure is written as μ_C.

The relational aspect inherited from a simple graph structure allows us to define the set of faces F. Euler's equality is respected so that $V_{\text{No.}} - E_{\text{No.}} + F_{\text{No.}} = 1$. To each edge e we associate the set $V(e)$ of its extremities in V, and to each point $v \in \pi_C$ we associate $E(v)$ to the set of edges that pass through it. $N(v)$ is the degree of a vertex.

We partition $V = V_1 \cup V_2 \cup V_+$, where V_1 contains all vertices of degree 1, V_2 of degree 2, and V_+ of higher degrees. Vertices in V_1, V_2, and V_+ are, respectively, terminations, junctions, and intersections. Elements of V_2 are seen as sampling artifacts used to fit the curved geometry of the city. Elements of E are called street segments, those of H streets.

III. CITY SPACE FEATURES

Owing to their spatial constraints, real geometrical graphs do not behave as classical complex networks (small-world or scale-free networks) [9]. Real cities display structural, geometrical, and functional features. For instance, a real city aims both at lodging its inhabitants and at providing them with efficient access to geographical and human resources. These constraints logically affect the structure of a city graph. The purpose of this section is to define some mathematical tools that will quantitatively measure structural differences between city graphs. Classical measures from a complex network theory (efficiency, robustness, centrality, degree correlation) have been investigated in Refs. [11–13]. The measures presented below are dedicated to cities. Each one is illustrated with the city of Amiens in France: the entire city and its center (Fig. 5). The main properties are recapitulated in Sec. III E for ten French towns and their centers.

We have imported vector maps in a calculus framework and successively rectified them by taking care of conserving the planarity and angles at the intersections, and we used $R_{\pi/10}$ to obtain an hypergraph structure.

A. First-order topology

Let $C = [(V,E),H)]$ be a city and $N(k) = \{v \in C, N(v) = k\}_{\text{No.}}$ be the number of vertices of degree k in C and $\bar{N}(k) = N(k)/\sum N(i)$. The set V_2 (junctions) should not be taken into account because it only represents sampling artifacts to preserve the shape of streets. In Ref. [8] the histogram of N is studied by means of an exponential tail of distribution. Nonetheless, this distribution (Fig. 4) is very peaked in 3 or in 4. It is sufficient to describe the histogram N by the organic ratio

$$r_N = \frac{N(1) + N(3)}{\sum_{j\neq 2} N(j)}, \tag{7}$$

which allows to discriminate quickly whether the city had been planned ($r_N \simeq 0$ in the limit case) or not. Indeed, a planned city is filled with a homotopy of a rectangular grid [only $\bar{N}(4) \neq 0$]. This is clearly useful to settle buildings but also sticks to a human perception of space because we have the intuition of left and right and front and behind. In unplanned cities (organic will be the dedicated word in Sec. IV to emphasize the comparison with living systems), i.e., created by the interaction between nonconcerted settlements, there is little probability for street segments to be coherent and thus $r_N \simeq 1$.

Following Ref. [13], we characterize the topology of a city by its "meshedness coefficient." It is easy to count v and e, and then f is deduced from Euler's formula. Given V, the maximum number $2s - 5$ of faces is obtained by the Greedy triangulation algorithm [13]. So the quantity $M = (e - v + 1)/(2v - 5)$ is equal to 0 if the city is a tree and is close to 1 if it is a highly connected graph. For real cities [12], M typically ranges between 0.08 and 0.35.

To be coherent with our preceding remark, we should not take junctions into account. They have no incidence on the numerator but we have to change $2v - 5$ into $2v(1 - \bar{N}(2)) - 5$:

$$M_3 = \frac{e - v + 1}{2v(1 - \bar{N}(2)) - 5}. \tag{8}$$

M_3 is quite small because of the general lack of triangles in the topology of a city. As a trapezoid contains two triangles, we rescale it in $M_4 = 2M_3$, whose maximum is hit when the considered city contains the maximal number of trapezoids. Amiens appears as an "average" organic city, with a meshedness coefficient of 0.41 (0.54 when restricted to the center) and $r_N = 0.79$ (0.68 in the center).

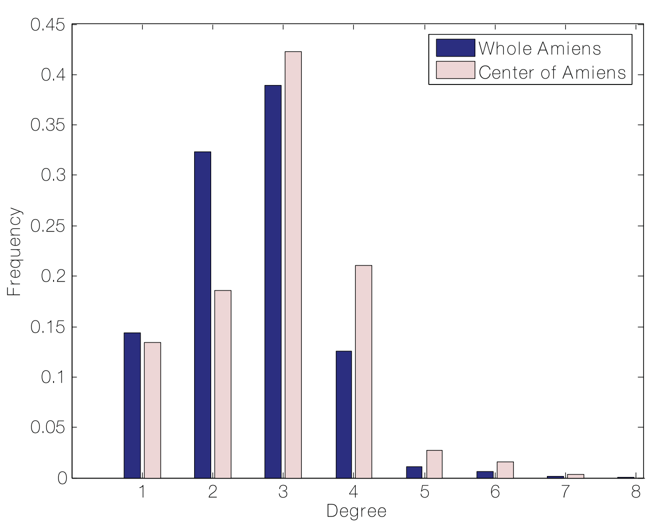

FIG. 4. (Color online) The histograms of degrees' distribution in Amiens (entire city and center). We observe that both distributions are peaked in 3 even in the more regular city center, where the number of connections 4 is still half. The large number of 2 comes from straightening, especially in the suburbs' curved streets, and should not be taken into account.

B. Second-order topology

The streets H induce a particular topology. In Ref. [12] its study is referred as the "dual approach." We use here the expression "second-order topology" because it is a derived topology. Moreover, as faces are the dual of vertices, the dual of a space is a space containing the same information. We prefer to retain the word "duality" to express this idea: "the mass of a city (buildings, houses, parks) is the dual of the street system," which could refer to the work of Refs. [1] and [2].

Let us call the topological distance of C the function $d_C^{\text{topo}} : H \times H \longrightarrow \mathbb{N}$ that satisfies

$$d_C^{\text{topo}}(h_1,h_2) = \begin{cases} 0, & \text{if } h_1 = h_2, \\ \min\limits_{h\in H, h\cap h_2 \neq \emptyset} d_C^{\text{topo}}(h,h_1) + 1, & \text{otherwise} \end{cases} \tag{9}$$

The topological distance counts the number of times one needs to turn to go from a street to another one. The topological average distance of a street h_0 is

$$\bar{d}_C^{\text{topo}}(h_0) = \frac{1}{H_{\text{No.}}} \sum_{h\in H} d^{\text{topo}}(h,h_0). \tag{10}$$

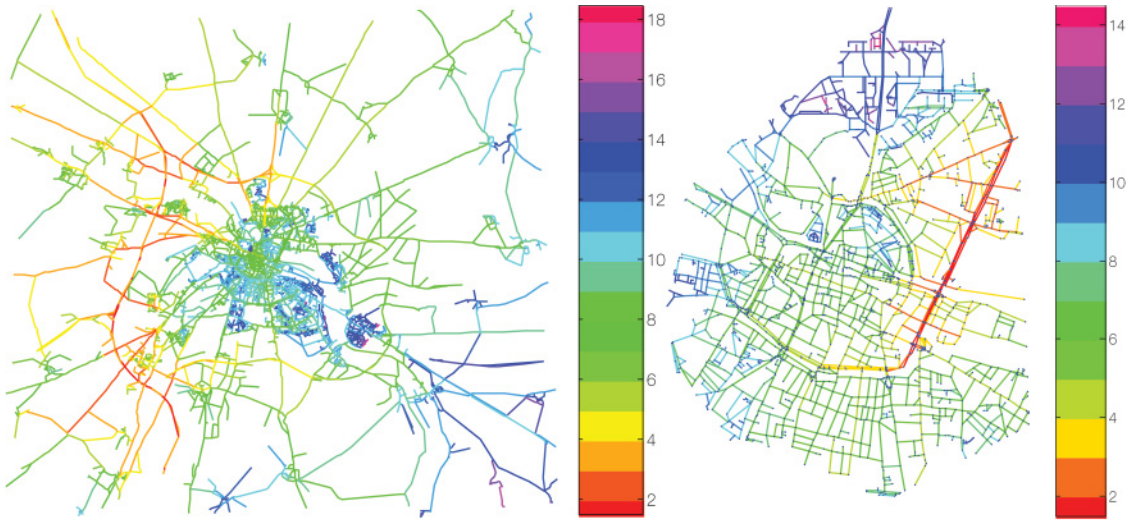

FIG. 5. (Color online) . The "topological maps" of Amiens (left-hand side) and its center (right-hand side). In each map, the red street is the topological center and the color of each street refers to its distance to that center. The maximum distance to the center is the radius of the map (18 in the entire city and 16 in the center). It is striking that this radius increases slowly with the size of the considered street system. We can even infer that the construction of the surrounding highway belt (the found center) is made precisely to keep the topological radius of the city small.

This formula defines a new centrality measure on the map similarly to those studied in Ref. [11]. A street that minimizes $\bar{d}_C^{\text{topo}}$ is then called a center. One drawback of this wholly topological definition is that, because topological distances are integers, several streets can be defined simultaneously as central streets. Common sense would then be to take all of these streets as simultaneous central streets, and to calculate the distance of any street as the minimum of the distance to any of these streets. Another way is to weight by the street's length:

$$\bar{d}_C^{\text{topo leng}}(h_0) = \frac{1}{\mu_C(C)} \sum_{h\in H} ||h|| d^{\text{topo}}(h,h_0) \tag{11}$$

($||h||$ is the length of the street h). From this, a unique center h_c is defined if the city is not too regular. The topological radius of the city is then defined by $r_C^{\text{topo}} = \max d_C^{\text{topo}}(h,h_c)$ and its diameter by

$$\text{diam}_C^{\text{topo}} = \max_{h_1,h_2\in H} \quad d_C^{\text{topo}}(h_1,h_2), \tag{12}$$

with $r_C^{\text{topo}} \leqslant \text{diam}_C^{\text{topo}} \leqslant 2r_C^{\text{topo}}$ that make these measures be equivalent. Figure 5 plots in a color map the distance of each street to the topological center of Amiens that ends up to be a part of its highway belt. This map gives a hierarchical vision of the space. There is no radial component of the increase of the topological distance: A scale of long streets serves the entire city, allowing the variation of the topological distance to be mainly local.

Added to that, the topological radius of the city grows very slowly with the size of the city (14 in the center of Amiens, 18 in the entire city that is eight times bigger).

The topological efficiency $1/\bar{d}_C^{\text{topo leng}}(h_0)$ is defined for each street, and we can consider that it is defined on each point of the city, being constant almost everywhere in a street. It defines a new centrality [11] on the network.

C. Anisotropy

In order to grant an efficient access to physical resources, the street system locally tends to be perpendicular to *structuring elements* as, for example, rivers or older streets. Then a city is not "isotropic."

Let $\vec{u}_0 \in \mathbb{R}^2$ be an arbitrary vector, taken as an angular origin. For angle $\alpha \in [0,\pi]$,

$$\Psi^*(\alpha) = \mu_C(c \in C, \measuredangle(c,\vec{u}_0) \in [0,\alpha]), \tag{13}$$

where $\measuredangle(\cdot,\cdot)$ is the angle measure between two vectors in $[0,\pi]$. It is a measure of the "total length" of streets in C that are oriented in directions $[0,\alpha]$; in the special case of straight graphs, the impact of each street segment is proportional to its length. From this, we define the angular density by $\Psi(\alpha) = \frac{d\Psi^*(\alpha)}{d\alpha}$, whose representation describes the anisotropy of the city graph (Fig. 6). We notice a fuzzy symmetry around the first bisectrix as a result of the streets' local perpendicularity. For an isotropic city, the angular density Ψ_I would be a continuous and uniform density $\Psi_I(\alpha) = \frac{1}{\pi}$.

It would be useful to sum up this angular density as a single normalized indicator. We looked for a bound distance measure between Ψ and Ψ_I. Because the observed distribution Ψ is discrete because of the limited number of street segments, measures such as $\int |f - \frac{1}{\pi}|^n$ depends highly on the size of the bean chosen to estimate the integral.

Because the angle is defined modulo π, we can "fold" Ψ: $\Psi^{(2)}(\theta) = \Psi(\theta)e^{i2\theta} - \int \Psi(u)e^{i2u}du$. The inertia matrix

$$\begin{pmatrix} \int \text{Re}(\Psi^{(2)})^2 & -\int \text{Re}(\Psi^{(2)})\text{Im}(\Psi^{(2)}) \\ -\int \text{Re}(\Psi^{(2)})\text{Im}(\Psi^{(2)}) & \int \text{Im}(\Psi^{(2)})^2 \end{pmatrix}$$

is symmetric and positive (from Cauchy-Schwartz's inequality) with two eigenvalues $\lambda_1 > \lambda_2$, such that

$$\mathcal{A} = 1 - \frac{\lambda_2}{\lambda_1} \tag{14}$$

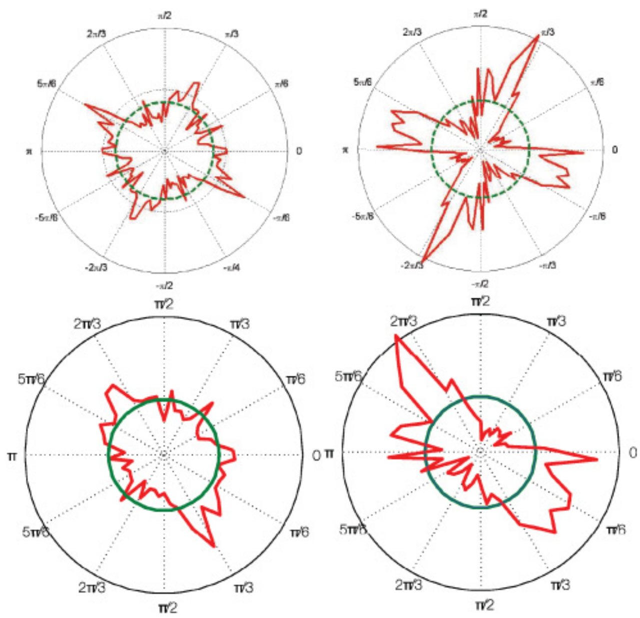

FIG. 6. (Color online) Angular distribution of Amiens (top left: entire city; top right: center). We notice in both distributions a fourfold symmetry. The doubled angular distributions (bottom) appear as ellipsoïds, which allows to define the anisotropy coefficient as the ratio of the eigenvalues of the inertia matrix: $\mathcal{A} = 0.42$ in the entire city and $\mathcal{A} = 0.71$ in the center.

correctly defines an anisotropy coefficient. As for Amiens, its anisotropy coefficients varies from 0.42 in the entire city to 0.71 in the center.

D. Street length

The second-order topology leads us to think that a city organizes into a hierarchical way with long streets and shorter streets. Consequently we do not expect an exponential decay in the distribution of street lengths. In Amiens, the distribution of street lengths is L. Figure 7 is well fitted by a mixture of log-normal laws:

$$\ln L \sim p_{-}\mathcal{N}(m_{-},\sigma_{-}) + (1 - p_{-})\mathcal{N}(m_{+},\sigma_{+}), \qquad (15)$$

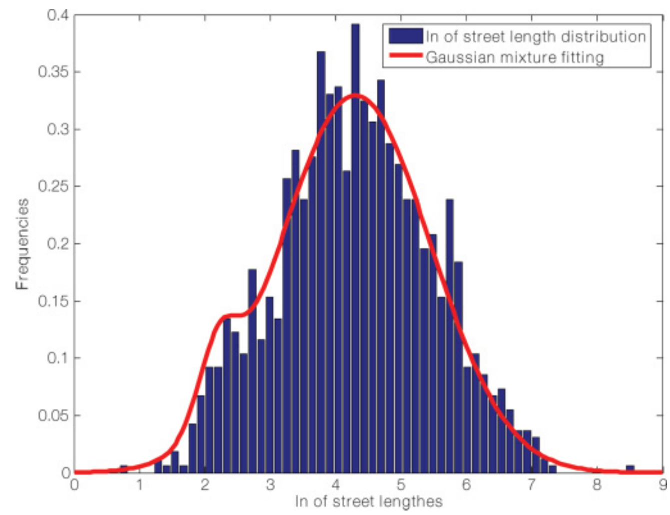

FIG. 7. (Color online) The histogram represents the distribution of the logarithm of street length in the center of Amiens. The red plot is the fitting of this histogram with a gaussian mixture whose parameters are $p_{-} = 0.5, m_{-} = 2.2, \sigma_{-} = 0.3; m_{+} = 4.3, \sigma_{+} = 1.2$.

with $m_{-} < m_{+}$. The identification of this model is performed with an expectation minimization algorithm.

The log scaling reveals that there is no evident length scale in a city, and there is no preexistent typical street. A city autoscales: Its dynamics is purely multiplicative and could be the result of new streets cutting through former blocks or extending space at the exterior of the city.

As for the bimodality, a town-planning explanation is that several transportation modes follow each other through time and their superposition creates modes in the distribution of L.

E. Synthesis: The city as a space division process

Table I summarizes the main indicators presented through this section for ten French cities and their extracted centers. For French towns the organic coefficient r_N may seem surprisingly high, even for Lyon's center, which is visually gridlike. The preponderance of degree 3 intersections speaks in favor of seeing a city as the result of a division process. The cutting of

TABLE I. Features of 10 French cities and their centers. r_n is the organic coefficient, M_4 the meshedness coefficient, $\mathcal{A}$ the anisotropy, $\mu_C(C)$ the total street length, $H_{\text{No.}}$ the number of streets in the hypergraph generated wit $R_{\pi/10}$, r^{topo} the topological radius of that hypergraph, and res the root mean square between the street length distribution and its bi-log-normal fitting.

	Whole cities							Centers						
City C ↓	r_N	M_4	$\mathcal{A}$	$\mu_C(C)$	$H_{\text{No.}}$	r^{topo}	res	r_N	M_4	$\mathcal{A}$	$\mu_C(C)$	$H_{\text{No.}}$	r^{topo}	res
Angoulme	0.80	0.28	0.46	300	1628	14	0.21	0.73	0.37	0.29	39	261	7	0.2
Avignon	0.85	0.23	0.59	625	3348	13	0.12	0.82	0.30	0.73	240	1607	9	0.12
Caen	0.79	0.29	0.53	485	3045	11	0.08	0.76	0.34	0.67	128	797	9	0.22
Carcassonne	0.86	0.20	0.66	483	1997	20	0.17	0.72	0.38	0.89	66	296	6	0.42
Dijon	0.75	0.33	0.39	558	2605	14	0.14	0.66	0.42	0.74	149	860	8	0.33
Grenoble	0.74	0.32	0.61	361	1638	10	0.17	0.70	0.40	0.74	119	576	6	0.29
Lyon	0.66	0.47	0.53	837	3606	15	0.17	0.53	0.51	0.93	183	606	6	0.24
Rennes	0.82	0.26	0.79	625	3538	15	0.20	0.77	0.30	0.80	192	1041	7	0.17
Rouen	0.71	0.38	0.69	348	1770	17	0.19	0.66	0.43	0.85	141	788	7	0.13
Troyes	0.81	0.28	0.87	230	1079	9	0.17	0.67	0.42	0.92	48	248	6	0.41

TABLE II. Variation of the meshedness coefficient $M_4 = 2(e - v + 1)/[2v(1 - \bar{N}(2)) - 5]$ for the 16 simulated cities. Each result is the averaging of 30 simulations. The variation for each case is almost constant and equal to 0.04. M_4 is an increasing function of both P_e and ω.

Ω P_e	0	0.5	0.8	0.99999
1	0.37	0.43	0.46	0.48
0.6	0.26	0.31	0.33	0.36
0.3	0.14	0.16	0.18	0.20
0	0	0	0	0

a block creates new intersections of degree 3 even if the block was formed of degree 4 vertices.

The meshedness coefficient M_4 is strongly correlated to r_N. In fact, it is a second-order refinement of r_N. We will show that it describes the shape of city graphs when r_N is constant or equal to 1 (shown in Table II).

The anisotropy coefficient $\mathcal{A}$ is the most discriminating indicator: It ranges from 0.29 to 0.93. It also points out that the street system within a city is not homogenous: for instance, $\mathcal{A}$(Lyon) = 0.53 and $\mathcal{A}$(Lyon Center) = 0.93.

The total street length $\mu_C(C)$ and the number of streets $H_{\text{No.}}$ measure the size of a city. In fact, the ratio between $\mu_C(C)$ and $H_{\text{No.}}$ provides an indication of the straightness of the city, which is geometrical information.

The topological radius r^{topo} is very small compared to a measure of the city size: $r^{\text{topo}} \ll H_{\text{No.}}$. r^{topo} may scale as the logarithm of $H_{\text{No.}}$, as a small-world complex network. This scaling may also be the consequence of a division process and/or of an upper system of long streets that covers the entire city's stretch, constraining the topological distance's variations to be mainly local. However, the topological radius is an indicator of the transportation performance of a given city: Carcassonne and Caen have almost the same total street length but the topological radius of Carcassonne is roughly twice Caen's.

res, the relative root mean square between the street length distributions and their bi-log-normal fitting, is $\lesssim$0.2 with two exceptions (Carcassonne and Troyes's centers), $\sim$0.4. The log-normal scaling of street lengths is relevant in most of the cases. This also calls for a division (multiplicative) modeling of the city, but a division process's length distribution would only have a single tail.

In the following section (Sec. IV) we present a morphogenesis model for the organic city consisting in the duality between division and extension of space. In Sec. V we will show that this model reproduces the features we have presented above: a high organic coefficient, a small topological radius, and a local variation of the topological distance, plus a log-normal distribution for streets' lengths.

IV. A STREET-BASED DYNAMICAL MODEL

This section presents a model of the growth and development of a town. The town is reduced to its streets and we build a dynamical model, allowing to add street segments one after another. As discussed in the previous sections, the spatial extension of the town and the geometry of the streets is of prime interest. As pointed out in Ref. [14], a city is above all an out-of-equilibrium system—that is to say, a dynamical system observed at a random time of its development.

Our model is based on three assumptions, two principles (installation and connection), and a few parameters, with the entire model giving a coherent and consistent vision of the problem. We aim at building a model that can reproduce several limit cases of urban growth but can also point out continuity between them. The principles and parameters we use are meaningful, and expressed in an interface language that allows the mathematical and physical communities to exchange with town planners, architects, and social scientists.

This section develops the model in the quite general case of organic development of the city on flatlands, which we can easily translate our assumptions into analytical procedures.

A. Hypothesis

As a dynamical system and a geometrical graph, we will see a city as a function $C : \mathbb{R}_+ \longrightarrow \mathcal{G}_C \subset \mathcal{C}_d$, with $C(t) = \{V(t), E(t)\}$. Then we make the following postulates on the evolution of C:

P1: A city is the result of a sequence of operations occurring at increasing times $(t_i)_{i\in\mathbb{N}}$ such that

$$C(t) = C(t_i) \quad \forall t \in [t_i, t_{i+1}[.$$

P2: Infrastructures are conserved:

$$C(t_1) \subseteq C(t_2) \quad \text{if } t_1 \leqslant t_2.$$

P3: There exist two functions P_t (price) and V_t (potentiality) such that the city is a compromise between them:

$$C(t + \Delta_t) = \underset{\substack{c \subset C(t) \\ P_t(c - C(t)) \leqslant 0}}{\operatorname{argmin}} V_t(C(t), c).$$

Functions C and P are not obvious to define. They should be in a "microscopic" point of view aggregation of economical parameters. We can avoid developing them if we observe a city's growth is determined by "macroscopic" insights:

Its planning: A city may be organic (the sum of local and independent phenomena; streets are added independently with no visibility on a global planning) or centralized (a global authority decides on the coherent and simultaneous addition of several streets on a large surface).

Its construction: The capacity to add new elements to the map and to build new streets.

Its organization: From a random settlement to a highly structured one.

Its sprawling: A city has to make a compromise between its inner development and outer growth.

We will consider here the case of organic growth.

B. Organic growth of a city

The algorithm below simulates a city's growth within the individual settlement hypothesis. Under this assumption, each settlement (a generic term to designate a commercial infrastructure, a private individual, etc.) is added at a given time and at a given location, then connects to the existing infrastructures.

The main idea is that the city $C(t)$ induces a spatial potential field describing the attractiveness of any point of the available space. A new settlement (either an individual settler or a facility) has its own policy (see Sec. IV B 3) of choice with respect to this potential. After having chosen its location, it connects to the existing street system. This model explicitly decouples the problems of positioning and of connecting.

In this section we will model how the geometry of the current city induces a potential field, how a new settlement can be connected to the city, and at last how we can tune the behavior of settlements by a few parameters.

1. *Potential field*

For each point x in the available space, the potential $P_{C\to}$ quantifies to what extent x is a good choice to locate a new center. This potential should mimic the following ideas:

(i) A large-scale behavior such that the global attraction of a part of the city should be proportional to the global mass of infrastructures in place and slowly decrease with some distance d: $P_{C\to x} \propto -\frac{\int d\mu_C}{d^\gamma(x,C)}$.

(ii) A very short-scale behavior that forbids a new center to be located on existing infrastructures: $P_{C\to x} = +\infty$.

(iii) A medium scale deduced from the two previous ones, which should display some local minimum.

Thus among several possible fields we choose (Fig. 8)

$$P_{C\to x} = \left(\frac{\alpha}{d_{\min}(x,C)} - \frac{\beta}{\sqrt{d_{\min}(x,C)}}\right)\int \frac{d\mu_C}{\sqrt{d_\perp(c,x)}}. \quad (16)$$

$d_{\min}(x,C) = \min_{c\in C}(x,c)$ is used in Eq. (16) so that the rejecting zone is hard: There is a tube around the city where new settlements are impossible. The radius of this tube is $\lambda_0 = (\alpha/\beta)^2$.

$d_\perp(x,c)$ is the $||\cdot||_1$ norm in the local basis formed by the unitary tangent and the normal to C in c. The use of such a distance simplifies the integral calculus compared to the Euclidian distance. Toward ∞, $P_{C\to x} \sim \frac{\beta}{d(x,C)}$, and between those two extreme positions, interferences between street segments produce local minima. To choose parameters α and β, one sets λ_0, the hard rejection radius, and β, the long-range influence. The choice of β influences the local geometry of the city but will not be discussed here.

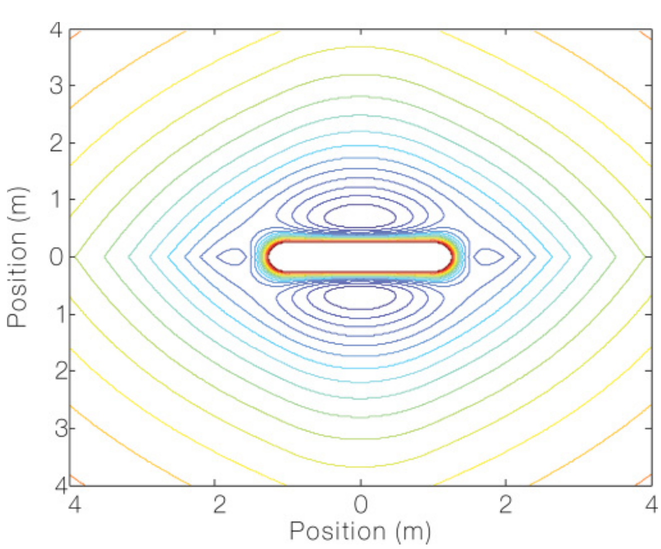

FIG. 8. (Color online) The level lines of the potential field for a city reduced to a single segment of length 1 with $\lambda_0 = 1$ and $\beta = 10$.

Among all possible potentials, we picked one that fulfills the conditions we set and that allows an explicit calculus of the integral. Further discussions would address the choice of the used distance and the decay exponent γ.

2. *Connection*

Once a settlement is added in a location x, it links to the existing network C. Not all connections are eligible.

From a point x we define the visible set of points,

$$V_{x|C}\{x \in C, [c\ x] \cap C = \{c\}\}, \quad (17)$$

and the optimal set of points from x of a part E of C,

$$\dot{E}_x = \{e \in E\ \exists\epsilon \mid \forall e' \in E \cap c \oplus B_\epsilon,\ d(x,e) \leqslant d(x,e')\}. \quad (18)$$

New connections are made between x and points in the optimal visible set $\dot{V}_{x|C}$. This is a finite set included in $C \cup (x\perp C)$, where $x\perp C$ is the set of orthogonal projections of x on the city. To avoid connections too close to each other, we introduce the relative neighborhood. In a general way, if P is a point and E a point set, then $s \in E$ is said to be in the relative neighborhood of P ($s \in RN[E||P]$) if and only if [14]

$$\forall u \in S,\ d(P,s) \leqslant \max\{d(u,P), d(u,s)\}, \quad (19)$$

i.e., there is no point both closer to s and to P. All candidates to become new connections are segments from x to $RN(\dot{V_{x|C}}||x)$: $[x, RN(C||x)]$.

3. *Parameters*

The tuning of those parameters will be discussed in Sec. V.

Organization. The global field induces minima. These minima represent points where it is the most interesting to settle.

The question is to find a parameter P_e that describes whether the city is organized or not. The idea is that when, a city is organized, it sticks strictly to optimal settlement places, and when it is purely unorganized, new settlements are added at random without any influence of the potential field.

Then a new settlement is selected by a Monte Carlo method with a number n of iterations, and the new point is be chosen as $X = \text{Argmin}_n P(X_i)$. For random cities n is close to 1 and for an organized city it is much higher.

Let W be the area of a part of the plan that contains the current city, and let $X_1,\ldots,X_n$ be n points on this part, uniformly and independently chosen. And let $X = \text{argmin}\,P(X_i)$. Then let

$$P_e = \mathbb{P}(|X - \text{Argmin}\,P)| \leqslant e) \quad (20)$$

represent the probability that the Monte Carlo method throws a point in a radius e of a local minimum.

We want to give P_e as an input parameter and traduce it into an iteration number. Let N be the number of local minima.

If e is quite small,

$$P_e \approx 1 - \left(\frac{W - N\pi e^2}{W}\right)^n, \quad (21)$$

$$n \simeq \frac{\ln(1 - P_e)}{e^2}\frac{W}{N\pi}. \quad (22)$$

n is estimated roughly by noticing that a local minimum is often owing to the interaction between two close streets segments: $n \sim 3v$ because 3 is roughly the average connectivity number of intersections.

Connection and construction. There are typically approximately four or five streets segments in $[x, RN(V_+||x)]$ for a new center x. If the city shapes as a slum, it would be treelike, so we link the number of street segments indeed added with the construction $\omega \in [0,1]$ of the city. We sort segments in $[x, RN(V_+||x)]$ by increasing length: $(s_1, \ldots, s_n)$. s_1 is drawn with probability 1. $n' \sim \mathcal{B}(\omega, n-1) + 1$ and segments $s_2, \ldots, s_{n'}$ are also added. If $\omega = 1$, every admissible segment is added, and if $\omega = 0$, only the shortest one is added.

Sprawling. When constructing with a rejection radius λ_0, the city gets a typical mesh width. If at a particular urban operation a potential field with a rejection radius of $K\lambda_0$ with $K > 1$ is considered, then the city's inner meshes will appear as filled up with the rejection zone of this potential field and new points of interest will position outside of the city.

With this observation we will consider that in a proportion f_{ext} centers are added with respect to a potential of rejecting radius $K_{\text{ext}}\lambda_0$. This creates foils at the outskirt of the city and thus an extension of the city that represents, for instance, an industrial zone that needs a large surface.

Refinements. To enhance the realism of this model, some empirical parameters are added. The length of a new street segment is bounded to $l_{\max} = k_{l_{\max}}\lambda_0$. This can avoid too long and costly connections (possibly $l_{\max} = \infty$).

The windows W out of where new settlements are chosen can be definitely set (Fig. 9) or dynamically change with the overall city (Figs. 11 and 13).

Because a Monte Carlo method is used to pick new centers, there is very little probability that a new settlement is added in line with an existing street. If geometrically this does not have many consequences, it may strongly change the local topology. That is why, if an orthogonal projection is in a radius $c\lambda_0$ with typically $c \simeq 0.3$ of an intersection that is visible from the center, then this orthogonal projection is removed. This raises the vertices degree and allows longer streets.

V. A FEW SIMULATIONS

To summarize an individual simulation of the city growth, we need to provide our algorithm with several parameters:
(1) The number of settlements: N.
(2) The organization probability P_e.
(3) The radius of the rejecting tube: λ_0.
(4) The long-scale influence: β.
(5) The construction: ω.
(6) The sprawling factor K_{ext} and the sprawling probability f_{ext}.

Notice that only four parameters will actually shape the simulated city (P_e, β, ω, f_{ext}), the others being scaling parameters and that the influence of β will not be disscussed here.

A. Simulations with constant parameters

Figure 9 shows the result of 16 simulations. The organization probability P_e and the construction ω are varying jointly when the same number of operations $N = 80$, the same rejecting radius $\lambda_0 = 10$ m, the same available space (a square with an area of 1.6 km^2), the same initial city (a segment of length 20 m at the center of the available space), and the same extension probability of O are used. The first result is that this model is able to reproduce very different types of growth with very few "physical" parameters.

We observe on this matrix representation that the meshedness M_4 (see Sec. III) is an increasing function of both P_e and ω (Table II). This result has been obtained by averaging the meshedness coefficient of 30 simulations for each couple (P_e, ω). Added to that, the standard deviation of M_4 for each couple is of 4% so that this coefficient is characteristic of the conditions of simulation. Contrary to that, the anisotropy coefficient $\mathcal{A}$ is almost the same in each case (between 0.31 and 0.46), with a large standard deviation of 20%. This $\mathcal{A}$ is quite large in the absolute: For the first iterations some directions have to be arbitrary chosen, which creates favored directions, but it is the same order of weight as for the most isotropic French towns. Of course, the organic ratio r_N is in every case close to 1.

When $\omega \simeq 0$, the resulting simulations are to be compared to the Saffman-Taylor instability. It seems when $\omega \simeq 0$ that only a bounded number of ramifications are possible from the

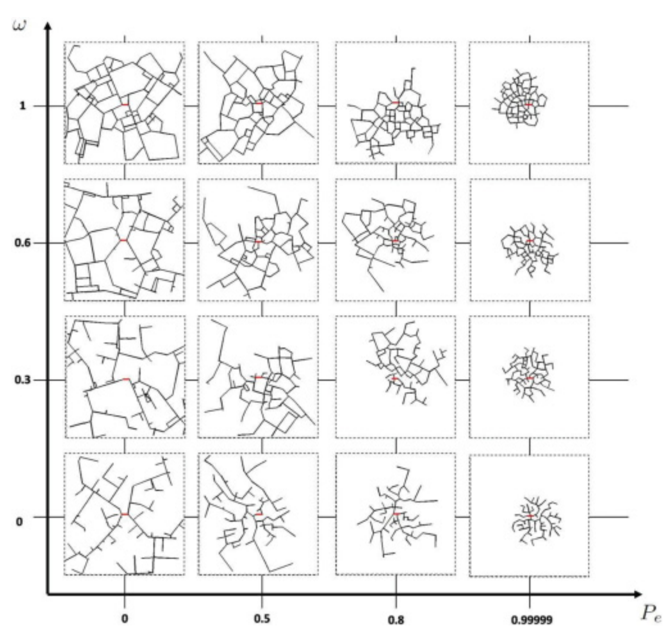

FIG. 9. (Color online) Simulations of the morphogenesis model with constant parameters with variation of the organization P_e and the construction ω. In each thumbnail, the rejecting radius is $\lambda_0 = 10$m, and there is no sprawling: $k = 0$, the number of settlements is $N = 80$, and the available space is bound in a square with sides of 400 m. The red and bold segment represents an initial street segment and a scale of 20 m.

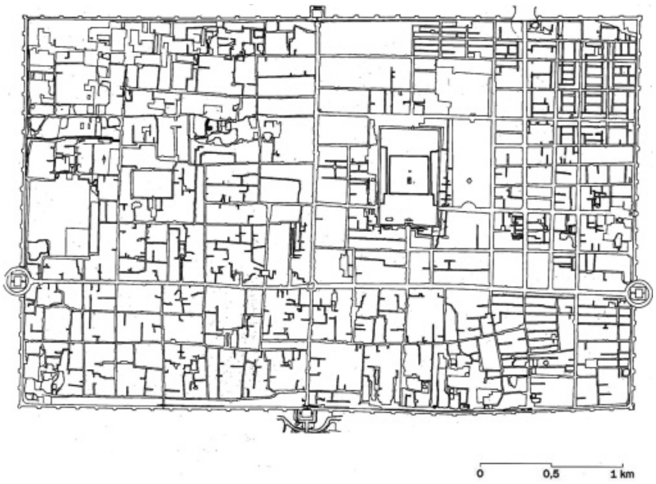

FIG. 10. The Chinese town of Xi'an in 1949, whose various subdivision patterns inside a regular grid recall variations in the parameters of the morphogenesis model.

initial segment (4 in this figure) as if the first created branches shielded the initial center from newer ones. When $\omega > 0$, the resulting cities are to be compared to crack patterns: Their dynamics follows a logic of division and subdivision of space.

Interestingly, the Chinese town of Xi'an (Fig. 10) has grown on a regular grid with a large mesh length, with several populations that have different characteristics. The result is a gradient of meshedness coefficient, from almost 0 in the southwest to almost 1 in the top-right-hand side. Figure 11 presents a city evolving with $\lambda_0 = 10$ m, $K = 10$, $K_f = 0.1$, $P_e = 0.8$, and $\omega = 0.7$. The regular need for larger surface for activities such as industries, big institutions, etc., is well reproduced here. During its history, as the development of the city center progresses, it eventually absorbs the peripheral larger surfaces, splitting them into smaller surfaces, with thus new larger places appearing at the new periphery. This reproduces and explains the situation of economical zones always outside at the periphery of towns. It explains as well the successive subdivision of space that leaves so many traces, first in the log-normal distribution of street lengths but also in the hierarchical distributions of streets (Fig. 12). For this

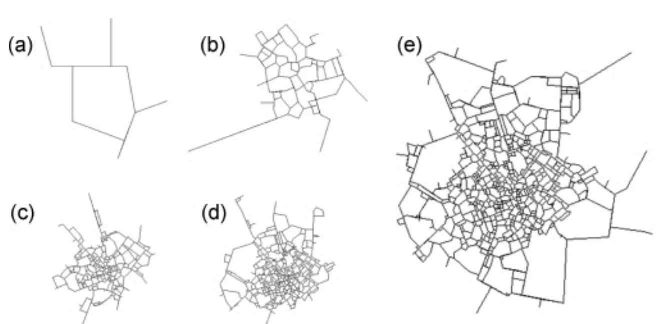

FIG. 11. (a), (b), (c), (d) Four steps in the development of the city (e) with 600 urban operations. For this simulation, $\lambda_0 = 10$ m, $K = 10$, $K_f = 0.1$, $P_e = 0.8$, $\omega = 0.7$, and the windows are adapted to the size of the current city. The main phenomenon at work is the dynamics between the inner development and the extension of the city that creates two hierarchical scales.

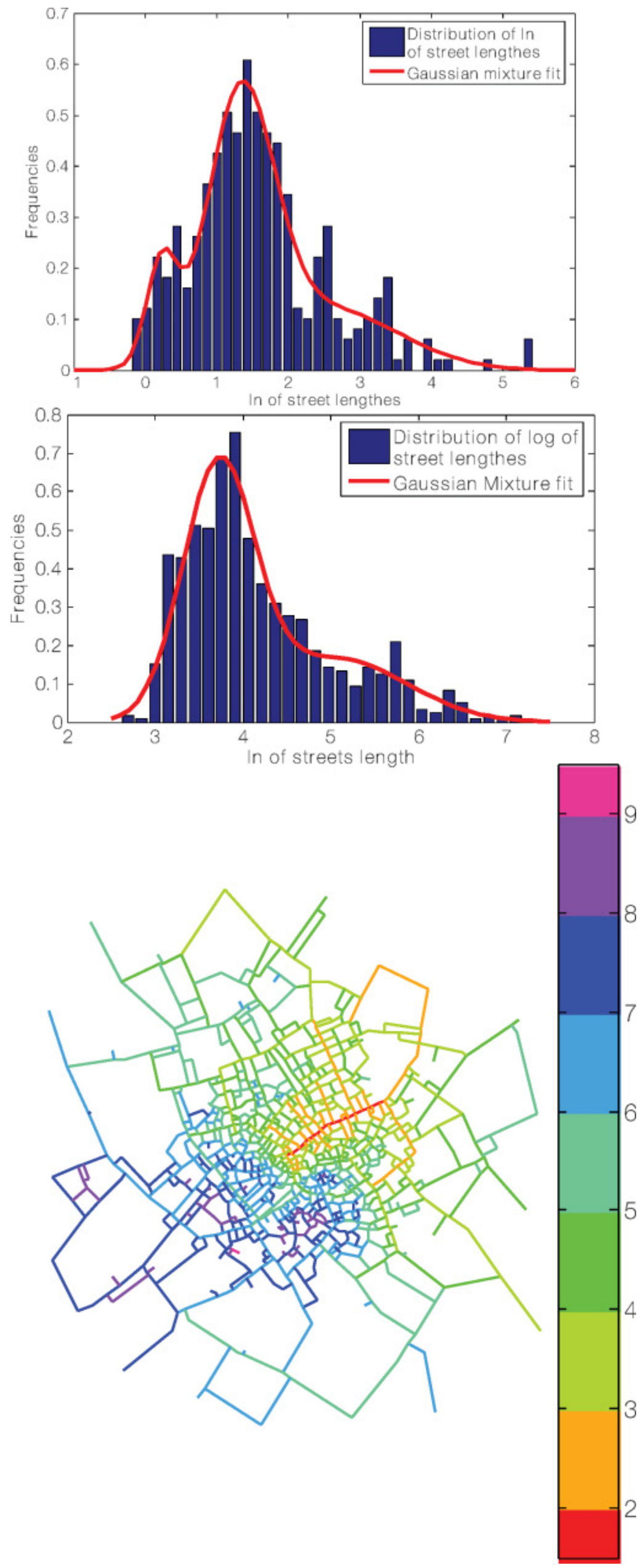

FIG. 12. (Color online) The distribution of the logarithm of street lengths for two synthetic cities (top) and the second-order topology (bottom) for the resulting city of simulation (see Fig. 11). As for real cities (Amiens), the second-order topology presents a bounded hierarchical representation of the city and the street length is well fitted by a mixture of log-normal random variables.

simulation, the ratio r_N is equal to 0.93, so that the term "organic" fits. The meshedness coefficient $M_4 = 0.48$ is quite close to Amien's (between 0.41 and 0.54) as the anisotropy (0.69 to be compared to 0.71 in the center of Amiens).

Figure 12 shows that the morphogenesis reproduces the small topological radius and the log scaling of street lengths observed in real cities (see Sec. III E). We have run 20 simulations with the same parameters. The root mean square distance between the length distribution and its best bi-log-normal fitting ranges between 0.19 and 0.26. This, compared to the result of the fittings for real French towns (less than 0.2 when the model is good and more than 0.4 when it is false), permits to claim that the model reproduces the street length scaling.

B. Simulations with varying parameters: The city's history

Constant parameters are not realistic to model a real city, this one being shaped by its history, which is from the morphogenetic point of view a variation of input parameters. We represent the history of a city by a piecewise constant function $t \to (P_e, \omega, l_{\max}, f_{\text{ext}}, K, \beta, \lambda_0)$.

For instance, the city shown in Fig. 13 has been obtained by simulating at first a city with a low construction and no sprawling ($\omega = 0.2$ and $f_{\text{ext}} = 0$) and then changed to a sprawling and constructed city ($\omega = 0.8$ and $f_{\text{ext}} = 0.15$). The simulation starts with two perpendicular streets with a length 20 times larger than $\lambda = 10$ m. These preexisting streets are structuring elements, which could be a river. This kind of variation in parameters recalls a Kasbah in Morocco, where the historical center of the city is a souk. This model is the first step of a very simple model. We can see that, with only one type of event (new settlement) and a few parameters, a great variety of structures can be obtained. More refinements can be added. The first one is the planned creation of a highway belt as in Amiens, or an enclosure using the punctual addition of the convex hull of the current city. The second possibility would be to add also the "Hausmann" effect, allowing splitting preexisting streets with new street patterns. We could also consider distinct populations with interaction rules that build a city in the same time. These three points are going to be developed in a second version of the morphogenesis model.

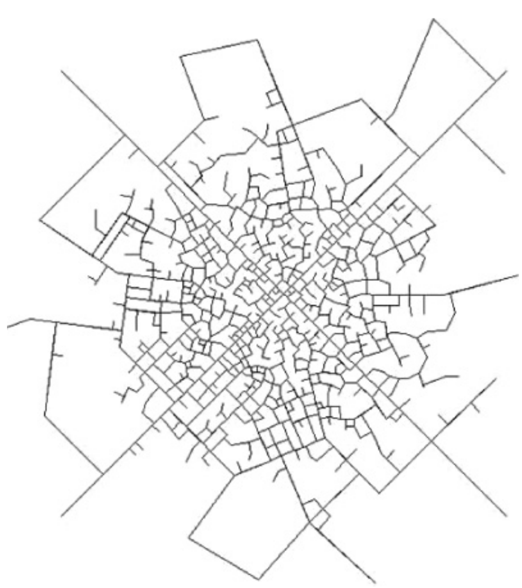

FIG. 13. A city developing with varying parameters from two perpendicular, long structuring elements. The historic center of the city has been built with a low construction parameter and no sprawling ($\omega = 0.2$ and $f_{\text{ext}} = 0$) to recall the tree aspect of a central souk in the Kasbah. Then parameters are changed to $\omega = 0.8$ and $f_{\text{ext}} = 0.15$, which produces an industrial crown.

VI. CONCLUSION

We have reduced cities to the map of their street segments and shown that much information can be deduced from this representation without additional data such as population dispersion, width of streets, and ground-specific use. For instance, we can find backstreets, characterize topology and shape, or define a centrality. To this we have introduced the notion of a geometrical and straight graph with a canonical hypergraph structure to define the difference between street segments and streets. A measure allows seeing a city's map as a "continuous graph" or likewise as an object both relational and geometrical (see Sec. II).

While Ref. [3] considered a city as a pure graph and Ref. [11] took into account its spatiality, we have explored the geometrical aspect of the city, its topology being only the skeleton that holds it up. From this point of view we have shown that, despite an evident diversity on their overall shapes (anisotropy first-order topology), a few fundamental rules can explain the cities' general morphogenesis (see Sec. IV). The urban infrastructure differentiates to adapt to the local geography and to fulfill the constraints of lodging people while maintaining their efficient transportation. The model developed in Sec. IV and illustrated in Sec. V is based on the division and extension of space principle observed for French towns in Sec. III. It shows that structural properties of cities stand out of the local constraints and behaviors that define the dynamics of cities. For instance, the log scaling of a resulting street system is a global, nontrivial property that validates the model as well as the small observed topological radius. Even in the organic case, when there is no global and coherent plan, the topological radius of the city increases slower than the size of the city.

To simulate the cities' dynamics, we have uncoupled the space potential induced by the current infrastructure, the policy of connections, and the freedom a new settlement (a generic term to refer to a commercial infrastructure, a private individual, etc.) ought to take on the two previous rules.

Further than the interest of a physical modeling of their object of study, our work can be applied by town planners and social scientists: The morphogenesis can help in semisupervised planning, and the analysis of cities allows detecting abnormalities on a map (the second-order topology representation is particularly effective). It can also be applied in engineering to test a technology that strongly depends on the urban infrastructures. This morphogenetic model calls for many outcomes: the study of several potential fields, a comparison to a large data basis of existing cities, and the following of the evolution of parameters as the city grows. The important element is that morphogenesis implements a space division and extension process. Because we used arbitrary tunings and yet obtained realistic results, we have shown that this class of process is robust to model cities, whether it be in the centralized case or in the organic one.

In the centralized case, it shows that the reason one wants a street system to be the homotopy of a square grid is to adapt to the local geography and infrastructure, which can be seen afterward as the result of a correlated division process. Indeed, in the organic case, the city's layout comes from the duality between the city's expansion and the graining of former large cadastres by new settlements. Also, from this local division process emerge some nontrivial global phenomena (log scaling of streets, low topological radius). One can thus say that the division of space is a natural response of cities to fulfill their functional goals.

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