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Analytical calculation of fragmentation transitions in adaptive networks

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In adaptive networks, fragmentation transitions have been observed in which the network breaks into disconnected components. We present an analytical approach for calculating the transition point in general adaptive network models. Using the example of an adaptive voter model, we demonstrate that the proposed approach yields good agreement with numerical results.

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In the past decade, networks have proven to offer a metaphor for describing and analyzing complex systems in different fields ranging from technological to biological and social systems [1–4]. By conceptually reducing a complex system to a set of discrete *nodes* connected by *links*, a simplification is achieved that often enables deep insights into the structure and dynamics of the system.

In network physics, dynamics can refer to two different concepts. First, the dynamics *of* networks describes the temporal evolution of the topology, the specific pattern of nodes and links. Second, the dynamics *on* networks refers to the evolution of internal properties in the network nodes, which are coupled according to the (typically static) topology. Systems combining both types of dynamics are called adaptive or coevolutionary networks [5,6].

Adaptive networks are presently studied in several different disciplines [7] and are known to exhibit unique phenomena, including robust self-organization to critical states [8], emergence of distinct classes of nodes from an initially homogeneous population [9], formation of complex hierarchical topologies [10,11], and complex network-level dynamics and phase transitions [12].

One exciting recent discovery in adaptive networks is the existence of a generic scenario for fragmentation transitions (FTs) [13]. In many adaptive networks, dynamics is contingent on the presence of so-called *active links* that connect nodes in different states. Absorbing states are therefore encountered either when the network becomes polarized, so that all nodes are in the same state, or when the network fragments such that nodes separate into disconnected components, which are internally state uniform.

FTs are frequently observed in simulations [13–17]. However, for a detailed understanding of dynamics in more complex future models, analytical approaches to FTs will be instrumental. Existing approaches that faithfully capture other transitions [12,17–23] yield only rough approximations for the fragmentation threshold, overestimating the actual value by 150%–200% in some examples [17,22,23]. In this paper, we propose an analytical approach for the FT allowing accurate prediction of transition points.

Below, we illustrate our results by the example of the adaptive voter model, a paradigmatic model of opinion formation in networked populations [14–17,22,23]. The original voter model [24] describes a network in which the nodes represent agents and the links represent social contacts. Each agent can hold either of two opinions. In time, the opinions are updated by either (a) selecting a random agent and letting it adopt the opinion of a randomly chosen neighbor (direct voter model), (b) selecting a random agent and copying its opinion to a randomly chosen neighbor (reverse voter model), or (c) selecting a random link and letting one of the linked agents adopt the other's opinion (link-update voter model). Because in the original model the topology remains fixed, the dynamics continues until global consensus is reached.

Adaptive variants of the voter model take an additional process into account: At a certain rate, agents that are connected to agents of different opinion break the respective link and connect to a randomly chosen agent of their own opinion. The rewiring leaves the number of agents and links unchanged but alters the structure of the network. Specifically, it can cause a fragmentation of the network such that both opinions survive in disconnected network components that are internally in consensus [14,16,25,26].

Here, we consider an adaptive voter model with link update in which rewiring events occur at a rate p and opinion updates occur at a rate $1-p$. The challenge that we address is computing the fragmentation threshold, i.e., the critical value of p at which the active links disappear. While details of the computation depend on the specific update mechanism, the proposed approach is applicable to a wide range of models employing different update rules.

For computing the FT, we follow an approach inspired by the computation of epidemic thresholds: We consider a situation in which the network is almost fragmented so that there are two almost disconnected clusters of different opinions, with few remaining active links between the clusters. We then derive dynamical equations capturing the net change in the density of active links. If this change is negative, then the number of active links declines exponentially, leading eventually to the fragmented state. If the balance is positive, then active links proliferate, preventing the network from reaching the fragmented state.

We have to take into account that a single opinion adoption event will create several active links that are connected to the same node. Even in the limit of low average active link density, we can not treat such links as independent because they all become inert at once if the focal node reverts to its original opinion. This correlation may explain the observed inaccuracy of mean-field and pair approximations.

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Predictions can be improved by using a basis that accounts for multiple active links connecting to the same node. In a first approximation, we use q-fan motifs, which we define as a subgraph consisting of one node holding a given opinion and q neighbors of the node holding the opposite opinion. The node holding the solitary opinion is denoted as the *base node* of the fan, whereas the other nodes are denoted as the *fringe nodes*. For instance, a 4-fan contains a base node and four fringe nodes, which are connected to the base node by active links.

For simplicity, let us first assume that the network is *degree-regular*, so every node has exactly the same number of neighbors k. The dynamics of active links for the special case of $k = 3$ is illustrated in Fig. 1. We start from a single active link (left half of figure). In an update event, with probability p, the active link is rewired becoming inert (not shown) or, with probability $1 - p$, one of the nodes adopts the other's opinion. In the adoption event, the original active link becomes inert, but the two other links of the adopting agent become active, forming a 2-fan. We continue by studying how updates affect this 2-fan (Fig. 1, right half). If an update is a rewiring event (probability p), then it decreases the width of the fan, turning the 2-fan into a single active link. If the update is an opinion adoption event (probability $1 - p$), then there are two possible scenarios occurring with equal probability. In the first scenario, the node at the base of the fan changes its opinion. In this case, the 2-fan becomes inert, but one new active link is formed at the base of the fan. In the second scenario, one of the fringe nodes of the fan adopts the base node's opinion; in this case, the width of the fan is reduced by one, but an additional 2-fan is activated. Because the two active motifs in the latter scenario are now separated by an inert link, they can be assumed to be independent to good approximation.

In finite networks, the dynamics of q-fans can be formulated as a Markov chain. However, for estimating the fragmentation transition in large networks, it is reasonable to move to a continuous time framework, where the density of q-fans is described by a system of differential equations. For the example of degree-regular networks with $k = 3$, we thus obtain (see Fig. 2, upper panel)

$$\begin{aligned} \dot{\{1\}} &= -\{1\} + 2\{2\}, \\ \dot{\{2\}} &= -2\{2\} + (1-p)\{1\} + (1-p)\{2\}, \end{aligned} \tag{1}$$

where $\{q\}$ is the q-fan density, i.e., the number of q-fans normalized by the total number of links L.

The systems of equations obtained by the approximation are linear and can therefore be solved straightforwardly. Even for nonlinear systems, the stability of the fragmented state can be tested by a local linearization given by the system's Jacobian matrix $\mathbf{J}$, with $J_{ij} = \partial\dot{\{i\}}/\partial\{j\}$, e.g., for $k = 3$,

$$\mathbf{J} = \begin{pmatrix} -1 & 2 \\ 1-p & -1-p \end{pmatrix}. \tag{2}$$

The fragmented state $\{i\} = 0$ is stable if all eigenvalues of the Jacobian matrix have negative real parts. For linear systems, this state is then also globally attractive. The FT occurs in the bifurcation where eigenvalues cross the imaginary axis. For $k = 3$, this transition occurs at $p = 1/3$.

We note that the stability of global consensus states, which are also characterized by $\{i\} = 0$, is not captured by the same Jacobian because these states violate our assumption

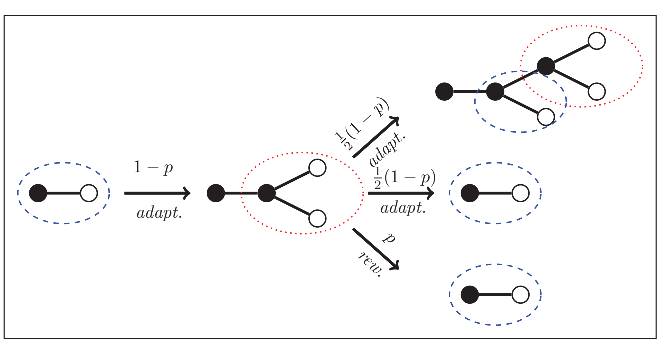

FIG. 1. (Color online) Illustration of the evolution of active links in a degree-regular network with degree $k = 3$. Agents are depicted as nodes that are open or solid depending on their opinion. Shown is the network in the neighborhood of an active link connecting clusters of different opinions. Arrows correspond to adaptation and rewiring events and are labeled with the corresponding transition rates. Depending on the parameters, the updates lead to proliferation or decline of isolated active links (encircled by dashed lines) and 2-fan motifs (encircled by dotted lines).

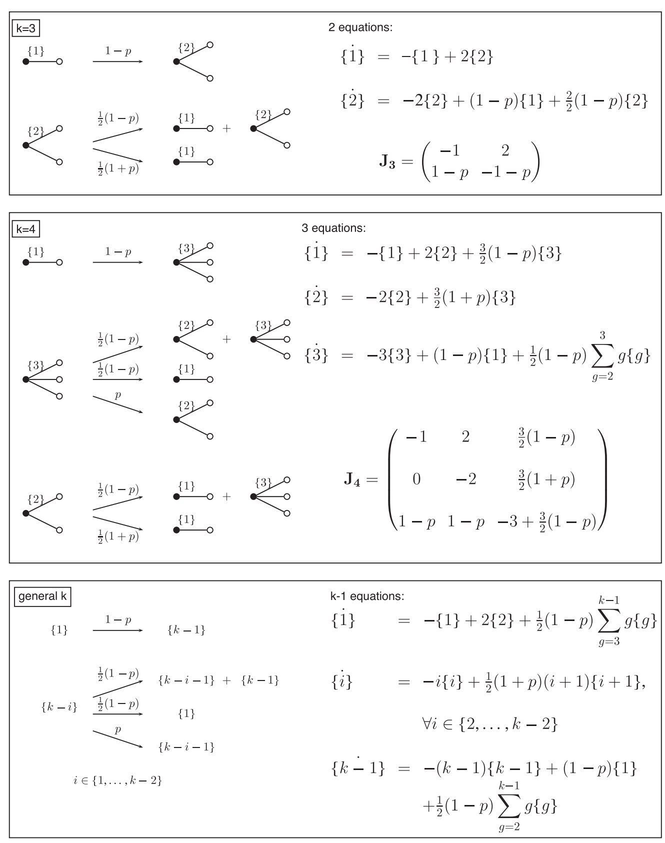

FIG. 2. Illustration of the transitions for the q-fan motif $\{q\}$ in degree-regular networks with degree k. Motifs and transitions are depicted as in Fig. 1. Additionally, the corresponding systems of differential equations and the Jacobians are shown. Note that the prefactor q for each contribution from a q-fan accounts for the q possibilities of choosing an active link.

of the presence of two almost disconnected clusters. Although adaptations for other transitions may be possible, the method, as proposed here, only captures transitions to the fragmented state.

Degree-regular networks with $k > 3$ can be treated analogously to the $k = 3$ example. The corresponding equations for $k = 4$ and the generalization to arbitrary k is shown in Fig. 2. An update affecting an active link deactivates the link and activates a $(k-1)$-fan with probability $1-p$. An update affecting a q-fan (a) deactivates the fan and activates a single link [probability $(1-p)/2$], (b) decreases the width of the fan by one, turning the q-fan into a $(q-1)$-fan (probability p), or (c) decreases the width of the fan by one and activates a new $(k-1)$-fan [probability $(1-p)/2$].

From the Jacobian, we can obtain the transition points either by numerical computation of eigenvalues or analytically by construction of test functions [27]. This procedure yields a more precise estimate of the FT than pair approximations (Fig. 3).

So far, we assumed degree regularity and did not account for inert links. Even if present in the initial network, degree regularity is destroyed by adaptive rewiring. Further, one can suspect that active regions of the network have a decreased density of inert links because of past rewiring events. For improving the prediction, we therefore introduce *spider* motifs, which consist of one central base node connecting to m nodes of its own opinion and l nodes of the opposing opinion. The $\{m,l\}$-spider thus holds m inert links and l active links, leading to a total degree $k = m + l$. As before, we do not account for all motifs in the network, but consider only active spiders, i.e., $l > 0$.

The effects of updates on a spider motif are shown schematically in Fig. 4. In a rewiring event, either the rewired link is kept by the fringe node and the $\{m,l\}$-spider is turned into a $\{m,l-1\}$-spider or the rewired link is kept by the base node turning the $\{m,l\}$-spider into a $\{m+1,l-1\}$-spider. In

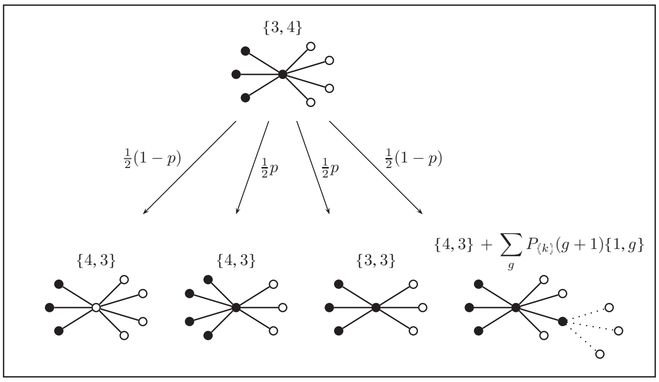

FIG. 4. Example for the transition probabilities of a {3,4}-spider motif. Diagrams and arrows represent motifs and transitions in analogy to Fig. 2. The dashed links indicate the number of next-nearest neighbors, which is drawn from the excess degree distribution (here, Poisson distribution).

an opinion adoption event, either the base node is convinced, which turns all active links into inert links and vice versa, leading to a $\{l,m\}$-spider, or one of the fringe nodes is convinced by the base node, giving rise to a new $\{1,g\}$-spider, while in the focal spider one active link turns into an inert link. In the latter case g, the number of active links of the newly activated spider is given by the excess degree distribution $P(g+1)$ [28]. The evolution of spider densities thus follows:

$$\{m,l\} \begin{cases} \xrightarrow{\frac{1}{2}(1-p)} \{l,m\}, \\ \xrightarrow{\frac{1}{2}(1-p)} \{m+1,l-1\} + \sum_{g=1}^{k_{\max}-1} P(g+1)\{1,g\}, \\ \xrightarrow{\frac{1}{2}(p)} \quad \{m+1,l-1\}, \\ \xrightarrow{\frac{1}{2}(p)} \quad \{m,l-1\}. \end{cases}$$

In an adaptive network, the excess degree distribution is often unknown because it is reshaped by the rewiring events. However, the success of the degree-regular approximation suggests that good results can be obtained if reasonable distributions are used. The results in Fig. 3 were obtained by assuming a Poissonian degree distribution with mean degree $\langle k \rangle$, which implies that the excess degree distribution is also Poissonian with the same mean degree [28]. Further, we considered only spiders up to a maximal degree of $k_{\max} = 50$. For simplicity, we constructed the Jacobian matrix by computer algebra and determined the FT from numerical computation of eigenvalues. Figure 3 shows that the results from the refined procedure are in good agreement with numerical values.

In summary, we proposed two approaches for computing fragmentation thresholds in adaptive networks. The simpler approach allows for a quick analytical estimation of the threshold with higher accuracy than previously proposed approaches. The refined approach yields numerical predictions with high accuracy. In particular, it works well for low $\langle k \rangle$, where pair and mean-field expansions yield poor results. For high $\langle k \rangle$, there is a small discrepancy, which may be due to

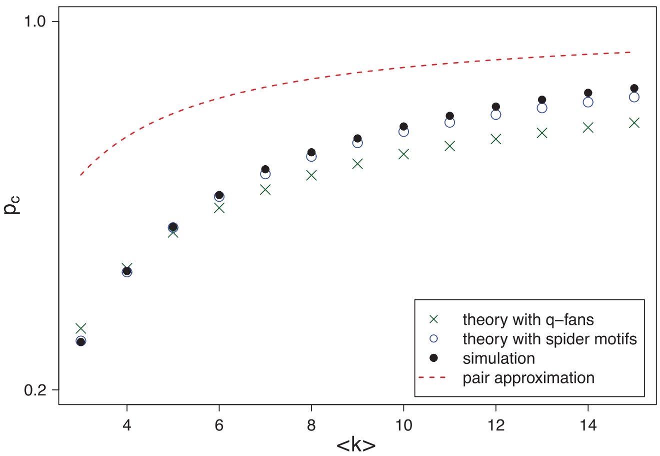

FIG. 3. (Color online) Fragmentation threshold in the adaptive voter model. Shown are numerical results from agent-based simulation (black dots), pair approximation (dashed line), the analytical approach proposed here (crosses), and its refined variant (circles). The proposed approaches yield a better match with the numerical results than the established procedure. Numerical simulations used $N = 10^6$ nodes.

long-range correlations or inaccuracies in the assumed degree distribution. In contrast to other techniques, the proposed approaches lead to linear ordinary differential equations because they assume a low density of active links. These equations are valid in the limit of infinite network size, where noise can be neglected. In small networks, the fan and spider expansions, proposed here, can be used for constructing Markov chains capturing the effect of randomness in the updates. A difficulty not addressed in this paper is obtaining the degree distribution for an adaptive network without explicit simulation. In principle, the relevant information for applying the refined procedure proposed here could be obtained by third-order moment expansions [17], which can not predict the fragmentation transition precisely, but allow for a relatively faithful estimation of the width of the degree distribution. In practice, it may be simpler to use statistical fits of distributions observed in a small set of exploratory simulation runs. Further, using a Poisson distribution should yield good results for all networks with exponentially decaying degree distributions. We therefore believe that both the simple and the refined procedures, proposed here, can be applied with relative ease in practice and will be instrumental to exploring fragmentation transitions in future adaptive network models.

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