Coupled quantum Otto cycle

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We study the one-dimensional isotropic Heisenberg model of two spin-1/2 systems as a quantum heat engine. The engine undergoes a four-step Otto cycle where the two adiabatic branches involve changing the external magnetic field at a fixed value of the coupling constant. We find conditions for the engine efficiency to be higher than in the uncoupled model; in particular, we find an upper bound which is tighter than the Carnot bound. A domain of parameter values is pointed out which was not feasible in the interaction-free model. Locally, each spin seems to cause a flow of heat in a direction opposite to the global temperature gradient. This feature is explained by an analysis of the local effective temperature of the spins.

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I. INTRODUCTION

Quantum generalizations of classical heat cycles have now been studied for some years. When the working medium is a few-level quantum system, new lines of enquiry open up due to additional features like discreteness of states, quantum correlations, quantum coherence, and so on [1–7]. Many models have served to investigate the validity of the second law of thermodynamics in the quantum regime [8,9]. The possibility of small scale devices and information processing machines [10] has generated further interest in the fundamental limits imposed on the heat generation, cooling power, and thermal efficiencies achievable with these models [11–13]. Quantum analogs of Carnot cycles, Otto cycles, and other Brownian machines have been analyzed [14,15]. Further, both infinite [2–4] and finite-time [16–22] thermodynamic cycles have attracted attention.

The quantum Otto cycle which occupies our interest here consists of a working substance with Hamiltonian H and initial density matrix ρ being manipulated between two heat reservoirs (the reservoir temperatures satisfy $T_1 > T_2$) under two adiabatic and two isochoric branches. On the adiabatic branches, the system is assumed to follow the quantum adiabatic theorem and thermodynamic work is defined in terms of the change in energy levels at given occupation probabilities. If the Hamiltonian is changed from H_1 to H_2 by controlling an external parameter then the work performed is defined as $\mathrm{Tr}[\rho(H_2 - H_1)]$. On the other hand, while traversing the isochoric branches, heat is exchanged with the reservoirs. Thus if the density matrix of the system changes from ρ_1 to ρ_2 for a given Hamiltonian H, then the heat exchanged is $\mathrm{Tr}[(\rho_2 - \rho_1)H]$. As an example, for an effectively two-level system whose energy splitting can be varied from E_1 to E_2, the Otto efficiency has been found to be $1 - E_2/E_1$, which is bounded from above by the Carnot value due to the condition $E_2/E_1 > T_2/T_1$ [2].

Recently, the role of quantum interactions using spin-1/2 particles has been adrressed for a quantum Otto cycle [3–5]. In particular, the role of quantum entanglement has been conjectured using measures like concurrence and the second law has been shown to hold in such models. In this paper, we investigate a coupled Otto engine using a one-dimensional (1D) Heisenberg model with isotropic exchange interactions between two spin-1/2 particles [see Eq. (1) below]. In earlier studies [3], during the adiabatic steps the exchange constant J was altered between two chosen values ($J_1 \rightarrow J_2 \rightarrow J_1$), while keeping the external magnetic field B at a fixed value. From an experimental point of view, it is also interesting to investigate the Otto cycle where the exchange constant is fixed and only the magnetic field is varied during the adiabatic steps. Note that the Carnot cycle gives a higher efficiency than an Otto cycle, but operating an adiabatic process within a Carnot cycle would imply changing both J and B simultaneously, which is more demanding than changing only B. Further, the uncoupled model cycles considered earlier in the literature can be taken as a benchmark with which to compare the engine performance of the coupled model.

The paper is organized as follows. In Sec. II, we present the quantum model of our working medium, enumerating the energy eigenstates and eigenvalues. In Sec. II A, the various stages of the heat cycle are described and expressions for heat exchanged with reservoirs and work delivered are calculated. It is instructive to develop the engine operation based on a local description. It is shown that all the work is done locally by each spin. In Secs. III A and III B we develop two cases: (i) $B_1 > B_2$ and (ii) $B_2 > B_1$. The latter case is possible only in the presence of interactions. It is observed for case (ii) that the heat exchange at the local scale seems to be counter to the global temperature gradient. General conditions are derived when the efficiency is higher than in the noninteracting model. We also present an upper bound for efficiency which is lower than the Carnot bound. The proof is sketched in the Appendix. In Sec. III C, we interpret some nontrivial features of the engine operation in terms of local spin temperatures. The final section summarizes our findings.

II. THE COUPLED QUANTUM HEAT ENGINE

The working medium for our quantum heat engine (QHE) consists of two spin-1/2 particles within the 1D isotropic

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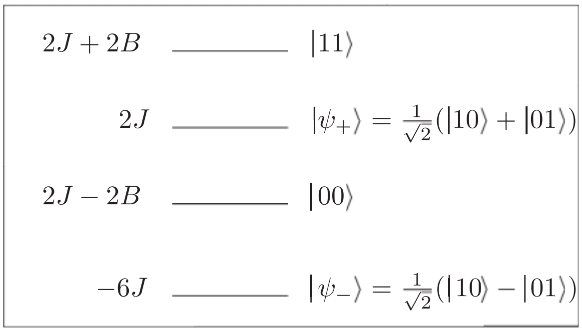

FIG. 1. Energy eigenvalues and eigenstates of two-spin entangled model system.

Heisenberg model [3,23]. The Hamiltonian is given by

$$H = J(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} + \vec{\sigma}^{(2)} \cdot \vec{\sigma}^{(1)}) + B\big(\sigma_z^{(1)} + \sigma_z^{(2)}\big), \quad (1)$$

where $\vec{\sigma}^{(i)} = \{\sigma_x^{(i)},\sigma_y^{(i)},\sigma_z^{(i)}|i = 1,2\}$ are the Pauli matrices; $J = J_x = J_y = J_z$ is the exchange constant and B is the magnetic field along the z axis. The cases $J > 0$ and $J < 0$ correspond to antiferromagnetic and ferromagnetic interactions, respectively. In this paper, we consider the antiferromagnetic case only. The energy eigenvalues of H are $-6J$, $(2J - 2B)$, $2J$, and $(2J + 2B)$ as shown in Fig. 1. If $|0\rangle$ and $|1\rangle$ represent the states of the spin along and opposite to the direction of the magnetic field, respectively, then in the natural basis $\{|11\rangle,|10\rangle,|01\rangle,|00\rangle\}$, we can write the density matrix as

$$\rho = P_1|\psi_-\rangle\langle\psi_-| + P_2|00\rangle\langle 00| + P_3|\psi_+\rangle\langle\psi_+| + P_4|11\rangle\langle 11|, \quad (2)$$

where $|\psi_\pm\rangle = (|10\rangle \pm |01\rangle)/\sqrt{2}$ are the maximally entangled Bell states. The occupation probabilities of the system in the thermal state at temperature T are given by

$$P_1 = \frac{e^{8J/T}}{Z}, \quad (3)$$

$$P_2 = \frac{e^{2B/T}}{Z}, \quad (4)$$

$$P_3 = \frac{1}{Z}, \quad (5)$$

$$P_4 = \frac{e^{-2B/T}}{Z}, \quad (6)$$

where $Z = (1 + e^{8J/T} + e^{2B/T} + e^{-2B/T})$ is the normalization constant.

A. The heat cycle

The four stages involved in our quantum Otto cycle are described below.

Stage 1. The system with the external magnetic field at B_1 attains thermal equilibrium with a bath of temperature T_1. Let the occupation probabilities be p_1, p_2, p_3, and p_4 as tabulated above with $T = T_1$ and $B = B_1$. *Stage 2*. The system is isolated from the hot bath and the magnetic field is changed from B_1 to B_2 by an adiabatic process. According to the quantum adiabatic theorem, the process should be slow enough to maintain the individual occupation probability of each energy level. *Stage 3*. The system is brought into thermal contact with a cold bath at temperature T_2. Upon attaining equilibrium with the bath, the occupation probabilities become p_1', p_2', p_3', and p_4', corresponding to the thermal state with $T = T_2$ and $B = B_2$. On the average, the system gives off heat to the bath. *Stage 4*. The system is removed from the cold bath and undergoes another quantum adiabatic process which changes the magnetic field from B_2 to B_1 but keeps the probabilities p_1', p_2', p_3', and p_4' unaffected. Finally, the system is brought back to touch the hot bath. On the average, heat is absorbed from the bath and the system returns to its initial state.

The heat transferred in stages 1 and 3 of the cycle respectively is

$$Q_1 = \sum_i E_i(p_i - p_i') \quad (7)$$

$$= 8J(p_1' - p_1) + 2B_1(p_2' - p_2 + p_4 - p_4') \quad (8)$$

and

$$Q_2 = \sum_i E_i'(p_i' - p_i) \quad (9)$$

$$= -8J(p_1' - p_1) - 2B_2(p_2' - p_2 + p_4 - p_4'). \quad (10)$$

In the above, E_i and E_i' $(i = 1,2,3,4)$ are the energy eigenvalues of the system in stages 1 and 3, respectively. $Q_1 > 0$ and $Q_2 < 0$ correspond to absorption of heat from the hot bath and release of heat to the cold bath, respectively. Comparing the equations for heat transfer between the system and the reservoirs, Eqs. (8) and (10), the quantity of heat $8J(p_1' - p_1)$ appears in both the equations. Obviously, this term is absent in the uncoupled case for which $J = 0$. As will be shown below, the sign $(\pm)$ of this term determines whether the efficiency in the coupled case will be higher or lower than the uncoupled case.

The work is done in stages 2 and 4 when the energy levels are changed at fixed occupation probabilities. The net work done by the QHE is

$$W = Q_1 + Q_2 = 2(B_1 - B_2)(p_2' - p_2 + p_4 - p_4'). \quad (11)$$

Note that $W > 0$ corresponds to work performed by the system.

III. THE LOCAL DESCRIPTION

In this section, we discuss how the individual spins in the system undergo the cycle. Again, let ϱ_{12} and ϱ_{12}' represent the thermal states in the natural basis when the two-spin system is in thermal equilibrium in stages 1 and 3, respectively. Explicitly, the density matrices are

$$\varrho_{12} = \begin{pmatrix} p_4 & 0 & 0 & 0 \\ 0 & \frac{p_1+p_3}{2} & \frac{p_3-p_1}{2} & 0 \\ 0 & \frac{p_3-p_1}{2} & \frac{p_1+p_3}{2} & 0 \\ 0 & 0 & 0 & p_2 \end{pmatrix}, \quad (12)$$

$$\varrho_{12}' = \begin{pmatrix} p_4' & 0 & 0 & 0 \\ 0 & \frac{p_1'+p_3'}{2} & \frac{p_3'-p_1'}{2} & 0 \\ 0 & \frac{p_3'-p_1'}{2} & \frac{p_1'+p_3'}{2} & 0 \\ 0 & 0 & 0 & p_2' \end{pmatrix}. \quad (13)$$

Let ϱ_1 and ϱ_2 be the reduced density matrices in stage 1 for the first and the second spin, respectively. Then from the normalization constraints, $\sum_i p_i = \sum_i p_i' = 1$, we get

$$\varrho_1 = \varrho_2 = \begin{pmatrix} \frac{1}{2} - \frac{(p_2-p_4)}{2} & 0 \\ 0 & \frac{1}{2} + \frac{(p_2-p_4)}{2} \end{pmatrix}. \quad (14)$$

Similarly, in stage 3, the reduced density matrices for the first and second spins are

$$\varrho_1' = \varrho_2' = \begin{pmatrix} \frac{1}{2} - \frac{(p_2'-p_4')}{2} & 0 \\ 0 & \frac{1}{2} + \frac{(p_2'-p_4')}{2} \end{pmatrix}. \quad (15)$$

Since the applied magnetic field is the same for each spin, their local Hamiltonian is also same. Let H_l and H_l' be the local Hamiltonians for individual spins with eigenvalues $(B_1, -B_1)$ and $(B_2, -B_2)$ in stages 1 and 3 respectively. The heat transferred locally between *one* spin and a reservoir is defined as $q_1 = \mathrm{Tr}[(\varrho_1 - \varrho_1')H_l]$ and $q_2 = \mathrm{Tr}[(\rho_1' - \rho_1)H_l']$. The explicit expressions are given as

$$q_1 = B_1(p_2' - p_2 + p_4 - p_4'), \quad (16)$$

$$q_2 = -B_2(p_2' - p_2 + p_4 - p_4') \quad (17)$$

for the hot and the cold reservoir, respectively. So we get the net work done by an individual spin as

$$w = q_1 + q_2 = (B_1 - B_2)(p_2' - p_2 + p_4 - p_4'). \quad (18)$$

From Eqs. (18) and (11)

$$W = 2w. \quad (19)$$

Thus the total work performed is the sum of work obtained from the two spins locally. Now if $B_1 > B_2$, the (local) efficiency of the spin based on the heat absorbed and the work done by it is given by $\eta_l = w/q_1 = 1 - B_2/B_1$. Note that the expression for η_l is identical to the efficiency η_0 for uncoupled spins. Later on, we also discuss the case $B_2 > B_1$, which does not yield the engine operation in the absence of interactions under the given setup ($T_1 > T_2$). It will be seen then that the local efficiency for $B_2 > B_1$ is $\eta_l = 1 - B_1/B_2$.

So the total heat absorbed by the system can be written as

$$Q_1 = 8J(p_1' - p_1) + 2q_1, \quad (20)$$

and similarly the heat released to the cold bath is

$$Q_2 = -8J(p_1' - p_1) + 2q_2. \quad (21)$$

Clearly, we can define a global heat exchange (Q_i) as well as a local heat exchange (q_i) where $i = 1,2$. Because work is being done only due to change in the local Hamiltonians, so in our opinion, it is reasonable to assume that only part of the heat which is absorbed locally by a spin is involved in energy conversion. Clearly, some part of the heat exchanged by the total system, which is present due to interaction between the spins, cannot *potentially* be converted into work. However, even if the operating conditions for the engine imply $Q_1 > 0$, and $Q_2 < 0$, it is possible that $q_1 < 0$ and $q_2 > 0$. This means that locally a spin can give off heat at the hot bath contact as well as absorb heat at the cold bath contact. In this paper, we highlight such an unexpected operation of the coupled-spin engine.

In the following, we consider two cases whereby magnetic field may be decreased or, alternately, increased in stage 2. It will be seen that the second case is feasible only in the presence of interactions, $J \neq 0$. In the first case when $J = 0$, the above equations go back to those for Kieu's model with two uncoupled spins where an engine operation is obtained given $T_1 > T_2$ and $B_1 > B_2$ with the additional condition $B_2/T_2 > B_1/T_1$.

A. The case $B_1 > B_2$

From Eq. (11), the condition that the work performed be positive ($W > 0$) is given by

$$(p_2' - p_4') > (p_2 - p_4). \quad (22)$$

Second, for the heat to be absorbed from the hot bath ($Q_1 > 0$), from Eq. (8) we have one of the following two possibilities: (i) $p_1' > p_1$ or (ii) $p_1' < p_1$. Along with the possibility (ii), we must also have $(p_2' - p_2 + p_4 - p_4') > (4J/B_1)(p_1 - p_1')$. Now we rewrite Eq. (8) as

$$Q_1 = 8J(p_1' - p_1) + \frac{WB_1}{(B_1 - B_2)}, \quad (23)$$

or $8J(p_1' - p_1) = Q_1(1 - \eta/\eta_0)$, where $\eta = W/Q_1$ is the efficiency of the coupled engine and $\eta_0 = (B_1 - B_2)/B_1$ is the efficiency of the uncoupled, i.e., $J = 0$, case which is the same as the local efficiency. Thus for $J > 0$, if $p_1' > p_1$, then $\eta < \eta_0$, or in the presence of coupling between the spins the efficiency is lower than η_0. This is shown in Fig. 2 as the region below the horizontal line. The global efficiency is equal to the local efficiency in two situations, when $J = 0$ or $p_1 = p_1'$.

On the other hand, if $p_1' < p_1$, then it is possible that the efficiency of the coupled engine can be higher than in the

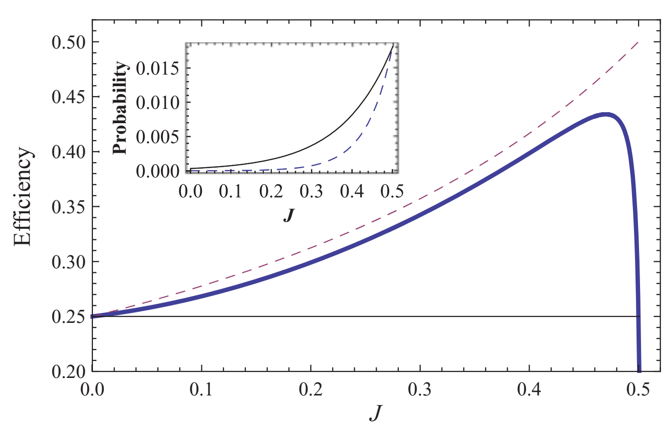

FIG. 2. (Color online) Efficiency versus the coupling constant J, for the $B_1 > B_2$ case, for values $B_1 = 4$, $B_2 = 3$, $T_1 = 1$ and $T_2 = 0.5$. The uncoupled model efficiency corresponds to $\eta_0 = 1 - B_2/B_1 = 0.25$, which is shown as the reference horizontal line. Case (i) $p_1' > p_1$, corresponds to efficiency below this line, while case (ii) $p_1 > p_1'$ gives a higher efficiency. The dashed curve denotes the bound for efficiency from Eq. (26). The inset shows the behavior of p_1 (solid line) and p_1' (dashed line) vs J.

uncoupled case. Using the latter condition with Eq. (22), we have

$$\frac{(p_2' - p_4')}{p_1'} > \frac{(p_2 - p_4)}{p_1}. \tag{24}$$

From the explicit expressions for the probabilities, the above inequality can be simplified to give

$$\frac{B_2}{T_2} > \frac{B_1}{T_1}, \tag{25}$$

for a given J value. Thus we see that the above condition which is necessary to extract work in the $J = 0$ model is *also* the condition for the coupled case to obtain an efficiency higher than η_0.

It is interesting to know how much maximum gain in efficiency is possible for a given set of parameters. We have proved an upper bound for the global efficiency, given by

$$\eta \leqslant \frac{1 - B_2/B_1}{1 - 4J/B_1} < \eta_c, \tag{26}$$

where $\eta_c = 1 - T_2/T_1$ is the Carnot bound. Also for $\eta > \eta_0$, we have the condition $B_1 > 4J$. This implies that the ordering of energy levels which gives an enhancement of efficiency (over the uncoupled model) is

$$(2J - 2B_1) < -6J < 2J < (2J + 2B_1), \tag{27}$$

and this after the first quantum adiabatic process becomes

$$(2J - 2B_2) < -6J < 2J < (2J + 2B_2). \tag{28}$$

The proof of Eq. (26) is given in the Appendix.

B. The case $B_2 > B_1$

In this case, during the first quantum adiabatic process, the magnetic field is *increased* from its value B_1 to B_2. If there is no interaction between the spins, the system cannot work as an engine in this case because the condition $W > 0$ will not be satisfied [2]. The conditions $T_1 > T_2$ and $B_2 > B_1$ directly lead to

$$p_4 > p_4', \tag{29}$$

$$p_3 > p_3'. \tag{30}$$

Further, the positive work condition implies $(p_2' - p_4') < (p_2 - p_4)$, which along with (29) gives

$$p_2 > p_2'. \tag{31}$$

The normalization of probabilities and the above three conditions Eqs. (29), (30), and (31) together give

$$p_1' > p_1. \tag{32}$$

These are the necessary conditions for the system to work as an engine given that $T_1 > T_2$ and $B_2 > B_1$. According to Eq. (18), the local work should be positive. This yields $q_1 < 0$ and $q_2 > 0$. This means that, locally, the heat is absorbed by a spin at the cold bath contact and heat is given off at the hot bath contact. The local efficiency defined as the ratio of work performed to the input heat is now given by

$$\frac{w}{q_2} = 1 - \frac{B_1}{B_2}. \tag{33}$$

Thus locally, the spins operate counter to the global temperature gradient present because $T_1 > T_2$. But globally we do have $Q_1 > 0$ and $Q_2 < 0$. Thus the function of the two-spin engine is consistent with the second law of thermodynamics, although locally we observe a flow of heat in a direction opposite to the "hot-to-cold" suggested by the baths. This apparent contradiction is resolved below using the concept of local effective temperatures.

C. Local temperatures

Now each spin in the two-spin system can be assigned a local effective temperature, corresponding to its local thermal state or the reduced density matrix [24–26]. This is true regardless of the state of the total system. Particularly, in stages 1 and 3 of the cycle, from Eqs. (14) and (15) along with the local Hamiltonian, we get the local temperatures as

$$T_1' = 2B_1 \left(\ln\left[\frac{2}{(1 + p_4 - p_2)} - 1 \right] \right)^{-1}, \tag{34}$$

$$T_2' = 2B_2 \left(\ln\left[\frac{2}{(1 + p_4' - p_2')} - 1 \right] \right)^{-1}. \tag{35}$$

The important fact is that in the presence of interactions, the local temperatures are different from the corresponding bath temperatures. Thus $T_1' \neq T_1$ and $T_2' \neq T_2$ if $J \neq 0$. Further, since the work in our heat cycle is done only locally, the total work by the system can be regarded as equal to the work by two independent spins operating between the highest and the lowest values of their effective temperatures (see Fig. 3).

(a) Engine working in $B_1 > B_2$. The positive work condition for a single spin is given by

$$\frac{B_2}{T_2'} > \frac{B_1}{T_1'}. \tag{36}$$

Since $B_1 > B_2$, we get

$$T_1' > T_2'. \tag{37}$$

After the first adiabatic process, the local temperature *decreases* from T_1' to $T_1'' = T_1'(B_2/B_1)$. Upon contact with the cold bath, the local temperature changes from T_1'' to T_2'. It can be shown that $T_1'' > T_2'$. Thus local temperature decreases

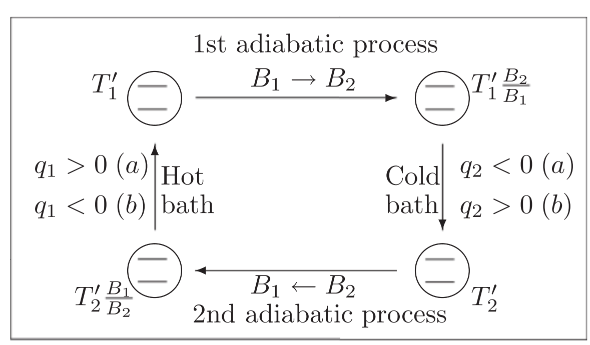

FIG. 3. Local effective temperatures of a spin (shown as a circle with two levels) during various stages of the heat cycle. Case (a) implies $B_1 > B_2$, while case (b) implies $B_1 < B_2$. Note that, in either case, the opposite signs of heat are exchanged locally upon contact with hot or cold baths.

upon attaching the whole coupled system with the cold bath. This means that locally heat is removed from a spin $q_2 < 0$. Thus both locally as well as globally, heat is given off at the cold bath contact. Similarly, one can explain $q_1 > 0$ in terms of change in the local temperatures at the hot bath contact.

(b) Engine working in $B_2 > B_1$. The situation is different in this case. Here, the positive work condition is satisfied only when

$$\frac{B_1}{T_1'} > \frac{B_2}{T_2'}. \tag{38}$$

Thus in this case $T_2' > T_1'$. Based on local temperatures, the counterintuitive mechanism which leads in case (ii) to $q_1 < 0$ and $q_2 > 0$ can be justified as follows. For $B_2 > B_1$, due to the first adiabatic process, the local temperature *increases* from T_1' to $T_1'' = T_1'(B_2/B_1)$. After contact with the cold bath, the new local temperature T_2' is more than the earlier value of $T_1'(B_2/B_1)$. Thus heat is expected to flow locally into the spin at the cold bath contact, or $q_2 > 0$.

IV. SUMMARY

A model of coupled spins with isotropic interactions is used as the working medium to realize a quantum Otto engine. The conditions for the efficiency to be higher than in the noninteracting case are found. A tighter upper bound for the efficiency is found which is lower than the Carnot value. The system can also work as a heat engine even if it undergoes an adiabatic compression ($B_2 > B_1$) in the second stage of the cycle. Here, using the reduced density matrix, we have observed an interesting mode of operation whereby each spin absorbs heat at the cold bath contact and rejects some heat at the hot bath contact, while performing a net work. However, globally the coupled system absorbs heat at the hot bath and rejects some heat at the cold bath. From the analysis of local effective temperatures of the spins, it becomes clear that the spin is actually operating as an engine between its highest local temperature T_2' and the lowest local temperature T_1' in one cycle. It will be interesting to extend the present cycle to models with anisotropic Heisenberg interactions and also to investigate if with such working media one can approach Carnot efficiency within an Otto cycle.

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APPENDIX: UPPER BOUND FOR GLOBAL EFFICIENCY

We consider the case of the engine working in the range $B_1 > B_2$. The condition to get a *higher* efficiency as compared to the uncoupled model is the case (ii) discussed in Sec. III A and is given by

$$p_1 > p_1'. \tag{A1}$$

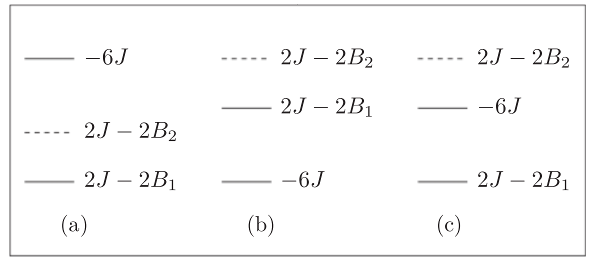

FIG. 4. Three possible configurations of energy levels with eigenvalues $-6J$ and $(2J-2B_1)$ and the level $(2J-2B_2)$ resulting from the first quantum adiabatic process whereby B_1 is changed to a lower value B_2. Only case (a) is possible, as discussed in the Appendix.

From the condition $B_2/T_2 > B_1/T_1$ [Eq. (25)], we get

$$p_3 > p_3', \tag{A2}$$

$$p_4 > p_4'. \tag{A3}$$

Then normalization of the probabilities gives

$$p_2' > p_2. \tag{A4}$$

From Eqs. (A1) and (A4), we have

$$\frac{p_2'}{p_1'} > \frac{p_2}{p_1}, \tag{A5}$$

which simplifies to

$$e^{(B_2-4J)/T_2} > e^{(B_1-4J)/T_1}. \tag{A6}$$

Figure 4 shows three possible ways of arranging the energy levels $(2J-2B_1)$ and $-6J$ relative to the level $(2J-2B_2)$ resulting from the first quantum adiabatic process. Equivalently, Eq. (A6) is of the form $e^x > e^y$, which may be satisfied in one of the following three ways:

Case (a) represents $y > 0, x > 0$ and so $x > y$. This implies $B_1 > 4J$ and $B_2 > 4J$.

Case (b) represents $x < 0$, $y < 0$ and $|x| < |y|$. This implies $B_1 < 4J$, $B_2 < 4J$, but due to the fact that $T_2/T_1 < 1$, we obtain $B_1 < B_2$ which leads to a contradiction.

Case (c) represents $y < 0$ and $x > 0$. This possibility is also similarly ruled out.

So the only possibility is case (a), representing the fact that the energy levels $(2J-2B_1)$ and $(2J-2B_2)$ lie below the level $-6J$ when the coupled engine gives a *higher* efficiency than in the uncoupled case.

When the inequality (A6) holds, we can write

$$\frac{B_2-4J}{T_2} > \frac{B_1-4J}{T_1}. \tag{A7}$$

Since $B_1 > 4J$, $B_2 > 4J$, and $T_1 > T_2$, we get

$$\frac{\eta_0}{1-4J/B_1} < \eta_c = 1 - \frac{T_2}{T_1}, \tag{A8}$$

where $\eta_0 = 1 - B_2/B_1$. Now the global efficiency defined as

$\eta = W/Q_1$, can be written as

$$\eta = \frac{\eta_0}{1 - \frac{4J(p_1 - p_1')}{B_1(p_4 - p_4' + p_2' - p_2)}}. \tag{A9}$$

From the inequalities between the probabilities [Eqs. (A1), (A3), and (A4)], it follows that $(p_1 - p_1') < (p_4 - p_4' + p_2' - p_2)$. Therefore, we finally obtain that when the efficiency is higher than in the uncoupled case (or the lower bound is η_0), then an upper bound for efficiency is given by

$$\eta < \frac{\eta_0}{1 - 4J/B_1} < \eta_c. \tag{A10}$$

When $J = 0$, we have $\eta = \eta_0$. A similar kind of proof can be constructed for the case $B_2 > B_1$. Interestingly, the same bound as Eq. (A10) is obtained.

[1] H. E. D. Scovil and E. O. Schulz-Dubois, Phys. Rev. Lett. **2**, 262 (1959); J. E. Geusic, E. O. Schulz-Dubois, and H. E. D. Scovil, Phys. Rev. **156**, 343 (1967).
[2] T. D. Kieu, Phys. Rev. Lett. **93**, 140403 (2004); Eur. Phys. J. D **39**, 115 (2006).
[3] T. Zhang, W.-T. Liu, P.-X. Chen, and C.-Z. Li, Phys. Rev. A **75**, 062102 (2007).
[4] G. F. Zhang, Eur. Phys. J. D **49**, 123 (2008).
[5] Hao Wang, Sanqiu Liu, and Jizhou He, Phys. Rev. E **79**, 041113 (2009).
[6] M. O. Scully, M. S. Zubairy, G. S. Agarwal, and H. Walther, Science **299**, 862 (2003).
[7] R. Dillenschneider and E. Lutz, Europhys. Lett. **88**, 5003 (2009).
[8] *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*, edited by H. S. Leff and A. F. Rex (Institute of Physics, Bristol, 2003).
[9] H. T. Quan, Y. D. Wang, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. **97**, 180402 (2006).
[10] Yun Zhou and Dvira Segal, Phys. Rev. E **82**, 011120 (2010).
[11] A. E. Allahverdyan, K. Hovhannisyan, and G. Mahler, Phys. Rev. E **81**, 051129 (2010).
[12] Noah Linden, Sandu Popescu, and Paul Skrzypczyk, Phys. Rev. Lett. **105**, 130401 (2010).
[13] Paul Skrzypczyk, Nicolas Brunner, Noah Linden, and Sandu Popescu, e-print arXiv:1009.0865.
[14] H. T. Quan, Yu-xi Liu, C. P. Sun, and F. Nori, Phys. Rev. E **76**, 031105 (2007).
[15] T. E. Humphrey, R. Newbury, R. P. Taylor, and H. Linke, Phys. Rev. Lett. **89**, 116801 (2002).
[16] E. Geva and R. Kosloff, J. Chem. Phys. **96**, 3054 (1992).
[17] C. Van den Broeck, Phys. Rev. Lett. **95**, 190602 (2005).
[18] Feng Wu, Lingen Chen, Shuang Wu, Fengrui Sun, and Chih Wu, J. Chem. Phys. **124**, 214702 (2006).
[19] A. E. Allahverdyan, R. S. Johal, and G. Mahler, Phys. Rev. E **77**, 041118 (2008).
[20] M. Esposito, K. Lindenberg, and C. Van den Broeck, Phys. Rev. Lett. **102**, 130602 (2009).
[21] M. Esposito, K. Lindenberg, and C. Van den Broeck, Europhys. Lett. **85**, 60010 (2009).
[22] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Phys. Rev. E **81**, 041106 (2010).
[23] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001)
[24] A. Garcia-Saez, A. Ferraro, and A. Acin, Phys. Rev. A **79**, 052340 (2009).
[25] J. Gemmer, M. Michel, and G. Mahler, *Quantum Thermodynamics* (Springer, Berlin, 2004).
[26] M. Hartmann, Contemp. Phys. **47**, 89 (2006).