Class of solvable nonlinear oscillators with isochronous orbits

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The nonlinear oscillator $\ddot{x} + (2m+3)x^{2m+1}\dot{x} + x + x^{4m+3} = 0$, with m a non-negative integer, is known to have a center in the origin, in a neighborhood of which are isochronous orbits, i.e., orbits with fixed period, not dependent on the amplitude. Here, we show that this oscillator can be explicitly integrated, and that its phase space can be completely characterized.

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I. INTRODUCTION

The Liénard equation (LE hereafter),

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

is a well-known equation of mathematical physics. Originally introduced to describe electrical circuits (Van der Pol [1], Liénard [2]), the LE, together with equivalent first order systems such as

$$\dot{x} = y,$$
$$\dot{y} = -f(x)y - g(x),$$

has been used in a variety of contexts, both in physics and biology. An extensive list of applications is given in [3], where the LE was also shown to be relevant in an epidemiologic context. In many applications, the coefficient $f(x)$ of the nonlinear damping term and the restoring force $-g(x)$ can be approximated by polynomials of x, and polynomial forms of f and g also arise quite naturally from the reduction of first order systems in which the right-hand sides are polynomials, as it often happens in ecological applications. As a consequence, LE's with polynomial coefficient functions have been widely used in the last decades to explore basic issues, such as the characterization of the period functions, the classification of the limit cycles, and the determination of the conditions allowing for isochronous solutions, i.e., oscillatory solutions with frequency independent from the oscillation amplitude.

An interesting example of LE with isochronous solutions has been given by Chandrasekar *et al.* [4], who showed that, when $\lambda > 0$,

$$\ddot{x} + 3kx\dot{x} + \lambda x + k^2x^3 = 0 \tag{2}$$

has the explicit solution

$$x(t) = \frac{A\sin(\omega t + \delta)}{1 - (k/\omega)A\cos(\omega t + \delta)}, \tag{3}$$

with $\omega = \sqrt{\lambda}$, and δ an arbitrary constant. For $0 < A < \omega/k$, this yields isochronous oscillations of frequency ω, the same frequency of the harmonic oscillator obtained for $k = 0$. To trace the reasons for this behavior, the authors of [4] have examined the transformation properties of the equation, and found that (2) can be mapped into the harmonic oscillator equation through a nonlocal, frequency preserving transformation, thus explaining the "unusual" frequency-amplitude relation. It is not clear, however, whether (2) represents an exceptional case, or belongs to a wider class of LE's admitting isochronous solutions.

The results of [4] can be put into a broader context by examining the related mathematical literature. Important works by Sabatini [5] and by Christopher and Devlin [6] have provided necessary and sufficient conditions for isochronicity—expressed as conditions on the coefficent functions $f(x)$ and $g(x)$—that define a wide class of isochronous LE's, to which (2) belongs. In particular, after an appropriate scaling, (2) is easily seen to be the simplest case (for $m = 0$) of the subclass

$$\ddot{x} + (2m+3)x^{2m+1}\dot{x} + x + x^{4m+3} = 0, \tag{4}$$

with m a non-negative integer, that was given in [5] as an explicit example of LE admitting isochronous orbits in a neighborhood of the origin.

It turns out that (4) is a particularly interesting example, since it can be explicitly solved. The main purpose of the present Brief Report is to derive such solution, building on the approach used in [5] to determine the isochronicity conditions. After reminding some basic facts about (2) in the next section, in Sec. III we derive the general solution of (4), together with a first integral not dependent on time, that allows us to characterize the orbits. Results are summarized in Sec. IV, where some open issues are also briefly indicated.

II. THE $m = 0$ CASE

The scaling $x \to (\omega/k)x, t \to (1/\omega)t$, transforms (2) into

$$\ddot{x} + 3x\dot{x} + x + x^3 = 0, \tag{5}$$

which contains no parameters, and manifestly corresponds to the case $m = 0$ of (4). The general solution of (5) can be written as

$$x(t) = \frac{\sin(t + \delta)}{C - \cos(t + \delta)}, \tag{6}$$

which, for $|C| > 1$, gives periodic solutions of unit frequency, with amplitudes growing without bounds when C approaches unity (correspondingly, as shown in Fig. 1, the solutions become more and more "nonlinear"). Thus, the restriction on C does not set limits to the amplitude of the oscillations, but only separates the periodic solutions from the singular ones, that blow up at finite times.

In [4], the ansatz $x \propto \dot{X}/X$ was used to map (5) into a third-order linear equation for X. An alternative path, suggested by

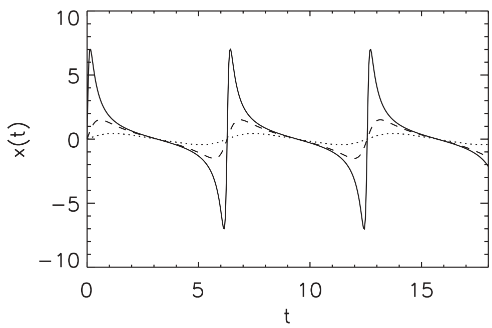

FIG. 1. Solution (6) for $C = 2.5$ (dotted), 1.2 (dashed), and 1.01 (solid).

the approach used in [5], relies on the analysis of the first order system

$$\dot{x} = y - x^2, \tag{7}$$

$$\dot{y} = -x - xy, \tag{8}$$

which reduces to (5) upon elimination of y [7]. It readily follows from (7) and (8) that

$$\frac{d}{dt}\left(\frac{x}{y}\right) = 1 + \left(\frac{x}{y}\right)^2, \tag{9}$$

which, as noted in [6], implies a constant angular velocity ($\dot{\theta} = -1$), and consequently isochronicity. In fact, solution of (9) gives

$$y = x/\tan(t+\delta), \tag{10}$$

with δ an integration constant, and placement of this expression in (7) yields a Bernoulli equation for x that is easily solved to find (6).

Elimination of time between x and $\dot{x}$ gives the first integral

$$C^2 = \frac{(\dot{x} + x^2 + 1)^2}{(\dot{x} + x^2)^2 + x^2}, \tag{11}$$

or, solving for $\dot{x}$,

$$\dot{x} = -x^2 + \frac{1}{C^2 - 1}\left[1 \pm C\sqrt{1 - (C^2 - 1)x^2}\right]. \tag{12}$$

This yields the phase space trajectories shown in Fig. 2 [8]. For $C^2 > 1$, one has closed orbits, corresponding to periodic isochronous solutions, that are located above the parabola $\dot{x} = -(x^2 + 1)/2$ (P), on which $C = 1$. The limiting x values for these orbits are given by $x = \pm 1/\sqrt{C^2 - 1}$, which diverge, as expected, when C approaches unity. For $1 < C < 2$, on each closed orbit there are four points on which both $d\dot{x}/dx$ and the acceleration vanish. Two of them are at $x = 0$, while the others, corresponding to minima of $\dot{x}$ along the orbit, are on the parabola $\dot{x} = -(x^2 + 1)/3$. Open trajectories, corresponding to singular solutions, are obtained for $C^2 < 1$, and lie in the negative $\dot{x}$ half-plane, below P, filling two distinct regions, separated by the parabola $\dot{x} = -x^2 - 1$, on which C vanishes.

Before moving to the general case, it is worth underlining that the choice of the numerical coefficients in (5) is the only one that yields isochronous solutions, since the necessary and sufficient condition for isochronicity given in [5] implies this choice.

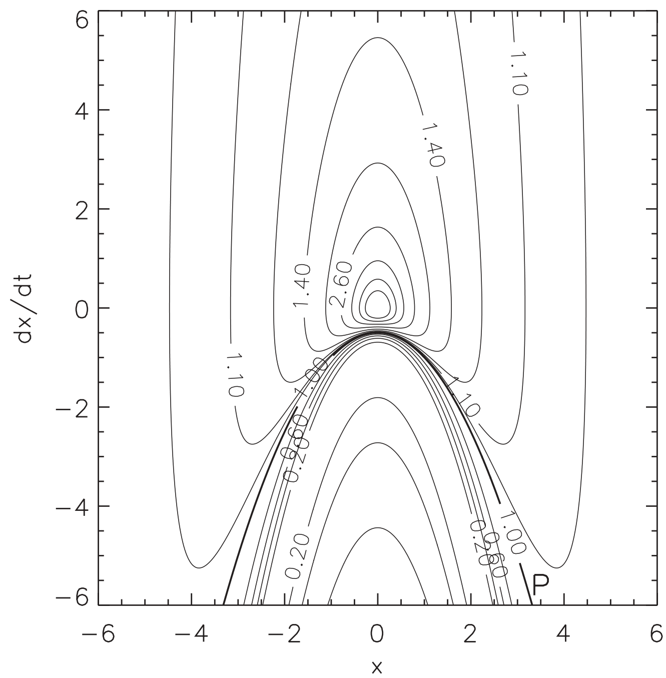

FIG. 2. Phase space of the nonlinear oscillator (5). The closed contours, lying above the parabola P, correspond to periodic, isochronous oscillations.

III. THE GENERAL CASE

The solution for general m can be obtained through a simple extension of the previous analysis. We now consider the system

$$\dot{x} = y - x^{2m+2}, \tag{13}$$

$$\dot{y} = -x - x^{2m+1}y. \tag{14}$$

It is easily verified that this reduces to (4) after elimination of y, and that (9) again holds, implying isochronicity. Placing (10) in (13) now gives

$$\dot{x} = \frac{\cos\tilde{t}}{\sin\tilde{t}}x - x^{2m+2}, \tag{15}$$

with $\tilde{t} = t + \delta$, that has the solution

$$x(t) = \frac{\sin\tilde{t}}{\left[C + (2m+1)\int dt \sin^{2m+1}\tilde{t}\right]^{1/(2m+1)}}. \tag{16}$$

Evaluation of the integral in the denominator (see, e.g., [10]) finally yields

$$x(t) = \frac{\sin\tilde{t}}{\left[C - \cos\tilde{t}\sum_{r=0}^{m} A_{mr}\sin^{2r}\tilde{t}\right]^{1/(2m+1)}}, \tag{17}$$

with

$$A_{mr} \equiv \frac{2^{2(m-r)}(m!)^2(2r)!}{(2m)!(r!)^2}, \tag{18}$$

which is the general solution we sought [9], and, moreover, a fully explicit one, that can be easily evaluated for any m. It is

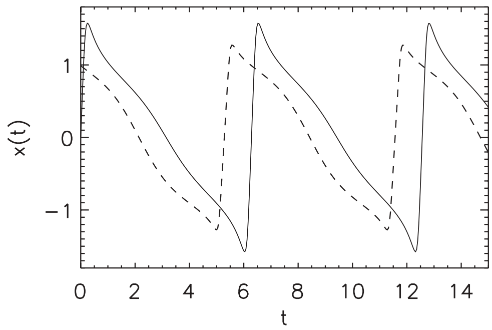

FIG. 3. Plots of $x(t)$, for $m = 1$ (solid), and $m = 2$ (dashed), for values of C just above the threshold (19) for periodic solutions.

clear from (16) that the condition for periodic solutions now is $|C| > A_{m0}$, or

$$|C| > \frac{2^{2m}(m!)^2}{(2m)!}. \tag{19}$$

Plots of the solution (17), for $m = 1$, and $m = 2$, for values of C close to this threshold, are shown in Fig. 3.

In order to characterize the orbits, we may derive a time independent first integral as follows. Rearranging terms in (15) and squaring, we get

$$\sin^2 \tilde{t} = \frac{x^2}{x^2 + (\dot{x} + x^{2m+2})^2}, \tag{20}$$

which allows us to express the sum in (17) in terms of x and $\dot{x}$. In the following, we denote this sum by S. Then, we rewrite (15) as

$$yS = \frac{x}{\sin \tilde{t}}(\cos \tilde{t} S - C) + \frac{x}{\sin \tilde{t}} C, \tag{21}$$

with y defined by (13). Using the solution (17) and (20), (21) yields

$$yS + \frac{1}{(x^2 + y^2)^m} = \frac{x}{\sin \tilde{t}} C. \tag{22}$$

Finally, squaring, and using (20) again, we obtain the first integral

$$C^2 = \frac{\left[1 + \sum_{r=0}^{m} A_{mr} y x^{2r} (x^2 + y^2)^{m-r}\right]^2}{(x^2 + y^2)^{2m+1}}. \tag{23}$$

It is readily verified that this reduces to (11) for $m = 0$, as it should.

We have used expression (23) to compute the phase space trajectories for the case $m = 1$, that are shown in Fig. 4. As expected, they are somewhat more complex than those of the $m = 0$ case. Consistently with Fig. 3, we now have closed orbits, close to the threshold (19), on which the acceleration vanishes at six different times. This happens for any finite m, since the points, with nonvanishing x, on which the acceleration vanishes lie on the curves defined by

$$\dot{x} = -\frac{1}{2m + 3}(x^{2m+2} + x^{-2m}). \tag{24}$$

These have maxima at $x = \pm[m/(m + 1)]^{1/(4m+2)}$, with the velocity decreasing on the two sides of each maximum, that

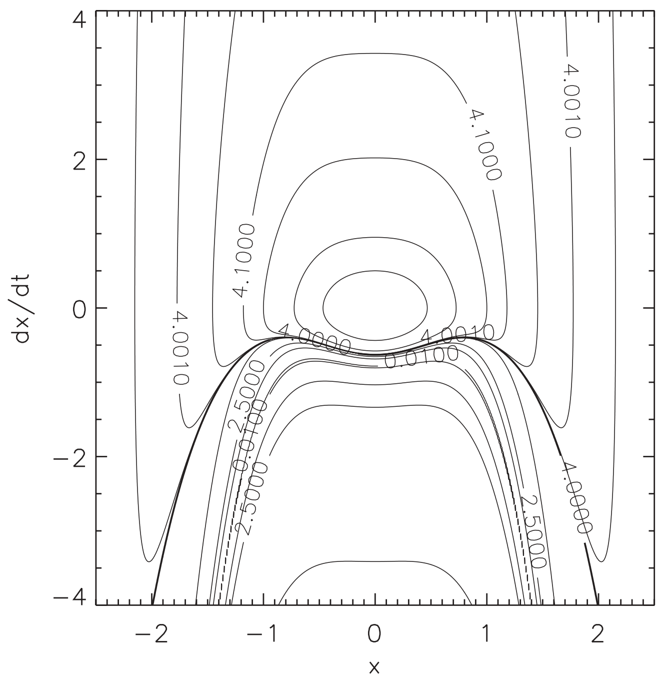

FIG. 4. Phase space of (4), with $m = 1$.

gives zeros of the acceleration for open and closed orbits, respectively.

Coming back to Fig. 4, we see that closed (open) orbits are obtained for $C^2 > 4$ ($C^2 < 4$), and are separated by the thick curve, on which C^2 equals 4. Two regions with singular orbits exist, separated by the curve corresponding to $C = 0$. The limiting x values for the periodic orbits are now given by

$$x = \pm 2^{-1/6} \frac{(C - \sqrt{C^2 - 4})^{1/6}}{(C^2 - 4)^{1/12}}, \tag{25}$$

which diverge— very slowly—when C^2 approaches 4.

The phase trajectories of the cases $m = 2$ and $m = 3$ (not shown) are similar, with the region of strong gradients more and more concentrated around $x = 0$. Thus, except for the differences highlighted, it can be said that the main qualitative features of the phase space of (4) with $m \neq 0$ look similar to those exhibited by the $m = 0$ case.

IV. CONCLUSION

In this Brief Report, we have studied the nonlinear oscillator (4), that was shown in [5] to admit isochronous orbits around the origin. We have derived the explicit solution of (4), and computed a first integral not dependent on time, that allows, in principle, a complete characterization of the phase space, for any m of interest. We have also pointed out the main features of the phase-space trajectories of the $m = 1$ case, and compared them with those of the $m = 0$ case, that had been previously discussed and solved in [4], even though it was not recognized as a member of (4).

The approach we have used can also be applied to the most general class of isochronous LE's found in [5] and [6], in which $g(x)$ is given by

$$g(x) = x + \frac{1}{x^3}\left(\int dx\, x f(x)\right)^2. \qquad (26)$$

In this case, the equivalent first order system is

$$\dot{x} = y - \frac{1}{x}\int dx\, x f, \qquad (27)$$

$$\dot{y} = -x - \frac{y}{x}\int dx\, x f. \qquad (28)$$

Clearly, (9) still holds, and use of (10) gives

$$\dot{x} = \frac{\cos\tilde{t}}{\sin\tilde{t}}x - \frac{1}{x}\int dx\, x f, \qquad (29)$$

which yields a time-dependent first integral for the problem, for a generic f. It is not clear whether there are other choices of $f(x)$, besides the one yielding (4), for which (29) can be integrated. Since there are so few solvable LE's, this seems a point worth of further examination.

The transformation properties of (4) should also be explored, since this could lead to the individuation of more general equations of the Liénard type admitting isochronous periodic solutions.

Finally, we note that (2) is the only LE with periodic solutions that can be linearized through a point transformation. This follows from the results of [11], where group theory was used to characterize the class of the LE's linearizable by a point transformation, and (2) was found to be the only member of this class that admits periodic solutions. Thus, the members of (4) with nonvanishing m cannot be linearized by a point transformation, even though the presence of isochronous orbits does imply linearizability around the center. Together with the fact that there is no bound to the amplitude of the periodic orbits, this indicates that the "obstruction" to linearization must be associated to properties of the singular orbits. Further analysis of this problem is under way.

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[7] Note that (8) implies the form of the basic ansatz used in [4], i.e., x proportional to $\dot{X}/X$.
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[9] The form of (16) suggests that the solution could also be obtained through a simple modification of the ansatz used in [4]; indeed, it is easily verified that the transformation $x^{2m+1} = (2m+1)^{-1}\dot{X}/X$ leads to a third-order ode for X that can be solved.
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