Exact traveling-wave and spatiotemporal soliton solutions to the generalized (3+1)-dimensional Schrödinger equation with polynomial nonlinearity of arbitrary order

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We obtain exact traveling wave and spatiotemporal soliton solutions to the generalized (3+1)-dimensional nonlinear Schrödinger equation with variable coefficients and polynomial Kerr nonlinearity of an arbitrarily high order. Exact solutions, given in terms of Jacobi elliptic functions, are presented for the special cases of cubic-quintic and septic models. We demonstrate that the widely used method for finding exact solutions in terms of Jacobi elliptic functions is not applicable to the nonlinear Schrödinger equation with saturable nonlinearity.

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I. INTRODUCTION

The generalized nonlinear Schrödinger equation (NLSE) is a generic model that appears in many fields of physics [1,2]. It is very important in NL optics, where it describes the full spatiotemporal optical solitons, or *light bullets* [3]. The *stable* exact soliton solutions to the NLSE with cubic nonlinearity in homogeneous media exist only in (1+1) dimensions [(1+1)D]. If the index of refraction is periodically modulated, then the (2+1)D soliton solutions are also found to be stable [4]. The stabilization of localized solutions by the management of dispersion and nonlinearity [5] has progressed far in recent years. Still, there are no known *exact* stable solitary solutions in higher dimensions.

The traveling wave and soliton solutions to the generalized NLSE in (3+1)D for third-order nonlinearity were developed in [6] for the anomalous dispersion and in [7] for the normal dispersion. In this paper we present *analytical* periodic traveling wave and soliton solutions for the polynomial Kerr nonlinearity of an arbitrarily high order. We still do not possess a formal proof of their stability; however, we found the solitons propagating in media with periodically modulated nonlinearities and dispersions to be stable in propagation over extended distances [6]. Criteria for the existence and stability of some analytical soliton solutions to the (1+1)D cubic-quintic NLSE have been discussed in [8]. A stability criterion for the dissipative multidimensional solitons in the complex cubic-quintic NLSE has been formulated in [9].

II. THE MODEL

Finding exact solutions to the NL-evolution partial differential equations (PDEs) is one of the essential tasks of NL mathematical physics. However, much of it is confined to (1+1)D and to constant coefficients in the equations. The objective here is to demonstrate traveling and solitary wave solutions to a multidimensional NLSE with variable coefficients that allow management. We are interested in the generalized NLSE in (3+1)D with distributed coefficients [10]:

$$i\partial_z u + \frac{\beta(z)}{2}\Delta u + \chi_1(z)|u|^2 u + \ldots + \chi_n(z)|u|^{2n}u = i\gamma(z)u, \tag{1}$$

which describes evolution of a slowly varying envelope $u(x,y,t;z)$ in a diffractive dispersive nonlinear medium, in the paraxial approximation. We separated the z variable in u from the rest, to distinguish the "marching" variable from the "transverse" variables x,y, and t. Hence, z is the propagation coordinate and $\Delta = \partial_x^2 + \partial_y^2 + \partial_t^2$ represents the generalized 3D transverse Laplacean, in which x and y are the transverse spatial coordinates, and t is the reduced time, i.e., the time in the frame of reference moving with the wave packet. All coordinates are made dimensionless by the choice of coefficients. The functions $\beta(z)$ and $\gamma(z)$ stand for the diffraction/dispersion and gain coefficients, respectively. Note that by choosing the positive sign of ∂_t^2 in the Laplacean, we choose the regime of anomalous dispersion. The functions $\chi_m(z)$ for $m = 1,2,\ldots,n$ stand for the nonlinearities of orders up to $2n+1$. For $n = 1$ one has the simple Kerr nonlinearity, for $n = 2$ the cubic-quintic, for $n = 3$ the septic, and so on.

The motivation to look for exact solutions of the generalized NLSE with high-order Kerr nonlinearity, *inter alia* comes from the fact that such a nonlinearity is an excellent approximation to the saturable nonlinearity $1/(1+sI) \approx 1 - sI + (sI)^2 - (sI)^3 + \ldots$, where $I = |u|^2$ is the beam intensity and s is the saturation parameter. The NLSE with saturable nonlinearity is an important generic model, for which unfortunately there are no known analytical solutions. In fact—and this is one of the major contributions of this paper—we establish here that it is *not* possible to obtain exact solutions to the NLSE with saturable nonlinearity by the F-expansion and balance principle method, or similar expansion methods [11]. Here "F" refers to the expansion functions, which in our case are Jacobi elliptic functions. A number of such methods have been applied to NLSE; of relevance to this work are the NLSE with power-law nonlinearity [12], time-dependent coefficients [13], and non-Kerr media [14].

III. THE METHOD OF SOLUTION

Following the standard procedure of the F-expansion and the balance principle [6,7], we write the complex field u of Eq. (1) in terms of its amplitude and phase:

$$u(x,y,t;z) = A(x,y,t;z)\exp[iB(x,y,t;z)]. \tag{2}$$

Substituting u into Eq. (1), the following coupled equations are obtained:

$$\partial_z A + \tfrac{1}{2}\beta[2\partial_x A\partial_x B + 2\partial_y A\partial_y B + 2\partial_t A\partial_t B + A\Delta B] = \gamma A, \quad (3)$$

$$-A\partial_z B + \tfrac{1}{2}\beta[\Delta A - A(\partial_x B)^2 - A(\partial_y B)^2 - A(\partial_t B)^2] + \chi_1 A^3 + \cdots + \chi_n A^{2n+1} = 0. \quad (4)$$

We apply the balance principle [15,16] and the F-expansion technique [17,18] as developed in [19], with modifications to account for the higher order nonlinearities. We seek the traveling wave solutions to Eqs. (3) and (4), and assume the A and B functions to be of the form

$$A = f_1(z)F^{\frac{1}{n}}(\theta) + f_2(z)F^{-\frac{1}{n}}(\theta), \quad (5)$$

$$\theta = k(z)x + l(z)y - \Omega(z)t + \phi(z), \quad (6)$$

$$B = a(z)(x^2 + y^2 + t^2) + b(z)(x + y + t) + e(z), \quad (7)$$

where f_1, f_2, k, l, Ω, ϕ, a, b, e are parameter functions to be determined, and F is a Jacobi elliptic function (JEF). These solutions resemble the solutions developed in [6], except for the power of the function F. Seemingly minor, the change in the power nonetheless is crucial, allowing for the establishment of new solutions. The power has to be such that the highest-order term from Laplacean matches the highest-order nonlinearity.

We substitute Eqs. (5)–(7) into Eqs. (3) and (4) and require that $x^j F^{\frac{2p-1}{n}}$, $y^j F^{\frac{2p-1}{n}}$, $t^j F^{\frac{2p-1}{n}}$, ($j = 0,1,2$, $p = -n,\ldots,n+1$), and $\sqrt{c_0 + c_2F^2 + c_4F^4}$ of each term be separately equal to zero. This is a rather formidable computational task, accomplished by the use of symbolic numerical packages. The real constants c_0, c_2, and c_4 are coefficients in the NL Jacobi elliptic *ordinary* differential equation

$$\left(\frac{dF}{d\theta}\right)^2 = c_0 + c_2F^2 + c_4F^4 \quad (8)$$

and are naturally related to the elliptic modulus M of JEFs [6]. After multiplying the expressions and factoring out common factors of f_1 and f_2, a system of first-order ordinary differential equations is obtained for the parameter functions:

$$\frac{df_j}{dz} + 3a\beta f_j - \gamma f_j = 0, \quad (9)$$

$$\frac{dk}{dz} + 2ka\beta = 0, \quad (10)$$

$$\frac{dl}{dz} + 2la\beta = 0, \quad (11)$$

$$\frac{d\Omega}{dz} + 2\Omega a\beta = 0, \quad (12)$$

$$\frac{d\phi}{dz} + \beta(k + l - \Omega)b = 0, \quad (13)$$

$$\frac{db}{dz} + 2\beta ab = 0, \quad (14)$$

$$\frac{da}{dz} + 2\beta a^2 = 0, \quad (15)$$

$$-\frac{de}{dz} - \frac{3}{2}\beta b^2 + qc_2 + \sum_{i=1}^{n}\overline{\chi_i}\binom{2i+1}{i+1} = 0, \quad (16)$$

where $j = 1,2$, $q = \frac{\beta(k^2+l^2+\Omega^2)}{2n^2}$, and by definition $\overline{\chi_m} = \chi_m f_1^m f_2^m$ $(m = 1,\ldots,n)$. A number of algebraic relations involving $\overline{\chi_m}$ are also obtained:

$$\sum_{i=1}^{n}\overline{\chi_i}\binom{2i+1}{i+p} = 0, \quad (17)$$

$$\overline{\chi_{n-1}} + (2n+1)\overline{\chi_n} - (n-1)qw = 0, \quad (18)$$

$$\overline{\chi_n} + (n+1)qw = 0, \quad (19)$$

where $w = c_0(\frac{f_1}{f_2})^n = c_4(\frac{f_2}{f_1})^n$ and $p = 2,\ldots,n-1$. Indeed, Eqs. (16), (18), and (19) are obtained from the terms next to $F^{\frac{1}{n}}$, $F^{2-\frac{1}{n}}$, and $F^{2+\frac{1}{n}}$, respectively. Note that Eq. (18) appears only for $n > 1$ and Eq. (17) only for $n > 2$. The binomial coefficient $\binom{2i+1}{i+p}$ is defined to be 0 for $i + p > 2i + 1$.

Equations (9)–(19) resemble a similar system of equations obtained in [6], the major difference being that the equations for $e(z)$ and $\chi_n(z)$ have to be generalized. By solving these equations self-consistently, one obtains a set of conditions on the coefficients and parameters necessary for Eq. (1) to have exact traveling wave solutions. We consider the most generic case, in which $\beta(z)$ and $\gamma(z)$ are arbitrary.

IV. RESULTS

We first solve Eqs. (9)–(15), to obtain expressions for f_1, f_2, k, l, Ω, ϕ, b, and a. Note that the equations for all these parameter functions depend explicitly or implicitly on a, while the equation for a, Eq. (15), depends only on the coefficient β. This testifies about the importance of the function a, which is known as the *chirp* function. Hence, one has to first solve Eq. (15) for a, and then find the rest of parameter functions. From the condition on w it follows that $f_2 = \epsilon f_1 \sqrt[2n]{\frac{c_0}{c_4}}$ and so $w = \epsilon^n\sqrt{c_0c_4}$, where $\epsilon = \pm 1$. We then proceed to solve for $\overline{\chi_m}$ recurrently, starting from $m = n$ and ending at $m = 1$. We easily obtain $\overline{\chi_m} = r_m qw$ and $\sum_{i=1}^{n}\overline{\chi_i}\binom{2i+1}{i+1} = rqw$, where the r parameters $r, r_1, \ldots, r_n$ are integer functions of n. Although it is difficult to find generic formulas for $r, r_1, \ldots, r_n$, in principle it is easy to calculate these parameters recurrently for any given n:

$$r_n = -(n+1), \quad (20)$$

$$r_{n-1} = (n-1) - (2n+1)r_n = 2n(n+2), \quad (21)$$

$$r_m = -\sum_{i'=m+1}^{n} r_{i'}\binom{2i'+1}{m+i'+1}, m = 1,\ldots,n-2, \quad (22)$$

$$r = \sum_{i=1}^{n} r_i\binom{2i+1}{i+1}. \quad (23)$$

In the end, the following set of exact solutions is found:

$$f_1 = (\alpha)^{3/2} f_0 e^{\int_0^z \gamma dz}, \qquad f_2 = \epsilon\sqrt[2n]{\frac{c_0}{c_4}} f_1; \quad (24)$$

$$k = \alpha k_0, \quad l = \alpha l_0, \quad \Omega = \alpha\Omega_0; \quad (25)$$

$$\phi = \phi_0 - \alpha(k_0 + l_0 - \Omega_0)b_0\int_0^z \beta dz; \quad (26)$$

$$a = \alpha a_0, \qquad b = \alpha b_0; \quad (27)$$

$$e = e_0 + \alpha\left[\frac{(k_0^2 + l_0^2 + \Omega_0^2)}{2n^2}(c_2 + r\epsilon^n\sqrt{c_0c_4}) - \frac{3b_0^2}{2}\right] \times \int_0^z \beta dz; \quad (28)$$

where $\alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1}$ is the normalized chirp function. The subscript 0 denotes the value of the given function at $z = 0$ and $f_0 = f_{10}$. The final form of the solution is thus

$$u = (\alpha)^{3/2} f_0 \exp\left(\int_0^z \gamma dz\right)\left[F^{1/n}(\theta) + \epsilon \sqrt[2n]{\frac{c_0}{c_4}}\frac{1}{F^{1/n}(\theta)}\right] \times \exp i[a(x^2 + y^2 + t^2) + b(x + y + t) + e], \quad (29)$$

where $\theta = \phi_0 + kx + ly - \Omega t - (k + l - \Omega)b_0 \int_0^z \beta dz$.

There are, however, a number of restrictions that have to be observed in order for the solutions to be valid. Since A has to be real, for even n we must have $F > 0$ at all times, so that $F^{1/n}$ is real. This restricts the range of allowed solutions in terms of JEFs. Another restriction involves the nonlinearity coefficients, which by the solution procedure are found related to β and γ:

$$\chi_m = \frac{\epsilon^{n+m} r_m \beta \alpha^{2-3m}}{2n^2 f_0^{2m}} \sqrt[2n]{c_4^{n+m} c_0^{n-m}} \exp\left(-2m \int_0^z \gamma dz\right), \quad (30)$$

where $m = 1,2,\ldots,n$. This relation should be understood as an integrability condition on Eq. (1) for finding solutions by the present method. Note that the nonlinearity coefficients χ_m are directly proportional to the r_m parameters, while the only parameter to explicitly appear in the solutions is r in Eq. (28).

We consider separately the case $f_2 = 0$. In this case the solutions are given in terms of single JEFs. We have one negative exponent of F in the term ΔA of Eq. (3), namely $(n-1)f_1 c_0 F^{-2+\frac{1}{n}}$. However, all other terms in Eq. (3) will be with a positive degree of F; hence for $n \neq 1$ we must have $c_0 = 0$. One finds $\chi_n = -q(n+1)c_4 f_1^{-2n}$ and all other $\chi_m = 0$ $(m = 1,\ldots,n-1)$; Eq. (1) then contains only the highest-order nonlinearity term. The correct solutions are still obtained from Eqs. (24)–(28), provided one takes $\epsilon = 0$. The case $n = 1$ was covered in [6]; the expression for χ there remains unchanged, c_0 need not be 0, and the correct expressions are again contained in Eqs. (24)–(28), as long as one takes $\epsilon = 0$.

All of these restrictions do not bode well for the application of the present solution method to the saturable Kerr-like nonlinearity, which was one of our original aims. Namely, one can understand the NLSE with saturable nonlinearity as the limiting case of the present model, in which all χ_m are expressed in terms of powers of the saturation parameter s and $n \to \infty$. However, as n increases from the solution procedure it is clear that for each new n one has to find new expressions for r and r_m coefficients, and correspondingly new expressions for χ_m, which cannot be presented as simple power functions of one variable s. There exist no limiting values for the r, r_m coefficients and no unified procedure that treats all values of n on the same footing. For the time being, the all-important model with saturable nonlinearity remains nonintegrable [20]. However, there exist exact solutions to the saturable *discrete* NLSE, also given in terms of JEFs [21].

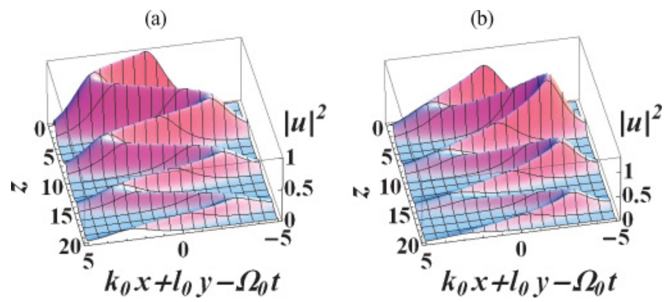

FIG. 1. (Color online) Soliton solutions for the cubic-quintic model $n = 2$ as a function of the propagation distance, for (a) $a_0 = 0$ (without chirp) and (b) $a_0 = 0.1$ (with chirp) for the $F = \text{sech}$ solution. Intensity $|u|^2$ is presented as a function of $k_0x + l_0y - \Omega_0t$ and z. Coefficients: $\beta(z) = \cos(z)$, $\gamma(z) = \gamma_0 = -0.05$, $M = 1$, $b_0 = 1$, $e_0 = 0$, $\epsilon = 0$, $k_0 = l_0 = -\Omega_0 = 1$, and $\phi_0 = 0$.

A way out, not only in the treatment of saturable models but in the solution of other multidimensional NL PDEs, is to consider generalizations of the F expansion and balance principle method, based on the generalized auxiliary *elliptic* equation

$$\left(\frac{dF}{d\theta}\right)^2 = \sum_{i=0}^{N} c_i F^i, \qquad c_N \neq 0. \quad (31)$$

Here the power N can vary, but most of the accounts in literature deal with the choice $N = 4$ [22]. A number of cases with $N = 6$ have been listed in [23], but many of the c_i coefficients are then equal to 0. The case $c_1 = c_3 = 0$ leads to JEFs and the rational forms of JEFs [24,25] as solutions; the case $c_2 = c_4 = 0$, with the notation $c_3 = 4$, $c_1 = -g_2$, and $c_0 = -g_3$, leads to the Weierstrass elliptic functions $p(\theta; g_2, g_3)$ as solutions [26]. Still, as exemplified in Ref. [22], most of the solutions of Eq. (31), including Weierstrass's elliptic function, can be expressed in terms of JEFs and their rational forms. When multiple JEFs are encountered as solutions of evolution PDEs, the model and the solution procedure are conveniently handled in terms of the projective Riccati equations [27,28].

We now apply the results obtained here to a few specific cases. For $n = 1$ we have $r_1 = -2$ and $r = -6$. Hence, Eqs. (24)–(28) reduce to the corresponding equations obtained in [6].

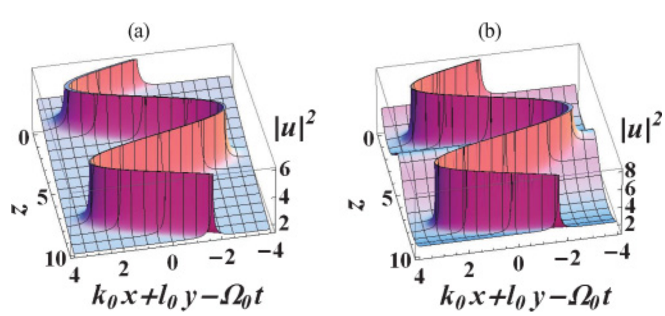

FIG. 2. (Color online) Soliton solutions for the septic model as functions of the propagation distance. The setup and parameters are the same as in Fig. 1, except for $n = 3$, $F = \tanh$, and $\epsilon = 1$.

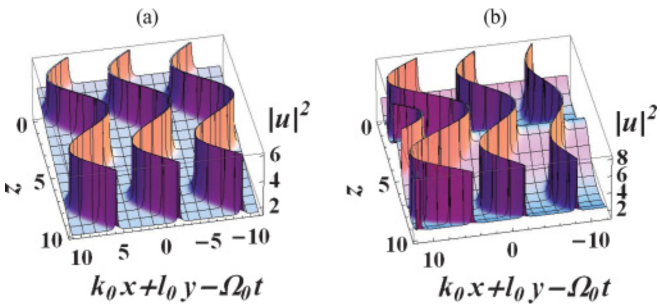

FIG. 3. (Color online) Periodic traveling wave solutions for the septic model as functions of the propagation distance. The setup and parameters are the same as in Fig. 2, except for $M = 0.99$ and $F = \text{sin}$.

For $n = 2$, corresponding to the cubic-quintic model, we obtain $r_1 = 16, r_2 = -3$, and $r = 18$. The only soliton solution found so far which satisfies both $F > 0$ and $c_0 = 0$ for $f_2 = 0$ is the bright soliton solution with $M = 1$ and $F = \text{sech}$. The solution is presented in Fig. 1.

For $n = 3$, corresponding to the septic model, we have $r_1 = -66$, $r_2 = 30$, $r_3 = -4$, and $r = -38$. We obtain the bright soliton solution for $f_2 = 0$, which looks very much like the solution seen in Fig. 1. The only noticeable difference is the transverse stretching of the wave. This is due to the fact that the function $F^{\frac{1}{n}}$ falls less rapidly as the argument decreases for larger n. For the septic model we also obtain solutions for $\epsilon = 1$. These correspond to the dark solitons, with $F = \tanh$. Since $c_0 = 0$ is no longer required, one can find both solitary ($M = 1$) and periodic ($M < 1$) traveling wave solutions. These solutions are shown in Figs. 2 and 3.

V. CONCLUSION

In conclusion, we have solved analytically the (3+1)D generalized nonlinear Schrödinger equation with distributed diffraction, dispersion, and gain, and with polynomial nonlinearity of an arbitrarily high order. A number of exact traveling wave and spatiotemporal soliton solutions are found. We established that the F-expansion and balance principle method cannot provide traveling wave and solitary solutions to the NLSE with saturable nonlinearity.

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