Fresnel integrals and irreversible energy transfer in an oscillatory system with time-dependent parameters

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We demonstrate that in significant limiting cases the problem of irreversible energy transfer in an oscillatory system with time-dependent parameters can be efficiently solved in terms of the Fresnel integrals. For definiteness, we consider a system of two weakly coupled linear oscillators in which the first oscillator with constant parameters is excited by an initial impulse, whereas the coupled oscillator with a slowly varying frequency is initially at rest but then acts as an energy trap. We show that the evolution equations of the slow passage through resonance are identical to the equations of the Landau-Zener tunneling problem, and therefore, the suggested asymptotic solution of the classical problem provides a simple analytic description of the quantum Landau-Zener tunneling with arbitrary initial conditions over a finite time interval. A correctness of approximations is confirmed by numerical simulations.

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I. INTRODUCTION

The problem of energy transfer is currently a topic of intense research with a broad spectrum of applications, from multibody systems [1–3] and waves in fluids and plasmas [4,5] to semiconductors [6,7] and nanocrystals with graphene layers [8], among other applications. A rich variety of examples in the diverse fields of applied mathematics, natural sciences, and engineering can be found in [1]. However, most of the results reported in the literature are related to energy exchange in systems with constant parameters. This work develops an analytical framework to investigate the dynamics of two weakly coupled oscillators with time-varying frequencies, with special attention to an analogy between the energy transfer in this classical oscillatory system and the quantum Landau-Zener tunneling.

The classic linear Landau-Zener problem [9–12] deals with a two-level system described by a Hermitian Hamiltonian depending linearly on time. Over last decades, growing attention has also been paid to non-Hermitian extensions of the classic theory, taking into account the effect of environment on two-level systems (e.g., [13–15] and references therein). Due to its generality, the Landau-Zener scenario has been applied to numerous problems in various contexts, such as laser physics [16], semiconductor superlattices [17], tunneling of optical [18] or acoustic [19,20] waves, and quantum information processing [21], to name just a few examples. Although a passage between two energy levels is an intrinsic feature of all above-mentioned processes, the demonstration of a direct connection between energy transfer in a classical oscillatory system with time-dependent parameters and nonadiabatic quantum Landau-Zener tunneling is a recent development. As shown in [22,23], the equations of the slow passage through resonance in a system of two weakly coupled pendulums with a time-dependent frequency are asymptotically identical to the equations of the Landau-Zener tunneling problem, i.e., there exists a profound analogy between irreversible energy transfer in the oscillatory system and *nonadiabatic* quantum tunneling. This conclusion may be treated as an extension of the previously found analogy between *adiabatic* quantum tunneling and energy exchange in a chain of weakly coupled oscillators with constant parameters [24,25].

While an exact solution to the Landau-Zener equation is well known [10], it is actually too complicated for any straightforward inferences about the system dynamics, and following the seminal Landau paper [9], attention has been focused on quasistationary solutions at infinitely large times (see, e.g., [26]). Recently, transient nonadiabatic tunneling has been studied asymptotically assuming quasistationary behavior of the system [27,28]. The purpose of the present paper is to formulate a rigorous asymptotic approach for studying the transient processes. We show that in some significant limiting cases, the second-order Landau-Zener equation can be approximately reduced to a first-order equation with a solution in the form of the Fresnel oscillations. The suggested approach, in addition to providing a simple explicit description of energy transfer in the oscillatory system, allows a straightforward extension to more complicated systems with a large number of degrees of freedom. Furthermore, it gives a simple and correct prediction of the transient Landau-Zener tunneling with arbitrary initial conditions over a finite time interval.

The paper is organized as follows. In Sec. II, we describe a model of two weakly coupled oscillators with the time-dependent frequency detuning. We transform the system of two differential equations into a single integro-differential equation for the coupled oscillator (the energy trap) and derive the evolutionary equations describing the slowly varying envelopes of near-resonance motion for both oscillators. We demonstrate that the second-order equation for the slow envelope of the trap oscillations is identical to that of the Landau-Zener problem. In Sec. III, we show that the latter equation can be reduced to the first-order equation in two special cases; in the first case, the mass of the excited oscillator far exceeds the mass of the coupled trap; in the second case, the coefficients of weak coupling are less than the detuning rate. In both cases, we find an explicit asymptotic solution in the form of the Fresnel integrals and then illustrate the theory by numerical simulations. In Sec. IV we compare the dynamics of the systems with the linear-in-time and quadratic-in-time frequency detuning. Auxiliary results are presented in Appendixes. In Appendix A, we discuss a connection of irreversible energy transfer in the oscillatory system with the Landau-Zener tunneling; in addition, we prove the convergence of the exact solution to the Fresnel oscillations in some limiting cases. In order to illustrate a correctness of the approximate solutions, in Appendix B we consider a system with a constant detuning possessing a simple exact solution.

II. MODEL AND MAIN EQUATIONS

A. Equations of motion

In this paper we investigate resonant energy transfer in a system of two weakly coupled linear oscillators. We suppose that the first oscillator of mass m_1 and stiffness c_1 is excited by an initial impulse V; the coupled oscillator of mass m_2 and time-dependent stiffness $c_2(t)$ is initially at rest; the oscillators are connected by linear coupling of stiffness c_{12}. The displacements and velocities of the oscillators are denoted by u_i and $V_i = du_i/dt,\ \ i = 1,2$. We will demonstrate that the second oscillator with a time-dependent frequency acts as an energy trap and ensures a visible reduction of oscillations of the excited mass.

In this notation, the kinetic energy T, the potential energy Π, and the total energy E of the system are written as

$$\begin{gathered} T = 1/2\big(m_1V_1^2 + m_2V_2^2\big),\\ \Pi = 1/2\big[c_1u_1^2 + c_2(t)u_2^2 + c_{12}(u_2 - u_1)^2\big],\\ E = T + \Pi. \end{gathered}$$

The dynamics of the system is described by

$$\begin{aligned} m_1\frac{d^2u_1}{dt^2} + c_1u_1 + c_{12}(u_1 - u_2) = 0,\\ m_2\frac{d^2u_2}{dt^2} + c_2\,(t)\,u_2 + c_{12}(u_2 - u_1) = 0, \end{aligned} \tag{2.1}$$

with the initial conditions $t = 0, u_1 = u_2 = 0;\ \ V_1 = V, V_2 = 0, i = 1,2$. We note that the initial conditions $u_2 = 0,\ V_2 = 0$ determine a so-called limiting phase trajectory corresponding to motion with a maximal possible energy transfer from the first to the second oscillator [29].

In this section we define the time-dependent stiffness as

$$c_2(t) = c_2 - (k_1 - k_2t), \quad k_{1,2} > 0.$$

Quasiresonance interactions between the oscillators imply that $(c_1/m_1)^{1/2} = (c_2/m_2)^{1/2} = \omega$; a likely small detuning may be included in the coefficient k_1. Assuming weak coupling and slowly varying detuning, we define the small parameter of the problem as $c_{12}/c_2 = 2\varepsilon \ll 1$. Then we introduce the dimensionless parameters,

$$\begin{gathered} c_{12}/c_r = 2\varepsilon\lambda_r, \quad r = 1,2; \quad \lambda_2 = 1; \quad k_1/c_2 = 2\varepsilon\sigma,\\ k_2/(c_2\omega) = 2\varepsilon^2\beta^2, \end{gathered} \tag{2.2}$$

and the dimensionless time scales $\tau_0 = \omega t, \tau_1 = \varepsilon\tau_0$. Now Eq. (2.1) can be rewritten in the dimensionless form

$$\begin{aligned} \frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) = 0,\\ \frac{d^2u_2}{d\tau_0^2} + u_2 + 2\varepsilon\lambda_2(u_2 - u_1) - 2\varepsilon\zeta(\tau_1)u_2 = 0, \end{aligned} \tag{2.3}$$

with the initial conditions $\tau_0 = 0, u_1 = u_2 = 0;\ v_1 = V/\omega = V_0, v_2 = 0, v_i du_i/d\tau_0$. The coefficient

$$\zeta(\tau_1) = \sigma - 2\beta^2\tau_1 \tag{2.4}$$

defines the detuning modulation; the coefficient β^2 characterizes the rate of the resonance crossing. It is necessary to note that system (2.3) can be considered as resonant only in a time interval wherein $|\zeta(\tau_1)| \sim 1$ in this interval the value of $\varepsilon\zeta(\tau_1)$ is small, and instant frequencies of the system remain close to ω.

In order to develop an effective asymptotic procedure, we express the solution u_1 of the first equation in (2.3) as

$$\begin{aligned} u_1(\tau_0) = {}& \omega_\varepsilon^{-1}V_0\sin\omega_\varepsilon\tau_0\\ &+ 2\varepsilon\omega_\varepsilon^{-1}\lambda_1\int_0^{\tau_0}u_2(s)\sin\omega_\varepsilon(\tau_0 - s)\,ds, \end{aligned} \tag{2.5}$$

where $\omega_\varepsilon = (1 + 2\varepsilon\lambda_1)^{1/2}$. Then we substitute (2.5) into the second equation of (2.3) to obtain the following system:

$$\begin{aligned} &\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) = 0,\\ &\frac{d^2u_2}{d\tau_0^2} + (1 + 2\varepsilon\lambda_2)u_2 - 2\varepsilon\zeta(\tau_1)u_2 = 2\varepsilon\omega_\varepsilon^{-1}\lambda_2V_0\sin\omega_\varepsilon\tau_0\\ &\quad + 4\varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\int_0^{\tau_0}u_2(s)\sin\omega_\varepsilon(\tau_0 - s)\,ds, \end{aligned} \tag{2.6}$$

with the initial conditions $\tau_0 = 0: u_1 = u_2 = 0;\ v_1 = V_0, v_2 = 0$. Hence, instead of two coupled second-order equations (2.3), we consider a single integro-differential equation for u_2. The variable u_1 is then calculated by (2.5).

B. Complex envelopes

The asymptotic analysis of Eq. (2.6) is performed with help of the so-called complexification-averaging technique based on the complexification of the dynamics and the separation of the fast and slow time scales [30]. Following [30], we introduce a pair of the complex-valued variables ψ and ψ^*:

$$\begin{gathered} \psi = v_2 + iu_2, \quad \psi^* = v_2 - iu_2,\\ u_2 = -\tfrac{i}{2}(\psi - \psi^*), \quad v_2 = \tfrac{1}{2}(\psi + \psi^*). \end{gathered} \tag{2.7}$$

Substituting (2.7) into (2.6), we derive the following equation for the variable $\psi(\tau_0,\varepsilon)$:

$$\begin{aligned} &\frac{d\psi}{d\tau_0} - i\psi - i\varepsilon[\lambda_2 - \zeta(\tau_1)](\psi - \psi^*)\\ &= 2\varepsilon\lambda_2V_0\omega_\varepsilon^{-1}\sin\omega_\varepsilon\tau_0 - 2i\varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\\ &\quad\times\int_0^{\tau_0}[\psi(s,\varepsilon) - \psi^*(s,\varepsilon)]\sin\omega_\varepsilon(\tau_0 - s)ds, \quad \psi(0) = 0. \end{aligned} \tag{2.8}$$

Using the same arguments as in [3], we can show that in the resonance case,

$$\int_0^{\tau_0} [\psi(s) - \psi^*(s)] \sin \omega_\varepsilon(\tau_0 - s)\, ds = o(\varepsilon^{-1}), \quad (2.9)$$

and therefore, the integral term on the right-hand side of (2.8) should be included in the first-order equation.

In order to separate resonance harmonics, we present the solution of (2.8) as

$$\psi(\tau_0,\varepsilon) = \varphi(\tau_0,\varepsilon)\, e^{i\omega_\varepsilon \tau_0}. \quad (2.10)$$

Using (2.8) and (2.10) to derive the equation for the complex amplitude φ, we obtain

$$\begin{aligned}&\frac{d\varphi}{d\tau_0} - i\varepsilon(\rho + 2\beta^2\tau_1)(\varphi - \varphi^* e^{-2i\omega_\varepsilon \tau_0}) + i\varepsilon\lambda_2\varphi^* e^{-2\omega_\varepsilon \tau_0}\\ &= -i\varepsilon\lambda_2 V_0 \omega_\varepsilon^{-1}(1 - e^{-2i\omega_\varepsilon \tau_0})\\ &\quad - \varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\left[\int_0^{\tau_0} \varphi(s,\varepsilon)(1 - e^{-2i\omega_\varepsilon(\tau_0 - s)})\, ds\right.\\ &\quad \left. + e^{-2i\omega_\varepsilon \tau_0}\int_0^{\tau_0} \varphi^*(s,\varepsilon)(1 - e^{-2i\omega_\varepsilon(\tau_0 - s)}) ds\right], \quad (2.11)\end{aligned}$$

where $\rho = \lambda_2 - \lambda_1 - \sigma$. We construct a solution of (2.11) in the form of the multiple-scales expansion [30,31]

$$\begin{aligned}\varphi(\tau_0,\varepsilon) &= \varphi_0(\tau_1) + \varepsilon\varphi_1(\tau_0,\tau_1) + \cdots\\ \frac{d\varphi}{d\tau_0} &= \frac{\partial\varphi}{\partial\tau_0} + \varepsilon\frac{\partial\varphi}{\partial\tau_1} + \cdots.\end{aligned} \quad (2.12)$$

Expansion (2.12) provides an accurate approximation of the exact solution with an error of $O(\varepsilon)$ for τ_1 of $O(\varepsilon)$ [31].

Substituting (2.12) into (2.11) and proceeding to the first-order approximation, we obtain

$$\begin{aligned}&\frac{\partial\varphi_0}{\partial\tau_1} + \frac{\partial\varphi_1}{\partial\tau_0} - i(\rho + 2\beta^2\tau_1)(\varphi_0 - \varphi_0^* e^{-2i\omega_\varepsilon \tau_0}) + i\lambda_2\varphi_0^* e^{-2i\omega_\varepsilon \tau_0}\\ &= -i\lambda_2 V_0(1 - e^{-2i\omega_\varepsilon \tau_0})\\ &\quad - \varepsilon\lambda_1\lambda_2\int_0^{\tau_0} [\varphi_0(\varepsilon s)(1 - e^{-2i\omega_\varepsilon(\tau_0 - s)})\\ &\quad + \varphi^*(\varepsilon s) e^{-2i\omega_\varepsilon \tau_0}(1 - s^{2i\omega_\varepsilon(\tau_0 - s)})]\, ds. \quad (2.13)\end{aligned}$$

In order to avoid the secular growth of φ_1 with respect to the fast time variable, i.e., to avoid a response not uniformly valid with increasing time, we need to eliminate nonoscillating terms from (2.13). This yields the following equation determining the leading-order term $\varphi_0(\tau_1)$:

$$\begin{aligned}&\frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\varphi_0\\ &= -i\lambda_2 V_0 - \lambda_1\lambda_2\int_0^{\tau_1} \varphi_0(r) dr, \quad \varphi_0(0) = 0. \quad (2.14)\end{aligned}$$

Equation (2.14) is equivalent to the second-order differential equation

$$\frac{d^2\varphi_0}{d\tau_1^2} - i(\rho + 2\beta^2\tau_1)\frac{d\varphi_0}{d\tau_1} + (\lambda_1\lambda_2 - i\beta^2)\ \varphi_0 = 0, \quad (2.15)$$

with the initial conditions $\tau_1 = 0: \varphi_0 = 0,\ d\varphi_0/d\tau_1 = -i\lambda_2 V_0$. The equivalence of Eq. (2.15) and the equation of the Landau-Zener transient tunneling problem is demonstrated in Appendix A.

Once the slow envelope $\varphi_0(\tau_1)$ is found, the leading-order approximations for u_2 and v_2 can be derived from (2.7) and (2.10). We obtain

$$\begin{aligned}u_{20}(\tau_0,\tau_1) &= -\tfrac{i}{2}[\varphi_0(\tau_1)e^{i\omega_\varepsilon \tau_0} - \varphi_0^*(\tau_1)e^{-i\omega_\varepsilon \tau_0}]\\ &= |\varphi_0(\tau_1)| \sin[\omega_\varepsilon \tau_0 + \alpha(\tau_1)],\\ v_{20}(\tau_0,\tau_1) &= \tfrac{1}{2}[\varphi_0(\tau_1)e^{i\omega_\varepsilon \tau_0} + \varphi_0^*(\tau_1)e^{-i\omega_\varepsilon \tau_0}]\\ &= |\varphi_0(\tau_1)| \cos[\omega_\varepsilon \tau_0 + \alpha(\tau_1)],\\ \alpha(\tau_1) &= \arg \varphi_0(\tau_1).\end{aligned} \quad (2.16)$$

Partial energy of the second oscillator is expressed as

$$e_{20}(\tau_1) = \tfrac{1}{2}\left(\langle u_{20}^2\rangle + \langle v_{20}^2\rangle\right) = \tfrac{1}{2}|\varphi_0(\tau_1)|^2, \quad (2.17)$$

where $\langle\cdot\rangle$ denotes the averaging over the "fast" period $T = 2\pi/\omega_\varepsilon$. It follows from (2.14) and (2.17) that for small τ_1

$$\varphi_0(\tau_1) = -i\lambda_2 V_0\tau_1, \quad e_{20}(\tau_1) = \tfrac{1}{2}(\lambda_2 V_0\tau_1)^2. \quad (2.18)$$

C. Calculation of u_1

Once u_{20} is determined, u_1 can be directly found from (2.5). However, in order to demonstrate an analogy between the dynamical model (2.5) and (2.6) and the Landau-Zener equations, we approximately calculate u_1 from

$$\begin{aligned}&\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1 u_1 = 2\varepsilon\lambda_1 u_{20},\\ &\tau_0 = 0: \quad u_1 = 0, \quad v_1 = V_0.\end{aligned} \quad (2.19)$$

By analogy with (2.7), we introduce the change of variables $y = v_1 + iu_1, y^* = v_1 - iu_1$ and then derive the following equation for the complex envelope y:

$$\begin{aligned}&\frac{dy}{d\tau_0} - i\omega_\varepsilon y + i\omega\lambda_1 y^* = -i\varepsilon\lambda_1[\varphi_0(\tau_1)e^{i\omega_\varepsilon \tau_0} - \varphi_0^*(\tau_1)e^{-i\omega_\varepsilon \tau_0}],\\ &y(0) = V_0.\end{aligned} \quad (2.20)$$

The substitution of

$$y(\tau_0,\varepsilon) = \eta(\tau_0,\varepsilon)e^{i\omega_\varepsilon \tau_0}. \quad (2.21)$$

into (2.20) yields

$$\begin{aligned}&\frac{d\eta}{d\tau_0} + i\varepsilon\lambda_1\eta^* e^{-2i\omega_\varepsilon \tau_0} = -i\varepsilon\lambda_1[\varphi_0(\tau_1) - \varphi_0^*(\tau_1)e^{-2i\omega_\varepsilon \tau_0}],\\ &\eta(0) = V_0.\end{aligned} \quad (2.22)$$

As above, we construct an approximate solution of Eq. (2.22) in the form of the multiple-scale expansion $\eta(\tau_0,\varepsilon) = \eta_0(\tau_1) + \varepsilon\eta_1(\tau_0,\tau_1) + \cdots$. As in the previous paragraph, we derive the following equation of the first-order approximation:

$$\begin{aligned}&\frac{\partial\eta_0}{\partial\tau_1} + \frac{\partial\eta_0}{\partial\tau_1} + i\lambda_1\eta_0^*(\tau_1)e^{-2i\omega_\varepsilon \tau_0}\\ &= -i\lambda_1[\varphi_0(\tau_1) - \varphi_0^*(\tau_1)e^{-2i\omega_\varepsilon \tau_0}], \quad \eta_0(0) = V_0. \quad (2.23)\end{aligned}$$

After eliminating nonoscillating terms from (2.23), the resulting system for the variables η_0,φ_0 becomes

$$\frac{d\eta_0}{d\tau_1} = -i\lambda_1\varphi_0(\tau_1), \quad \eta_0(0) = V_0,$$
$$\frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\varphi_0 = -i\lambda_2 V_0 - \lambda_1\lambda_2\int_0^{\tau_1}\varphi_{0(r)}dr, \tag{2.24}$$
$$\varphi_0(0) = 0.$$

It is easy to deduce that the main approximations of the solution u_1,v_1 takes the form

$$\begin{aligned} u_{10}(\tau_0,\tau_1) &= -\tfrac{i}{2}(y_0 - y_0^*) = -\tfrac{i}{2}(\eta_0 e^{i\omega_\varepsilon\tau_0} - \eta_0^* e^{-i\omega_\varepsilon\tau_0}) \\ &= |\eta_0(\tau_1)|\sin[\omega_\varepsilon\tau_0 + \delta(\tau_1)], \\ v_{10} &= \tfrac{1}{2}(y_0 + y_0^*) = |\eta_0(\tau_1)|\cos[\omega_\varepsilon\tau_0 + \delta(\tau_1)] \\ &= \arg[\eta_0(\tau_1)]. \end{aligned} \tag{2.25}$$

The partial energy of the first oscillator is calculated as

$$e_{10}(\tau_1) = \tfrac{1}{2}\left(\langle u_{10}^2\rangle + \langle v_{10}^2\rangle\right) = \tfrac{1}{2}|\eta_0(\tau_1)|^2. \tag{2.26}$$

In particular, for small values of τ_1 we obtain

$$\eta_0(\tau_1) = V_0\left(1 - \tfrac{1}{2}\lambda_1\lambda_2\tau_1^2\right), \quad e_{10}(t_1) = \tfrac{1}{2}V_0^2\left(1 - \lambda_1\lambda_2\tau_1^2\right). \tag{2.27}$$

It follows from (2.18) and (2.27) that on the initial time interval the energy of the excited oscillator decreases, while the energy of the second oscillator (the trap) increases. A time instant τ_1^* at which $e_1(\tau_1^*) = e_2(\tau_1^*)$ can be found from the equality $(\lambda_2 V_0\tau_1)^2 = V_0^2(1 - \lambda_1\lambda_2\tau_1^2)$; that is,

$$\tau_1^* = \frac{1}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}. \tag{2.28}$$

An increase in the coupling coefficients λ_1,λ_2 obviously entails a decrease in τ_1^*. This conclusion agrees with the experimental results of [22,23].

III. APPROXIMATE ANALYSIS OF ENERGY TRANSFER

The analysis of the full system (2.6) can be significantly simplified if the integral terms in (2.6) and (2.14) can be omitted. As shown in Appendixes A and B, the integral term on the right-hand side of (2.14) may be omitted if (i) $2\beta^2 \gg \lambda_1\lambda_2$ and/or (ii) $m_1 \gg m_2$. We consider these two cases in detail.

For the first case, suppose that $2\beta^2 \gg \lambda_1\lambda_2$. In this case, we introduce the small parameters ε by the equality $c_{12}/c_2 = 2\varepsilon^{3/2}$; the dimensionless initial impulse is $V/\omega = \varepsilon^{-1/2}V_0$; all other parameters are defined as in (2.2). For brevity, we take $\lambda_1 = \lambda_2 = \lambda$. Then we substitute $\varepsilon^{3/2}\lambda$ for $\varepsilon\lambda$ and $\varepsilon^{-1/2}V_0$ for V_0 in Eqs. (2.3) and (2.5). As in Sec. II, we obtain the dimensionless equations

$$\begin{aligned} &\frac{d^2u_1}{d\tau_0^2} + \omega_{1\varepsilon}^2 u_1 - 2\varepsilon^{3/2}\lambda u_2 = 0, \\ &\frac{d^2u_2}{d\tau_0^2} + \omega_{1\varepsilon}^2 u_2 - 2\varepsilon\zeta(\tau_1)u_2 = 2\varepsilon\omega_{1\varepsilon}^{-1}\lambda V_0\sin\omega_{1\varepsilon}\tau_0 \\ &\quad + 4\varepsilon^3\omega_{1\varepsilon}^{-1}\lambda^2\int_0^{\tau_0}u_2(s)\sin\omega_{1\varepsilon}(\tau_0 - s)\,ds, \end{aligned} \tag{3.1}$$

$$\tau_0 = 0, \quad u_1 = u_2 = 0; \quad v_1 = \varepsilon^{-1/2}V_0, \quad v_2 = 0,$$
$$v_i = du_i/d\tau_0,$$

where $\omega_{1\varepsilon} = (1 + 2\varepsilon^{3/2}\lambda)^{1/2}$. As shown in [3], the integral term of $O(\varepsilon^3)$ can be excluded from further analysis. This leads to the truncated system

$$\begin{aligned} &\frac{d^2u_1}{d\tau_0^2} + \omega_{1\varepsilon}^2 u_1 - 2\varepsilon^{3/2}\lambda u_2 = 0, \\ &\frac{d^2u_2}{d\tau_0^2} + \omega_{1\varepsilon}^2 u_2 - 2\varepsilon\zeta(\tau_1)u_2 = 2\varepsilon\lambda V_0\sin\omega_{1\varepsilon}\tau_0, \end{aligned} \tag{3.2}$$

with the same initial conditions as in (3.1). A change of variables similar to that in (2.7) and (2.10) leads to the following equation for the complex amplitude $\varphi(\tau_0,\varepsilon)$:

$$\frac{d\varphi}{d\tau_0} + i\varepsilon\zeta(\tau_1)(\varphi - \varphi^* e^{-2i\omega_{1\varepsilon}\tau_0}) = i\varepsilon\lambda V_0(1 - e^{-2i\omega_{1\varepsilon}\tau_0}), \tag{3.3}$$
$$\varphi(0) = 0.$$

As in Sec. II, $\varphi(\tau_0,\varepsilon)$ is constructed in the form of the multiple-scale expansion $\varphi(\tau_0,\varepsilon) = \varphi_0(\tau_1) + \varepsilon\varphi_1(\tau_0,\tau_1) + \cdots$. Reproducing the transformations of Sec. II, the equation for the slowly varying envelope $\varphi_0(\tau_1)$ is obtained as

$$\frac{d\varphi_0}{d\tau_1} - i(\rho_0 + 2\beta^2\tau_1)\varphi_0 = -i\lambda V_0, \quad \varphi_0(0) = 0, \tag{3.4}$$

where $\rho_0 = -\sigma$. The solution of Eq. (3.4) takes the form

$$\begin{aligned} \varphi_0(\tau_1) &= -i\lambda V_0 i(\tau_1) \\ i(\tau_1) &= \int_0^{\tau_1}\exp\left\{i\left[\rho_0(\tau_1 - s) + \beta^2\left(\tau_1^2 - s^2\right)\right]\right\}ds \\ &= e^{iB(\tau_1)}\int_0^{\tau_1}e^{-iB(s)ds}, \end{aligned} \tag{3.5}$$

where $B(s) = \beta^2 s^2 + \rho_0 s = (\beta s + \theta_0)^2 - \theta_0^2; \theta_0 = \rho_0/2\beta = -\sigma/2\beta$, and thus $e^{iB(\tau_1)} = e^{-i\theta_0^2}e^{i(\beta\tau_1+\theta_0)^2}$. This implies that

$$\int_0^{\tau_1}e^{-iB(s)}ds = ei\theta_0^2\int_0^{\tau_1}e^{-i(\beta s+\theta_0)^2}ds = \frac{1}{\beta}F(\tau_1,\theta_0)e^{i\theta_0^2},$$
$$\begin{aligned} F(\tau_1,\theta_0) = \int_{\theta_0}^{\beta\tau_1+\theta_0}e^{-ih^2}dh &= [C(\beta\tau_1 + \theta_0) - C(\theta_0)] \\ &- i[s(\beta\tau_1 + \theta_0) - s(\theta_0)], \end{aligned} \tag{3.6}$$

where $S(x)$ and $S(x)$are the cosine- and sine-Fresnel integrals defined in (A19). Finally, we write

$$\varphi(\tau_1) = -i\tfrac{\lambda V_0}{\beta}F(\tau_1,\theta_0)e^{i(\beta\tau_1+\theta_0)^2}. \tag{3.7}$$

If the envelope $\varphi_0(\tau_1)$ is known, the approximations u_{20} and u_{10} are calculated by (2.16) and (2.25), respectively; the envelope $\eta_0(\tau_1)$is expressed as

$$\eta_0(\tau_1) = \varepsilon^{-1/2}V_0 - i\varepsilon^{-3/2}\lambda\int_0^{\tau_1}\varphi_0(r_1)\,dr_1. \tag{3.8}$$

We evaluate the amplitude of oscillations in the following limiting cases.

(1) If $\beta\tau_1 \ll \sqrt{2}$, then we obtain from (3.4) that

$$|\varphi_0(\tau_1)| \approx \lambda V_0\tau_1. \tag{3.9}$$

(2) If $\beta\tau_1 \gg \sqrt{2}$, then the asymptotic representations (A20) hold, and therefore,

$$\varphi_0(\tau_1) \to \bar{\varphi}_0 = -i\frac{\lambda V_0}{\beta}\left\{\left[\sqrt{\frac{\pi}{8}} - C(\theta_0)\right] - i\left[\sqrt{\frac{\pi}{8}} - S(\theta_0)\right]\right\},$$
$$|\bar{\varphi}_0| = \frac{\lambda V_0}{\beta}\left\{\left[\sqrt{\frac{\pi}{8}} - C(\theta_0)\right]^2 + \left[\sqrt{\frac{\pi}{8}} - S(\theta_0)\right]^2\right\}^{1/2},$$
$$\text{as} \quad \tau \to \infty. \tag{3.10}$$

The energy of quasistationary oscillations of the trap is determined as $\bar{e}_{20} = 1/2|\bar{\varphi}_0|^2$; the residual energy of the first oscillator $\bar{e}_{10}$ can be calculated using the integral of motion (A9). As remarked in Sec. II, an analysis as $\tau_1 \to \infty$ is formally incorrect, but expression (3.10) can be considered as an illustration of a transition from the initial rest state to quasistationary oscillations.

Here we demonstrate and discuss the numerical solutions of the full system (2.3) and the truncated system (3.2). In computations, we use the following numerical values of the system parameters:

$$\varepsilon = 0.1\,36; \quad \varepsilon^{3/2} = 0.05; \quad \varepsilon^{-1/2}V_0 = 1;$$
$$\varepsilon\sigma = 0.1125; \quad (\varepsilon\beta)^2 = 0.025; \quad \lambda = 1. \tag{3.11}$$

Numerical simulations have been carried out in the interval $0 \leqslant \tau_0 \leqslant 80$; in this interval, we have $-3.375 \leqslant \zeta(\varepsilon\tau_0) \leqslant 1.125$. It is easy to calculate that the instantaneous partial frequency of the trap $\omega_2(\tau_1) = 1 + \varepsilon^{3/2}\lambda - \varepsilon\zeta(\tau_1)$ lies within the interval $0.994 \leqslant \omega_2(\tau_1) \leqslant 1.22$, whereas the constant partial frequency of the first oscillators is $\omega_1 = 1 + \varepsilon^{3/2}\lambda = 1.05$. This means that in the interval $0 \leqslant \tau_0 \leqslant 80$ the resonance mode of motion is preserved and the asymptotic approach is applicable.

We denote the solutions of the full system (2.3) and the truncated system (3.2) by u_i and $\widetilde{u_i}$, respectively. Numerical solutions u_i of the full system (2.3) are interpreted as exact solutions of the system under consideration; numerical solutions $\widetilde{u_i}$ of the truncated system (3.2) are interpreted as approximations of the exact solutions. In Figs. 1 and 2 one can observe a transition from the initial state to quasistationary motion with a decreasing amplitude of oscillations of the first oscillator and an increasing amplitude of the second oscillator (the energy trap).

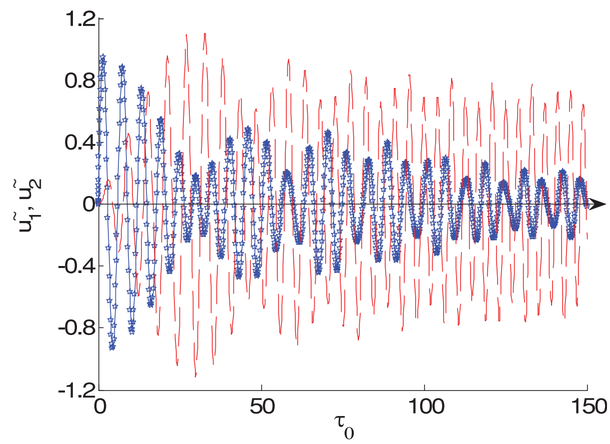

FIG. 2. (Color online) Solutions of the truncated system (3.2): $\tilde{u}_1$, starred line; $\tilde{u}_2$, dashed line.

Figures 3 and 4 prove a close proximity of exact solutions and their analytic approximations for each oscillator separately. Therefore, approximation (3.7) can be used instead of solving Eq. (2.3).

Figure 5 illustrates the occurrence of targeted energy transfer in the system. The energy of the trap is calculated by Eqs. (3.7) and (2.17); the energy of the first oscillator is calculated by Eqs. (2.24) and (2.26).

As seen in Fig. 6, $e_1 = e_2$ at $\tau_0^* \approx 15, \tau_1^* = 0.75$. It is important to note that formula (2.28) gives a close value of $\tau_1^* = \sqrt{1/2} = 0.71$.

For the second case, suppose that $m_2 = \varepsilon\delta m_1, \delta = O(1)$. Since $c_{20}/m_2 = c_1/m_1 = \omega^2$, then $c_1 = m_1 c_{20}/m_2 = c_{20}/\varepsilon\delta$ and $c_{12}/c_1 = 2\varepsilon^2\delta\lambda_2$. With these parameters, Eq. (2.6) is rewritten as

$$\frac{d^2u_1}{d\tau_0^2} + \omega_{2\varepsilon}^2 u_1 - 2\varepsilon^2\delta\lambda_1 u_2 = 0,$$

$$\frac{d^2u_2}{d\tau_0^2} + (1 + 2\varepsilon\lambda_2)u_2 - 2\varepsilon\zeta(\tau_1)u_2 = 2\varepsilon\omega_{2\varepsilon}^{-1}\lambda_2 V_0 \sin(\omega_{2\varepsilon}\tau_0)$$

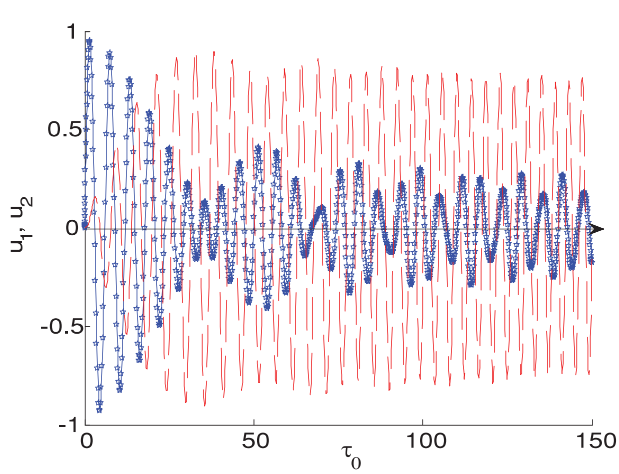

FIG. 1. (Color online) Solution of the full system (2.3): u_1, starred line; u_2, dashed line.

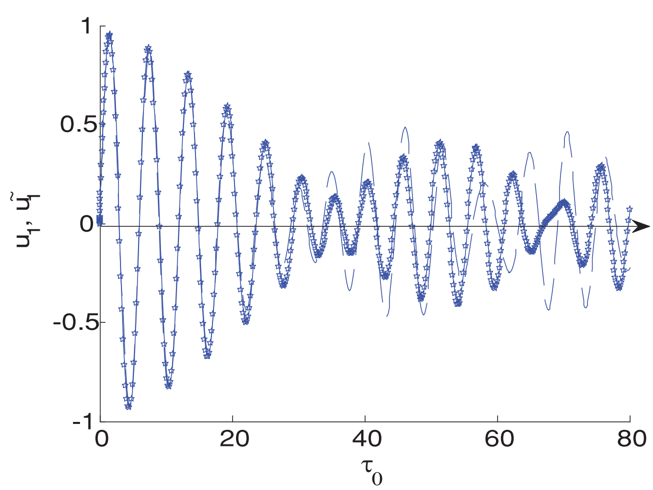

FIG. 3. (Color online) Exact solution u_1 (dashed line) and approximate solution $\tilde{u}_1$ (starred line).

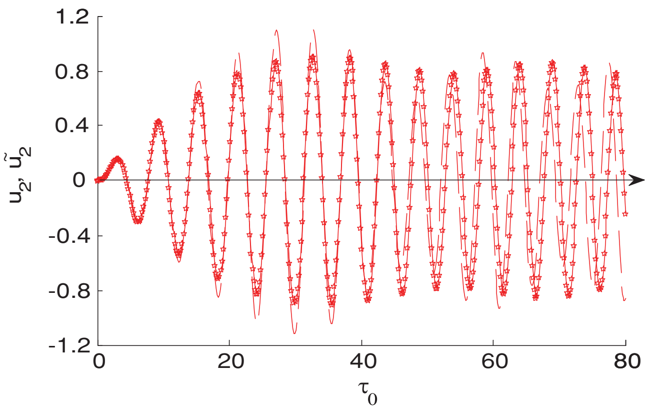

FIG. 4. (Color online) Exact solution u_2 (dashed line) and approximate solution $\tilde{u}_2$ (starred line).

$$+4\varepsilon^3\omega_{2\varepsilon}^{-1}\delta\lambda_2^2\int_0^{\tau_0}\sin\omega_{2\varepsilon}(\tau_0-s)u_2(s)\,ds, \tag{3.12}$$

$$\tau_0=0:\ u_1=u_2=0;\quad v_1=V_0,\quad v_2=0,$$

where $\omega_{2\varepsilon}=(1+2\varepsilon^2\delta\lambda_1)^{1/2}$. As in Eq. (3.1), we exclude the integral term of $O(\varepsilon^3)$ from consideration and replace (3.12) with the truncated system

$$\begin{gathered}\frac{d^2u_1}{d\tau_0^2}+\omega_{2\varepsilon}^2u_1-2\varepsilon^2\delta\lambda_1u_2=0,\\ \frac{d^2u_2}{d\tau_0^2}+(1+2\varepsilon\lambda_2)u_2-2\varepsilon\zeta(\tau_1)u_2=2\varepsilon\lambda_2V_0\sin\omega_{2\varepsilon}\tau_0\end{gathered} \tag{3.13}$$

with the initial conditions $\tau_0=0:\ u_1=u_2=0;\ v_1=V_0,v_2=0$. The approximation $u_{20}(\tau_0,\varepsilon)$ is then calculated by formula (2.16); the envelope of the process $u_{20}(\tau_0,\varepsilon)$ is defined by

$$\frac{d\varphi_0}{d\tau_1}-i(\rho_1+2\beta^2\tau_1)\varphi_0=-i\lambda_2V_0,\quad \varphi_0(0)=0, \tag{3.14}$$

where $\rho_1=\lambda_2-\sigma$. It is obvious that the solution of (3.14) is similar to (3.7), namely,

$$\varphi_0(\tau_1)=-i\frac{\lambda V_0}{\beta}F(\tau_1,\theta_1)e^{i(\beta\tau_1+\theta_1)^2},\quad \theta_1=\frac{\rho_1}{\beta}. \tag{3.15}$$

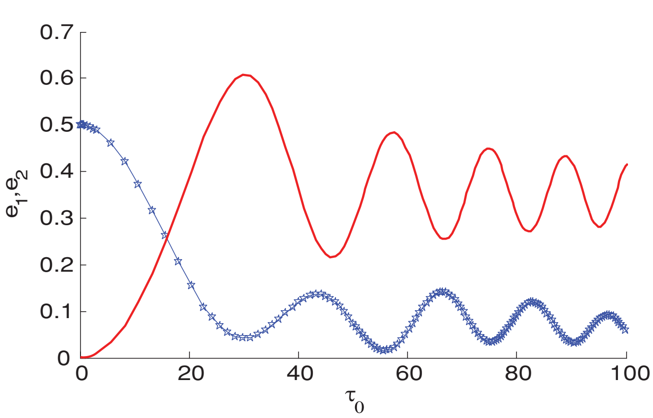

FIG. 5. (Color online) Energy of the oscillator (starred line) and the trap (solid line).

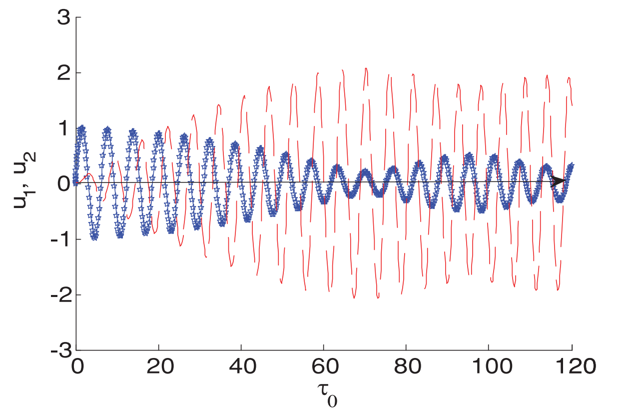

FIG. 6. (Color online) Solutions of the full system (2.3): u_1, blue starred line; u_2, red dashed line.

The asymptotic behavior of the complex envelope $\varphi_0(\tau_1)$ is described by expressions similar to (3.9) and (3.10). The envelope of process u_1 can be found from (2.26).

Figures 6 and 7 demonstrate the results of numerical simulation for systems (2.3) and (3.13) with the following parameters:

$$\begin{gathered}m_1=5m_2,\quad \varepsilon\delta=0.2;\quad \varepsilon=0.05;\quad V_0=1,\\ \varepsilon\sigma=0.1125,\quad \varepsilon\beta=0.1,\quad \lambda=1.\end{gathered} \tag{3.16}$$

The exact solutions u_1,u_2 of the full system (2.3) are depicted in Fig. 6; Fig. 7 demonstrates the solutions $\widetilde{u_1}$,$\widetilde{u_2}$ of the truncated system (3.13). Figures 6 and 7 indicate that the exact and approximate periods and amplitudes of slow and fast oscillations are almost identical for both oscillators up to an instant of transition to quasistationary motion, corresponding to a global minimum of the envelope of the first oscillator. The times of transition from the initial state to quasistationary oscillations (a counterpart of the Landau-Zener transition time) are equal to $\tau_0\approx 75$ and $\widetilde{\tau_0}\approx 80$ for the exact and approximate solutions, respectively; the difference is about 6%. After this moment, the slow envelope of the approximate solution $\widetilde{u_2}$ has a more distinctive minimum $\widetilde{A_2}\approx 1.5$ at $\tau_0\approx 95$ versus $A_2\approx 1.7$ at $\tau_0\approx 95$ for the exact solution u_2; the difference is about 12%. A corresponding maximum of the slow envelope

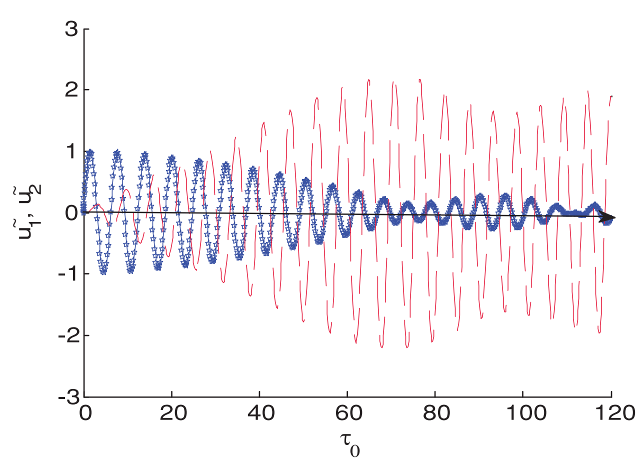

FIG. 7. (Color online) Solutions of the truncated system (3.13): $\tilde{u}_1$, starred line; $\tilde{u}_2$, dashed line.

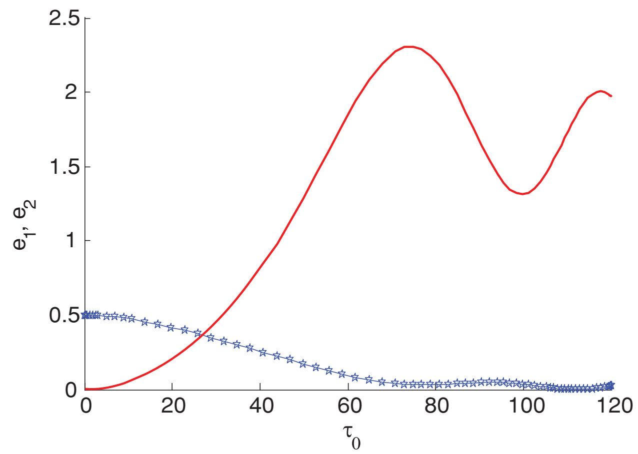

FIG. 8. (Color online) Energy of the oscillators (starred line) and the trap (solid line) calculated by (3.13).

of the approximate solution u_1 is equal to $A_1 \approx 0.48$ versus $A_1 \approx 0.5$ for the exact solution u_1; the difference is about 4%.

Figure 8 demonstrates irreversible energy transfer from the first oscillator to the trap. Figure 9 depicts the behavior of the full system (2.3) with the parameters in (3.11) but with different masses $m_1 = 5m_2$. A comparison of the results presented in Figs. 8 and 9 with those in Figs. 1 and 5 shows that energy transfer in a system with different masses is more intensive but requires an increased transition time against a system with equal masses. This stems from the fact that the minimal mass m_2 is able to take a large portion of energy from the maximal mass m_1 only during a protracted resonance interaction. This conclusion is consistent with relationship (2.28): A decrease in the parameter λ_1 increases the time τ_1^* and the total duration of the transient process.

IV. ENERGY TRANSFER IN THE SYSTEM WITH THE QUADRATIC-IN-TIME DETUNING LAW

In this section we briefly describe the analysis of system (2.3) with equal coupling $\lambda_1 = \lambda_2 = \lambda$ and with the quadratic-in-time detuning law (see also [23])

$$\zeta(\tau_1) = 2\left(\sigma - 2\beta_2^2\tau_1^2\right). \tag{4.1}$$

First, we consider the oscillators in which the rate of resonance crossing is large against weak coupling. In this case, the exact and approximate equations of motion are written in the forms (3.1) and (3.2), respectively; the slowly varying envelope $\varphi_0(\tau_1)$ satisfies an equation similar to Eq. (3.4); that is,

$$\frac{d\varphi_0}{d\tau_1} - i\left(-\sigma + 2\beta_2^2\tau_1^2\right)\varphi_0 = -i\lambda V_0, \quad \varphi_0(0) = 0. \tag{4.2}$$

Let us denote $2\beta_2^2\tau_1^2 = f(\tau_1), 2/3\beta_2\tau_1^3 = F(\tau_1)$. In this notation, the solution of Eq. (4.2) is written as

$$\varphi_0(\tau_1) = -i\lambda V_0 Y(\tau_1), \tag{4.3}$$

where

$$\begin{aligned} Y(\tau_1) &= \int_0^{\tau_1} \exp\{i[-\sigma(\tau_1 - s) + F(\tau_1) - F(s)]\}ds \\ &= e^{iB(\tau_1)}\int_0^{\tau_1} e^{-iB(s)}ds, \\ B(s) &= -\sigma s + F(s). \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\varphi_0(\tau_1) = -i\lambda V_0 Y_1(\tau_1)e^{iB(\tau_1)}, \quad Y_1(\tau_1) = \int_0^{\tau_1} e^{-iB(s)}ds. \tag{4.5}$$

The stationary phase method [32] allows us to approximate the solution by the Fresnel integrals. If $B(s)$ is a rapidly varying function of s, then significant contributions to the integral value take place in small intervals near the stationary phases θ_s such that $B'(\theta_s) = 0$ [32]. In the problem under consideration, a unique stationary phase θ_s is defined by

$$B'(\theta_s) = -\sigma + 2\beta_2^2\theta_s^2 = 0, \quad \theta_s = (\sqrt{\sigma/2})/\beta_2. \tag{4.6}$$

Expanding $B(s)$ in the Taylor series near $s = \theta_s$ and keeping only the first two nonzero terms, we obtain

$$B(s) \approx B(\theta_s) + 1/2k(s - \theta_s)^2. \tag{4.7}$$

where $k = B"(\theta_s) = 4\beta_2^2\theta_s$. Substituting (4.7) into (4.5), we obtain the following approximation:

$$Y_1(\tau_1) = \int_0^{\tau_1} e^{-iB(s)}ds \approx e^{-iB(\theta_s)}\int_0^{\tau_1} e^{-ik[(s-\theta_s)^2/2]}ds, \tag{4.8}$$

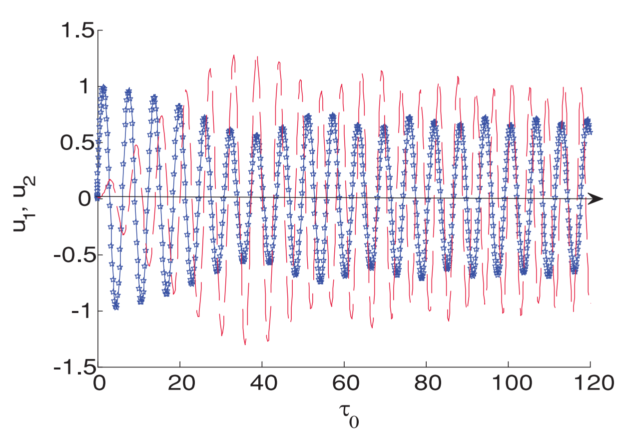

FIG. 9. (Color online) Solutions of system (2.3) with the parameters in (3.11) and $m_1 = 5m_2$: u_1, starred line; u_2, dashed line.

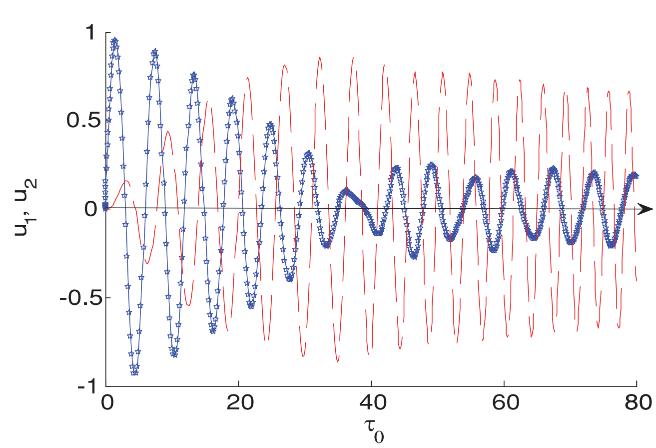

FIG. 10. (Color online) Solutions of the full system (2.3): u_1, starred line; u_2, dashed line.

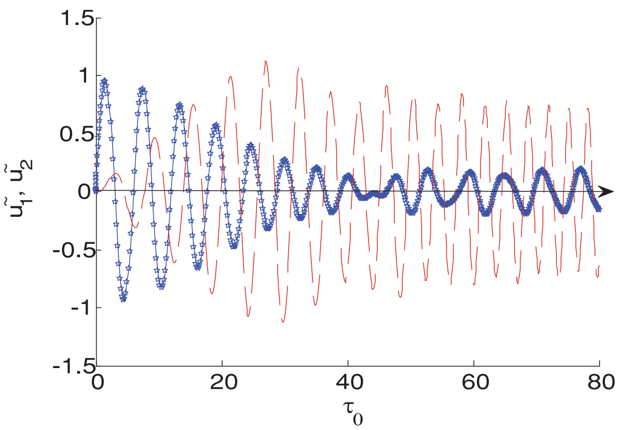

FIG. 11. (Color online) Solutions of the truncated system (3.2): $\tilde{u}_1$, starred line; $\tilde{u}_2$, dashed line.

and by analogy with (3.10),

$$Y_1(\tau_1) \to \bar{Y}_1 \approx \sqrt{\frac{\pi}{2k}} e^{-i[B(\theta_s)+\pi/4]} + O\left(\frac{1}{k}\right). \tag{4.9}$$

Formulas (4.8) and (4.9) demonstrate the convergence of highly oscillating integral (4.4) to a stationary value. As noted in Sec. III, this implies the occurrence of energy transfer from the first oscillator to the trap. Note that this formal approach gives a proper approximation if $k \gg 1$; however, numerical simulations show that irreversible energy transfer occurs for a wide range of parameters.

Using the stationary phase method, one can construct the solution of system (2.3) with the quadratic-in-time detuning in the same way as in Sec. II. We study a system with the parameters in (3.11), but we replace $\varepsilon^2\beta^2 = 0.025$ with $\varepsilon^3\beta_2^2 = 0.025$. A straightforward calculation gives $\theta_s = 0.2$, $k = 7.7$; that is, in principle, the stationary phase approximation is acceptable. However, for brevity, we omit a detailed derivation of the approximate analytic solution. We compute the numerical solutions of Eqs. (2.3) and (3.2) with detuning (4.1) and then compare the results with that of Sec. II.

Solutions of systems (2.3) and (3.2) with detuning (4.1) are shown in Figs. 10 and 11, respectively. A good agreement between the exact and approximate solutions is obvious.

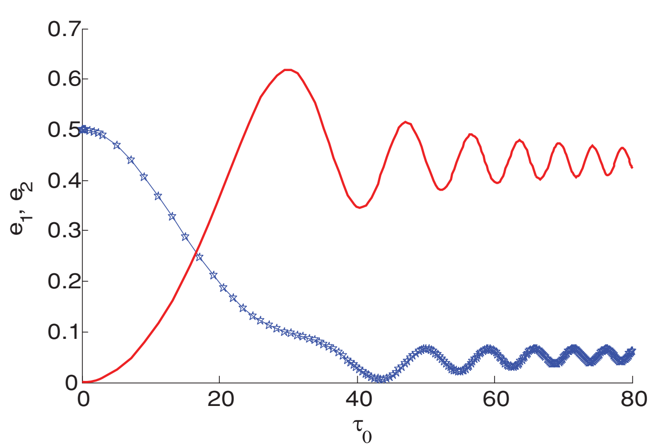

FIG. 12. (Color online) Energy of the oscillator (blue stars) and the trap (red line).

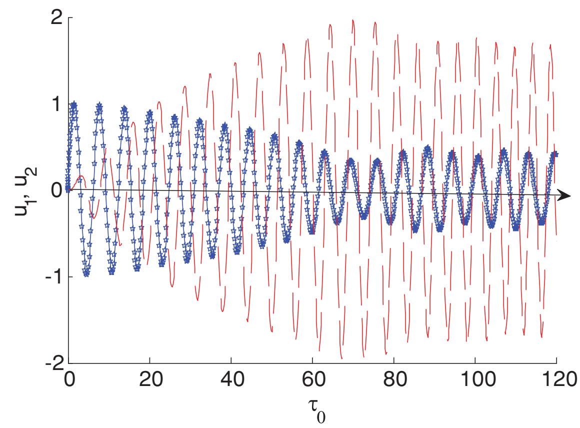

FIG. 13. (Color online) Solutions of the full system (2.3) with the parameters in (3.11) and $\varepsilon^3\beta_2^2 = 0.01$: u_1, blue starred line; u_2, red dashed line.

Figure 12 depicts energy transfer from the excited oscillator to the energy trap in the system with detuning (4.1).

Now we compare the results of simulation for the oscillators with the linear and quadratic-in-time detuning. For the first oscillator, the time to attain the first minimum of the slow envelope is $\tau_0 \approx 35$ in Figs. 1 and 5 versus $\tau_0 \approx 44$ in Figs. 10 and 12 However, as seen in Figs. 1 and 5, the system with the linear-in-time detuning demonstrates a sequence of beat oscillations before the transition to quasistationary oscillations. This means that the real transition time needed for the formation of quasistationary oscillations is about $\tau_0 \approx 100$. At the same time, the quadratic modulation suppresses beating and ensures a smooth transition from the initial state to small quasistationary oscillations. In addition, correlating the plots in Figs. 5 and 12, one can observe that the energy of stationary motion of the first oscillator in Fig. 5 is about 5 times less than the initial energy, but Fig. 12 demonstrates a tenfold decrease. This means that the system with the quadratic-in-time detuning is more effective than the model with the linear time dependence.

Next, we investigate the dynamics of the system with different masses. As in Sec. III, we consider the case $m_1 = 5m_2$. The theoretical analysis can be performed in the same way as in Sec. III, but here we demonstrate only the numerical results. The parameters of simulations are the same as in (3.16) with $\varepsilon^3\beta_2^2 = 0.01$ in place of $\varepsilon^3\beta^2 = 0.01$. A comparison of Figs. 6 and 13 shows that the transition time is $\tau_0 \approx 70$ in Fig. 13 versus $\tau_0 \approx 75$ in Fig. 6, but as in the previous case, the quadratic time dependence of the frequency suppresses beating, and the decaying transient process smoothly changes to quasistationary oscillations, while in the system with the linear-in-time detuning, beating oscillations precede quasistationary oscillations. This renders the system with the quadratic-in-time modulation more effective in suppressing undesired oscillations (cf. [23]).

V. CONCLUSIONS

The analytical study of irreversible energy transfer in a classical oscillatory system with time-dependent parameters has not been addressed thus far in the literature. This paper demonstrates a closed-form asymptotic solution of this problem for a system of two weakly coupled linear oscillators, in which the first oscillator with constant parameters is excited by an initial impulse, whereas the coupled oscillator with a time-dependent frequency is initially at rest but then acts as an energy trap. It has been shown that in physically meaningful limiting cases the problem of irreversible energy transfer from the excited oscillator to the trap is reduced to a first-order equation with the solution in the form of the Fresnel integrals. In view of a mathematical analogy between energy transfer in a classical oscillatory system with variable parameters and nonadiabatic quantum Landau-Zener transition, the results of this paper, in addition to providing an analytical framework for understanding the transient dynamics of coupled oscillators, suggest an approximate procedure for solving the linear Landau-Zener problem with arbitrary initial conditions over a finite time interval.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix we demonstrate a connection between the complex envelopes $\varphi_0(\tau_1)$ and $\eta_0(\tau_1)$ and the solution of the Landau-Zener tunneling problem. We consider again the system of the weakly coupled oscillators (2.3) with the parameters in (2.2):

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) = 0,$$
$$\frac{d^2u_2}{d\tau_0^2} + u_2 + 2\varepsilon\lambda_2(u_2 - u_1) - 2\varepsilon\zeta(\tau_1)u_2 = 0, \quad \text{(A1)}$$
$$\tau_0 = 0,\quad u_1 = u_2 = 0;\quad v_1 = V_0,\quad v_2 = 0,\quad v_i = du_i/d\tau_0.$$

As in [22,23], we introduce the change of variables

$$v_{1,2} + iu_{1,2} = a_{1,2}e^{i\tau_0}. \quad \text{(A2)}$$

The amplitudes $a_{1,2}$ are sought in the form of the multiple-scale expansions $a_r(\tau_0,\varepsilon) = a_{r0}(\tau_1) + \varepsilon a_{r1}(\tau_0,\tau_1) + \cdots, r = 1,2$. As shown in [22,23], the functions $a_{r0}(\tau_1)$ satisfy the equations

$$i\frac{da_{10}}{d\tau_1} = -\lambda_1(a_{10} - a_{20}),$$
$$i\frac{da_{20}}{d\tau_1} = \sigma(1 - \gamma\tau_1)a_{20} - \lambda_2(a_{20} - a_{10}), \quad \text{(A3)}$$
$$\tau_1 = 0 : a_{10} = V_0,\quad a_2(0) = 0,$$

describing a particular case of the linear quantum Landau-Zener problem. It is easy to deduce that the functions

$$\eta_0(\tau_1) = a_{10}(\tau_1)\,e^{-i\lambda_1\tau_1},\quad \varphi_0(\tau_1) = a_{20}(\tau_1)\,e^{-i\lambda_2\tau_1} \quad \text{(A4)}$$

satisfy the following equations:

$$i\frac{d\eta_0}{d\tau_1} = \lambda_1\varphi_0(\tau_1),\quad \eta_0(0) = V_0,$$
$$i\frac{d\varphi_0}{d\tau_1} = (\rho + 2\beta^2\tau_1)\varphi_0 + \lambda_2\eta_0(\tau_1),\quad \varphi_0(0) = 0, \quad \text{(A5)}$$

where $2\beta^2 = \sigma\gamma, \rho = \lambda_2 - \lambda_1 - \sigma$. Formally, system (A5) seems to be non-Hermitian, as its matrix has unequal antidiagonal terms [33]. However, the change of variables

$$\eta_0 = (\lambda_1/\lambda_2)^{1/4}x,\quad \varphi_0 = (\lambda_2/\lambda_1)^{\frac{1}{4}}y \quad \text{(A6)}$$

transforms (A5) into the Hermitian form

$$i\frac{dx}{d\tau_1} = \sqrt{\lambda_1\lambda_2}y,\quad \chi(0) = (\lambda_2/\lambda_1)^{\frac{1}{4}}V_0,$$
$$i\frac{dy}{d\tau_1} = (\rho + 2\beta^2\tau_1)y + \sqrt{\lambda_1\lambda_2}\chi,\ y(0) = 0. \quad \text{(A7)}$$

This implies a direct connection between systems (A3) and (A5) and the classic (Hermitian) Landau-Zener problem.

System (A7) conserves the integral of motion

$$|x(\tau_1)|^2 + |y(\tau_1)|^2 = |x(0)|^2 = (\lambda_2/\lambda_1)^{1/2}V_0^2, \quad \text{(A8)}$$

or by (A6),

$$|\eta_0(\tau_1)|^2 + (\lambda_1/\lambda_2\,|\varphi_0(\tau_1)|^2 = V_0^2. \quad \text{(A9)}$$

In the case $\lambda_1 = \lambda_2$, we obtain the well-known integral of energy $|\eta_0(\tau_1)|^2 + |\varphi_0(\tau_1)|^2 = V_0^2$. As remarked in Sec. II, system (A5) provides an approximate description of the full system (2.3) with an error of $O(\varepsilon)$ if τ_1 is of $O(1)$, and therefore, the conservation law (A9) is applicable only in this time interval. Simple algebra proves that the total energy of the initial system (2.1) and the corresponding quantity for the dimensionless system (2.3) increase as $\epsilon\tau_1$. This result is consistent with the theory.

Next, we show that Eq. (A5) is equivalent to the second-order Weber equation of the Landau-Zener problem [10], and then we prove the convergence of the exact solution to the Fresnel integrals as $\lambda_1\lambda_2/\beta^2 \to 0$.

It follows from (A5) that

$$\frac{d\varphi_0}{d\tau_1}i(\rho + 2\beta^2\tau_1)\varphi_0 = -i\lambda_2V_0 - \lambda_1\lambda_2\int_0^{\tau_1}\varphi_0(s)\,ds,$$
$$\varphi_0(0) = 0, \quad \text{(A10)}$$
$$\frac{d\eta_0}{d\tau_1} = -i\lambda_1\varphi_0(\tau_1),\quad \eta_0(0) = V_0.$$

It is obvious that (A10) is identical to system (2.24). Differentiation of the first equation in (A10) in τ_1 yields the second-order equation coinciding with (2.15),

$$\frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\frac{d\varphi_0}{d\tau_1} + (\lambda_1\lambda_2 - 2i\beta^2)\varphi_0 = 0,$$
$$\tau_1 = 0 : \varphi = 0, \frac{d\varphi_0}{d\tau_1} = -i\lambda_2V_0. \quad \text{(A11)}$$

Substituting $\varphi_0(\tau_1) = \phi(\tau_1)e^{i\psi(\tau_1)}, \psi(\tau_1) = \frac{1}{2}\int_0^{\tau_1}(\rho + 2\beta^2 s)\,ds$ transforms (A11) into

$$\frac{d^2\phi}{d\tau_1^2} + \left[\lambda_1\lambda_2 + \frac{1}{4}(\rho + 2\beta^2\tau_1)^2 - i\beta^2\right]\phi = 0, \quad \text{(A12)}$$

Then we introduce the independent variable $x = i^{-1/2}z, z(\rho + 2\beta^2\tau_1)/(\sqrt{2}\beta)$ and define the parameter $\Delta = \lambda_1\lambda_2(2\beta^2)$. As a result, we obtain

$$\frac{d^2\phi}{dx^2} + \left(i\Delta - \frac{1}{4}x^2 + \frac{1}{2}\right)\phi = 0, \tag{A13}$$

with the initial conditions $\tau_1 = 0 : x = i^{-1/2}\rho/(\sqrt{2}\beta), \phi = 0, d\phi/dx = -i^{1/2}\lambda_2 V_0/(\sqrt{2}\beta)$. Equation (A13) is identical to the Weber equation derived in [10]. The Weber functions $D_{-i\Delta-1}(\pm ix)$ are linearly independent particular solutions of (A13); their linear combination

$$\phi(x) = AD_{-i\Delta-1}(ix) + BD_{-i\Delta-1}(-ix) \tag{A14}$$

with a proper choice of constants A, B satisfies arbitrary initial conditions.

The analysis of Eq. (A13) can be simplified if $\Delta \ll 1$ or $\lambda_1\lambda_2 \ll 2\beta^2$. In this case,

$$\phi(x) \to \phi_0(x) = AD_{-1}(ix) + BD_{-1}(-ix), \tag{A15}$$

where [34]

$$D_{-1}(r) = e^{\frac{r^2}{4}}\sqrt{\frac{\pi}{2}}\left[1 - \Phi\left(\frac{r}{\sqrt{2}}\right)\right], \tag{A16}$$

$\Phi(r) = \frac{2}{\sqrt{\pi}}\int_0^r e^{-r^2}dr$ is the probability integral. Since $r = ix = i^{1/2}z$, then

$$\begin{aligned}\phi(z) &= AD_{-1}(i^{1/2}z) + BD_{-1}(-i^{1/2}z),\\ z &= i^{1/2}(\rho + \beta^2\tau_1)/\beta,\end{aligned} \tag{A17}$$

where

$$\begin{aligned}
D_{-1}(i^{1/2}z) &= \sqrt{\frac{\pi}{2}}\left[1 - \frac{2}{\sqrt{\pi}}\Phi_1\left(\frac{\sqrt{i}z}{\sqrt{2}}\right)\right]e^{\frac{iz^2}{4}},\\
D_{-1}(-i^{1/2}z) &= \sqrt{\frac{\pi}{2}}\left[1 - \frac{2}{\sqrt{\pi}}\Phi_1\left(\frac{z}{\sqrt{2i}}\right)\right]e^{\frac{iz^2}{4}},\\
\Phi_1\left(\frac{\sqrt{i}z}{\sqrt{2}}\right) &= \int_0^{\frac{\sqrt{i}z}{\sqrt{2}}} e^{-r^2}dr = i^{1/2}\int_0^{\frac{z}{\sqrt{2}}} e^{-is^2}ds\\
&= i^{1/2}\left[C\left(\frac{z}{\sqrt{2}}\right) - iS\left(\frac{z}{\sqrt{2}}\right)\right],\\
\Phi_1\left(\frac{z}{\sqrt{2i}}\right) &= \int_0^{\frac{z}{\sqrt{2i}}} e^{-r^2}dr = -i^{1/2}\int_0^{\frac{z}{\sqrt{2}}} e^{is^2}ds\\
&= -i^{1/2}\left[C\left(\frac{z}{\sqrt{2}}\right) + iS\left(\frac{z}{\sqrt{2}}\right)\right],
\end{aligned} \tag{A18}$$

$C(y)$ and $S(y)$ are the cosine- and sine-Fresnel integrals, and

$$C(y) = \int_0^y \cos(s^2)\,ds, \quad S(y) = \int_0^y \sin(s^2)\,ds. \tag{A19}$$

We recall that the following asymptotic representations hold [34]:

$$\begin{aligned}
C(y) &= \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + \frac{\sin y^2}{y}\right) + O\left(\frac{1}{y^2}\right),\\
S(y) &= \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} - \frac{\cos y^2}{y}\right) + O\left(\frac{1}{y^2}\right) \text{ if } y \gg 1.
\end{aligned} \tag{A20}$$

Formally, an analysis of the asymptotic stationary solution as $\tau_1 \to \infty$ or, equivalently, as $y \to \infty$ is incorrect, but expression (A20) can be considered an indication of the transition from the initial rest state to quasistationary oscillations.

It can be easily checked that as $\Delta \to 0$, the second-order equation (A11) is reduced to the following first-order differential equation:

$$\frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\varphi_0 = -i\lambda_2 V_0, \quad \varphi_0(0) = 0, \tag{A21}$$

with the solution (A17). A more detailed analysis of this equation is given in Sec. III.

APPENDIX B

In the time interval $\tau_1 \ll |\rho|/(2\beta^2)$, an increase in the frequency is negligible, and system (2.3) may be approximated by its conservative counterpart. An exact solution of the conservative system enables us to illustrate the accuracy of the approximation procedure for different relationships between the parameters.

We consider a conservative counterpart of system (2.3):

$$\begin{aligned}
&\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) = 0,\\
&\frac{d^2u_2}{d\tau_0^2} + u_2 + 2\varepsilon\lambda_2(u_2 - u_1) - 2\varepsilon\sigma u_2 = 0,
\end{aligned} \tag{B1}$$

$$\tau_0 = 0, \quad u_1 = u_2 = 0; \quad v_1 = V_0, v_2 = 0, \quad v_i = du_i/d\tau_0.$$

This system possesses the solution

$$\begin{aligned}
u_1(\tau_0,\tau_1) &= \frac{\lambda_2 V_0}{2\kappa}[\cos(\Omega_{2\varepsilon}\tau_0) - \cos(\Omega_{1\varepsilon}\tau_0)],\\
u_2(\tau_0,\varepsilon) &= \frac{\lambda_2 V_0}{2\kappa}\left[\Omega_{2\varepsilon}^{-1}\sin(\Omega_{2\varepsilon}\tau_0) - \Omega_{1\varepsilon}^{-1}\sin(\Omega_{1\varepsilon}\tau_0)\right],
\end{aligned} \tag{B2}$$

where

$$\begin{aligned}
\Omega_{i\varepsilon}^2 &= 1 + 2\varepsilon\rho_i, \quad \rho_{1,2} = -\mu \pm \mu^2 + \lambda_1\sigma)^{1/2},\\
\mu &= 1/2(\lambda_1 + \lambda_2 - \sigma),\\
2\kappa &= \rho_1 - \rho_2 = 2(\mu^2 + \lambda_1\sigma)^{1/2}\\
&= [(\lambda_2 + \lambda_1)^2 - 2\sigma(\lambda_2 - \lambda_1) + \sigma^2]^{1/2}.
\end{aligned}$$

The leading-order approximation of (B2) as $\varepsilon \to 0$ corresponds to beating oscillations

$$\begin{aligned}
u_{10}(\tau_0,\tau_1) &= \frac{\lambda_2 V_0}{\kappa}\sin(\kappa\tau_1)\sin(\tau_0 + \mu\tau_1),\\
u_{20}(\tau_0,\tau_1) &= \frac{\lambda_2 V_0}{\kappa}\sin(\kappa\tau_1)\cos(\tau_0 + \mu\tau_1),
\end{aligned} \tag{B3}$$

with the amplitudes $r_{10} = r_{20}(\tau_1) = \frac{\lambda_2 V_0}{\kappa}|\sin\kappa\tau_1|$. Next, we verify the correctness of the integral transformation. We transform Eq. (B1) into a form similar to (2.6):

$$\begin{aligned}
&\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) = 0,\\
&\frac{d^2u_2}{d\tau_0^2} + (1 + 2\varepsilon\lambda_2)u_2 - 2\varepsilon\omega_\varepsilon^{-1}\lambda_2 V_0\sin\omega_\varepsilon\tau_0
\end{aligned} \tag{B4}$$

$$+4\varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\int_0^{\tau_0}u_2(s)\sin\omega_\varepsilon(\tau_0-s)\,ds,$$

$$\tau_0=0:u_1=u_2=0;\quad v_1=V_0,\quad v_2=0,$$

where $\omega_\varepsilon=(1+2\varepsilon\lambda_1)^{1/2}$. Then we approximately solve Eq. (B4) using the complexification procedure of Sec. II. As a result, we obtain the approximate representation u_{20} in the form (2.16), where the complex envelope $\varphi_0(\tau_1)$ is defined by

$$\frac{d\varphi_0}{d\tau_1}-i\rho\varphi_0=-i\lambda^2V_0-\lambda_1\lambda_2\int_0^{\tau_1}\varphi_0(r_1)\,dr_1,\qquad \varphi_0(0)=0 \tag{B5}$$

or

$$\frac{d^2\varphi_0}{d\tau_1}-i\rho\frac{d\varphi_0}{d\tau_1}+\lambda_1\lambda_2\varphi_0=0,\qquad \tau_1=0:\varphi_0=0,\quad \frac{d\varphi_0}{d\tau_1}=-i\lambda_2V_0. \tag{B6}$$

It is easy to obtain from (B6) that

$$\varphi_0(\tau_1)=\frac{\lambda_2V_0}{p_2-p_1}(e^{ip_1\tau_1}-e^{ip_2\tau_1}),\qquad p_{1,2}=\frac{1}{2}[\rho\pm(\rho^2+4\lambda_1\lambda_2)^{1/2}]. \tag{B7}$$

Simple algebra shows that $(\rho^2+4\lambda_1\lambda_2)^{1/2}=[(\lambda_2+\lambda_1)^2-2\sigma(\lambda_2-\lambda_1)+\sigma^2]^{1/2}=2\kappa$. Hence, $p_{1,2}=1/2(\rho\pm2\kappa)$, and therefore,

$$\varphi_0(\tau_1)=-i\frac{\lambda_2V_0}{\kappa}e^{i\rho\tau_1/2}\sin\kappa\tau_1. \tag{B8}$$

Substituting (B8) in (2.16), we get

$$u_{20}(\tau_0,\tau_1)=-\frac{\lambda_2V_0}{2\kappa}\sin\kappa\tau_1\left(e^{\frac{i}{2}(\rho+\lambda_1)\tau_1}e^{i\tau_0}+e^{-\frac{i}{2}(\rho+\lambda_1)\tau_1}e^{-i\tau_0}\right). \tag{B9}$$

Note that $1/2\rho+\lambda_1=1/2(\lambda_2+\lambda_1-\sigma)=\mu$, and therefore,

$$u_{20}(\tau_0,\tau_1)=-\frac{\lambda_2V_0}{2\kappa}\sin\kappa\tau_1[e^{i(\tau_0+\mu\tau_1)}+e^{-i(\tau_0+\mu\tau_1)}]=-\frac{\lambda_2V_0}{\kappa}\sin\kappa\tau_1\cos(\tau_0+\mu\tau_1). \tag{B10}$$

Expression (B10) obviously coincides with (B2).

Now we suppose that the relationships between the system parameters allow us to ignore the integral term in Eqs. (B4) and (B5) and reduce (B5) to the first-order differential equation.

(1) Suppose that $m_1\gg m_2$; in this case, $m_2=\varepsilon\delta m_1$, $\delta=O(1)$. Since $c_{20}/m_2=c_1/m_1=\omega^2$, the resonance system, then $c_1=m_1c_{20}/m_2=c_{20}/\varepsilon\delta$, and $c_{12}/c_1=2\varepsilon^2\delta\lambda_2$. Hence, we obtain the dimensionless equations of motion in the form

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon^2\delta\lambda_2(u_1-u_2)=0,\qquad \frac{d^2u_2}{d\tau_0^2}+u_2+2\varepsilon\lambda_2(u_2-u_1)-2\varepsilon\sigma u_2=0, \tag{B11}$$

with the initial conditions $\tau=0,u_1=u_2=0;v_1=V_0,v_2=0,v_i=du_i/d\tau_0$. It follows from the first equation in (B11) that

$$u_1=\omega_{1\varepsilon}^{-1}V_0\sin\omega_{1\varepsilon}\tau_0+2\varepsilon^3\omega_{1\varepsilon}^{-1}\delta\lambda_2^2\int_0^{\tau_0}u_2(s)\sin\omega_{1\varepsilon}(\tau_0-s)\,ds,$$

where $\omega_{1\varepsilon}=(1+2\varepsilon^2\delta\lambda_2)^{1/2}$. Thus, Eq. (B4) takes the form

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon^2\delta\lambda_2(u_1-u_2)=0,$$

$$\frac{d^2u_2}{d\tau_0^2}+(1+2\varepsilon\lambda_2)u_2-2\varepsilon\sigma u_2=2\varepsilon\omega_{1\varepsilon}^{-1}\lambda_2V_0\sin\omega_{1\varepsilon}\tau_0+4\varepsilon^3\omega_{1\varepsilon}^{-1}\delta\lambda_2\int_0^{\tau_0}\sin\omega_{1\varepsilon}(\tau_0-s)u_2(s)\,ds. \tag{B12}$$

Referring to (2.9) and (2.14), we can show that the contribution of the integral term on the right-hand side of (B12) is of $O(\varepsilon^2)$. This implies that the integral term is not involved in the equation of the first-order approximation, and therefore, Eq. (B5) is reduced to

$$\frac{d\varphi_0}{d\tau_1}-i\rho_1\varphi_0=-i\lambda_2V_0,\quad \varphi_0(0)=0,\qquad \rho_1=\lambda_2-\sigma. \tag{B13}$$

This yields

$$\varphi_0(\tau_1)=\frac{\lambda_2}{\rho_1}(1-e^{i\rho_1\tau_1}). \tag{B14}$$

Compare the solutions (B7) and (B14). As $\lambda_1\lambda_2/\rho^2\to0$ and $\lambda_1/\lambda_2\to0$, then $p_1=\rho_1,p_2=0$. This means that the solution (B7) is transformed into (B14).

(2) Suppose that $k/c_{12}\gg1$. As in Sec. II, we introduce the dimensionless parameters $k/c_2=2\varepsilon\sigma$, but $c_{12}/c_i=2\varepsilon^{3/2}\lambda_i$, $V/\omega=\varepsilon^{-1/2}V_0$. In this case, the dimensionless equations of motion are given by

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon^{3/2}\lambda_1(u_1-u_2)=0,\qquad \frac{d^2u_2}{d\tau_0^2}+u_2+2\varepsilon^{3/2}\lambda_2(u_2-u_1)-2\varepsilon\sigma u_2=0, \tag{B15}$$

with the initial conditions $\tau_0=0,u_1=u_2=0;v_1=\varepsilon^{-1/2}V_0,v_2=0$. It is easy to obtain from the first equation that

$$u_1=\varepsilon^{-1/2}\omega_{2\varepsilon}^{-1}V_0\sin\omega_{2\varepsilon}\tau_0+2\varepsilon^{3/2}\omega_{2\varepsilon}^{-1}\lambda_2^2\int_0^{\tau_0}u_2(s)\sin\omega_{12\varepsilon}(\tau_0-s)\,ds, \tag{B16}$$

where $(\omega_{2\varepsilon}=1+2\varepsilon^{3/2}\lambda_1)^{1/2}$. Substituting (B16) into (B15), we derive the integro-differential equation for u_2:

$$\frac{d^2u_2}{d\tau_0^2}+(1+2\varepsilon^{3/2}\lambda_2)-2\varepsilon\sigma u_2=2\varepsilon\omega_{1\varepsilon}^{-1}\lambda_2 V_0\sin\omega_{1\varepsilon}\tau_0 + 4\varepsilon^3\lambda_1\lambda_2\omega_{1\varepsilon}^{-1}\int_0^{\tau_0}\sin\omega_{1\varepsilon}(\tau_0-s)u_2(s)\,ds. \tag{B17}$$

Transformations similar to those of Sec. II result in the following leading-order equation for the complex envelope $\varphi_0(\tau_1)$:

$$\frac{d\varphi_0}{d\tau_1}-i\rho_0\varphi_0=-i\lambda_2V_0,\quad \varphi_0(0)=0. \tag{B18}$$

It follows from (B18) that

$$\varphi_0(\tau_1)=\frac{\lambda_2}{\rho_0}(1-e^{i\rho_0\tau_1}),\quad \rho_0=-\sigma. \tag{B19}$$

It is easy to check that expression (B7) is transformed into (B19) in the limiting case $p_1=-\sigma$, $p_2=0$.

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