Validity criterion of the radiative Fourier law for an absorbing and scattering medium

Hector Gomart and Jean Taine*
Laboratoire EM2C, École Centrale Paris–UPR 288 Centre National de la Recherche Scientifique, Bâtiment Péclet, 92295 Châtenay-Malabry Cedex, France
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For radiative heat transfer applications, in particular in homogenized phases of porous media, an exhaustive and accurate validity criterion of the radiative Fourier law, depending only on the logarithmic derivative of the temperature field and an effective absorption coefficient, accounting for possible multiple scattering phenomena, has been established for a semitransparent medium. This effective absorption coefficient is expressed as a function of the absorption coefficient, the albedo, and the scattering asymmetry parameter. The criterion can be applied to semitransparent media that do not follow Beer's laws related to extinction, absorption, and scattering.

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I. INTRODUCTION

In many applications, semitransparent media are often optically thick at a spatial scale δ for which they are practically isothermal. In these conditions, the simplest and most common model to calculate the radiative flux and power field within medium is based on a radiative Fourier law, historically called the Rosseland approximation [1]; this model is widely applied in many fields [2], such as foams [3], insulate fibers [4,5], powder bed [6,7], and core nuclear reactor [8], without clear criterion on the temperature gradient field, to our knowledge.

This model, in particular, is often used for homogenized phases of porous media commonly assumed to follow Beer's laws related to extinction, absorption, and scattering. The radiative conductivity or conductivity tensor are then expressed as functions of extinction and scattering coefficients and a scattering phase function. The Fourier law is often obtained from simple linearization approaches; see, for instance, Ref. [9]. A physical perturbation approach of the radiative transfer equation (RTE) has been developed by Bellet *et al.* [10], and directly leads to the radiative conductivity model.

However, in the cases of most of the statistically anisotropic porous media [10,11] and of many statistically isotropic porous media of low or intermediate porosity [12], Beer's law is no longer valid. In these cases, a generalized radiative transfer equation (GRTE) has been recently developed by Taine *et al.* [11]. This equation degenerates into a classical RTE for media of large optical thicknesses (i.e., in the application conditions of the radiative Fourier law [11]).

The validity of the diffusion approximation applied to biomedical engineering applications has been studied by an electromagnetic approach, which is a similar problem [13,14] or has been limited to the calculation of the reflection properties of a body [15]. Other studies deal with the diffusion approximation in photonics, either for laser technology [16] or for nuclear applications [17]. To our knowledge, no clear and accurate criterion of validity of the Fourier law has been introduced for radiative heat transfer calculations (i.e., the determination of the radiative fluxes or radiative powers per unit volume in a semitransparent medium).

*jean.taine@ecp.fr

The aim of the present paper is to establish a simple and accurate validity criterion of the radiative Fourier law for radiative transfer, valid even for a non-Beerian semitransparent medium. This criterion depends on the temperature field and the physical parameters involved of the RTE (or GRTE). The radiative Fourier law is not valid in the boundary layer of the semitransparent medium since the intensity field strongly depends in this layer on the radiative boundary conditions applied to the whole medium [18]. The width of this layer also depends on these conditions [19] and can be determined case by case. Consequently, the present study is limited to the establishment of the validity criterion in the core of a semitransparent medium.

Section II deals with the physical analysis that allows us to introduce the framework of the criterion and to define a monodimensional benchmark case. The reference models and the model based on the Fourier law are developed in Sec. III. In Sec. IV, the validity criterion of the Fourier law is established. It is expressed versus both the albedo ω and the asymmetry parameter of scattering g of the medium.

II. PHYSICAL ANALYSIS

A. Perturbation method

In the general case of a homogenized medium that does not follow Beer's law (for instance, homogenized phases of many porous media), the variation of the intensity I_ν, in the direction (θ,φ) between the point M of abscissa s' and the point of abscissa $s'+ds'$, is given in the core of the medium by the GRTE [11]

$$\frac{dI_\nu}{ds'}(s',\theta,\varphi) = -\int_{s_\mathrm{b}}^{s'} S_\nu(s,\theta,\varphi)\frac{dG_{\mathrm{ext}\,\nu}}{ds'} \times (s'-s,\theta,\varphi)ds + S_\nu(s',\theta,\varphi). \qquad (1)$$

The first term of the second member of Eq. (1) corresponds to the extinction in the range $[s',s'+ds']$ of the sum of the contributions, from the boundary $s=s_\mathrm{b}$ to $s=s'$, of the source terms $S_\nu ds$ of the current ranges $[s,s+ds]$. $G_{\mathrm{ext}\,\nu}$ is the extinction cumulated distribution function from s to s' in the direction (θ,φ). The second term of the second member of Eq. (1) S_ν is the sum of an emission term S_ν^e and a scattering source term $S_\nu^{\rm sc}$, at the point M of abscissa s', in the direction (θ,φ),

$$S_\nu(s',\theta,\varphi) = S_\nu^{\rm e}(s',\theta,\varphi) + S_\nu^{\rm sc}(s',\theta,\varphi). \tag{2}$$

The emission term is given by

$$S_\nu^{\rm e}(s',\theta,\varphi) = K_\nu n_\nu^2 I_\nu^\circ[T(s',\theta,\varphi)], \tag{3}$$

where I_ν° is the equilibrium radiation intensity in vacuum, given by Planck's law, and K_ν is the generalized absorption coefficient at equilibrium; n_ν is the effective refractive index that depends on the direction (θ,φ) if the medium is stochastically anisotropic [11]. The scattering source term in the range $[s',s'+ds']$ at the point M of abscissa s' in the considered direction (θ,φ), given by

$$S_\nu^{\rm sc}(s',\theta,\varphi) = \int_0^{4\pi}\int_{s_{1\rm b}}^{s_1'} \frac{dP_{{\rm sc}\nu}}{ds_1'}(s_1'-s_1,\theta_1,\varphi_1) \times \frac{p_\nu(\theta_1,\varphi_1,\theta,\varphi)}{4\pi} S_\nu(s_1,\theta_1,\varphi_1)ds_1 d\Omega_1, \tag{4}$$

is due to the cumulated effect of radiations issued from any current direction (θ_1,φ_1) of the whole space; p_ν is the scattering phase function from the direction (θ_1,φ_1) to (θ,φ). From the point of view of the current incidence direction (θ_1,φ_1), the radiation scattered in (θ,φ) is issued from the contributions of all the source terms $S_\nu ds_1$ of the current ranges $[s_1,s_1+ds_1]$, from a boundary $s_1 = s_{1\rm b}$ to $s_1 = s_1'$; s_1' is the abscissa of the same point M in the direction (θ_1,φ_1). $P_{{\rm sc}\nu}$ is the scattering cumulated probability from s_1 to s_1' in the direction (θ_1,φ_1). Finally, the radiative flux at a point M of coordinate x_k, in tensorial notation, is obtained from the intensity field at M by

$$q_j(x_k) = \int_0^\infty \int_0^{4\pi} I_\nu(x_k,\theta,\varphi)u_j d\Omega\, d\nu, \tag{5}$$

where u_j is the unit vector associated with the direction i.

In the case considered here of a semitransparent medium that is optically thick on a spatial scale δ, considered as isothermal, it has been established that the GRTE degenerates into a classical RTE based on generalized extinction, absorption, and scattering coefficients at equilibrium, respectively called $B_\nu(\theta,\varphi)$, $K_\nu(\theta,\varphi)$, and $\Sigma_\nu(\theta,\varphi)$ [11]. Consequently, even in a case of a strongly anisotropic porous medium, we only use in the following a classical RTE. For the sake of simplicity we also consider that the extinction, absorption, and scattering coefficients, respectively called β_ν, κ_ν, and σ_ν, and the refractive index n_ν, are independent of the direction (θ,φ). Moreover, we assume that the scattering phase function p_ν only depends on the scattering angle cosine $\boldsymbol{u}'\cdot\boldsymbol{u}$, where $\boldsymbol{u}'$ and $\boldsymbol{u}$ are the unit vectors of the incidence and scattering directions, respectively. We will consider here the classical Henyey-Greenstein approximation. In the case of an anisotropic medium, the results associated with the three main directions have to be considered. The nondimensional RTE, associated with the previous assumptions, is

$$\frac{1}{\beta_\nu\delta}u_j\frac{\partial I_\nu}{\partial x_j^+}(x_k^+,\boldsymbol{u}) + I_\nu(x_k^+,\boldsymbol{u}) = (1-\omega_\nu)n_\nu^2 I_\nu^\circ[T(x_k^+)] + \frac{\omega_\nu}{4\pi}\int_{4\pi} p_\nu(\boldsymbol{u}'\cdot\boldsymbol{u})I_\nu(x_k^+,\boldsymbol{u}')\,d\Omega', \tag{6}$$

where $\omega_\nu = \sigma_\nu/\beta_\nu$ is the albedo, the nondimensional abscissa x_j^+ is equal to x_j/δ, and $u_j\partial I_\nu/\partial x_j^+$ stands for the tensorial notation of dI_ν/ds^+. It worth noticing that the quantity $(\beta_\nu\delta)^{-1}$ is the ratio of an extinction length $l_{{\rm ext}\,\nu}$ to the spatial scale δ. That is, in fact, an extinction Knudsen number

$$\mathrm{Kn}_\nu^{\rm ext} = \frac{l_{{\rm ext}\,\nu}}{\delta} = \frac{1}{\beta_\nu\delta}, \tag{7}$$

similar to the molecular Knudsen of the Boltzmann transport theory [20,21]. Since the medium is optically thick for the thickness δ, $\mathrm{Kn}_\nu^{\rm ext}$ is small in front of unity and the RTE can be solved by a perturbation method, as the similar Boltzmann equation. The intensity is then written

$$I_\nu(x_k) = I_\nu^{(0)}(x_k) + I_\nu^{(1)}(x_k) \quad \text{with} \quad I_\nu^{(1)}(x_k) \ll I_\nu^{(0)}(x_k), \tag{8}$$

where $I_\nu^{(0)}(x_k)$ is the solution of the RTE of zero order vs $\mathrm{Kn}_\nu^{\rm ext}$ (i.e., independent of $\mathrm{Kn}_\nu^{\rm ext}$), that is equal to the equilibrium intensity $n_\nu^2 I_\nu^\circ[T(x_k)]$ [18,22]. The contribution to the radiative flux of the zero-order solution is null, since it corresponds to the local thermal equilibrium (LTE) conditions of the radiative field, similar to the common LTE of a material system used in heat transfer.

The first-order solution $I_\nu^{(1)}(x_k)$, proportional to the perturbation parameter $\mathrm{Kn}_\nu^{\rm ext}$, is equal to

$$I_\nu^{(1)}(x_k,\boldsymbol{u}) = -\frac{n_\nu^2}{\beta_\nu}u_j\frac{\partial I_\nu^\circ}{\partial T}(T)\frac{\partial T}{\partial x_j}(x_k) + \frac{\sigma_\nu}{4\pi\beta_\nu}\int_{4\pi} I_\nu^{(1)}(x_k,\boldsymbol{u}')p_\nu(\boldsymbol{u}'\cdot\boldsymbol{u})\,d\Omega'. \tag{9}$$

It is worth noticing that the zero-order solution of the RTE has been used in the first term of the first member of Eq. (6); indeed, the contribution of the perturbation order larger than one is neglected. The only contribution to the radiative flux in Eq. (5) is issued from $I_\nu^{(1)}(x_k,\boldsymbol{u})$. The solution at first-order perturbation $I_\nu^{(1)}$ of the implicit Eq. (6) is proportional to the opposite of the first term of the first member of Eq. (6). Consequently the radiative flux, given by Eq. (5), can be written

$$q_j^{\rm F}(x_k) = -k\frac{\partial T}{\partial x_j}(x_k), \tag{10}$$

which is a radiative Fourier law, characterized by a radiative conductivity k. Under the previously considered assumptions, it is given by

$$k = \frac{4\pi}{3}\int_0^\infty \frac{n_\nu^2}{\kappa_\nu + \sigma_\nu(1-g_\nu)}\frac{\partial I_\nu^\circ}{\partial T}[T(x_k)]\,d\nu, \tag{11}$$

as established in Ref. [22] and Appendix B of Ref. [11]; g_ν is the asymmetry parameter of scattering defined by

$$g_\nu = \frac{1}{4\pi}\int_{4\pi} p_\nu(\boldsymbol{u}'\cdot\boldsymbol{u})\cos\theta\; d\Omega. \tag{12}$$

For determining the validity criterion of the radiative Fourier law in the core of the semitransparent medium, the study is limited in the following to a slab of very large absorption optical thickness, characterized by a monodimensional temperature field $T(x)$. The geometry is axisymmetrical of axis Ox; θ is the angle of a current direction $\boldsymbol{u}$ with Ox. The physical quantity that is useful in radiative heat transfer is the radiative power per unit volume, calculated from the Fourier law

$$P^{\mathrm{F}} = \frac{d}{dx}\left(k\frac{dT}{dx}\right). \tag{13}$$

B. Research of criterion

In the simple case of an emitting and absorbing, but not scattering, semitransparent medium, the physical validity condition of the Fourier law in any propagation direction is that the transport term of Eq. (8) (i.e., the first term of its first member) is small in front of the other terms, in particular the emission term; that is, in dimensional quantities

$$n_\nu^2 \cos\theta \frac{dI_\nu^\circ}{dT}(T)\frac{dT}{dx_j}(x_k) \ll \kappa_\nu n_\nu^2 I_\nu^\circ(T). \tag{14}$$

The corresponding hemispherical fluxes are obtained by multiplying each member of Eq. (14) by $\cos\theta$ and integrating them over the half space, associated with $\cos\theta > 0$; that is,

$$\frac{2\pi}{3}\frac{dI_\nu^\circ}{dT}(T)\frac{dT}{dx_j}(x_k) \ll \kappa_\nu \pi I_\nu^\circ(T). \tag{15}$$

By integrating over the whole spectrum, Eq. (15) becomes

$$\frac{1}{T}\frac{dT}{dx} \ll \overline{\kappa}^{\mathrm{P}}(T), \tag{16}$$

where $\overline{\kappa}^{\mathrm{P}}(T)$ is Planck's mean absorption coefficient [18,22].

We consider now an emitting, absorbing, and scattering semitransparent medium, of refractive index n equal to unity, because n does not play any role in Eqs. (15) and (16). The physical role of scattering is only indirect in the considered conditions. The emission-absorption phenomena, which still remain the keys of the local radiation thermalization, are now considered along all the path of a ray, which possibly undergoes multiple scattering phenomena, before total absorption. This broken path length, denoted l_{a}, is shown in Fig. 1. Due to scattering phenomena, the effective distance EA traveled up to total absorption is drastically reduced as shown in Fig. 1, and the size of the smallest effective optically thick volume element is now smaller than δ. We consider only monodimensional transfer between elementary layers k and q (see Fig. 2); therefore only the projection $l_{\mathrm{a}}^{\mathrm{eff}}$ of the distance EA over the x axis is considered. At this step, we estimate that an effective absorption coefficient κ^{eff} accounting for multiple scattering phenomena can be associated with an averaged value of $l_{\mathrm{a}}^{\mathrm{eff}}$.

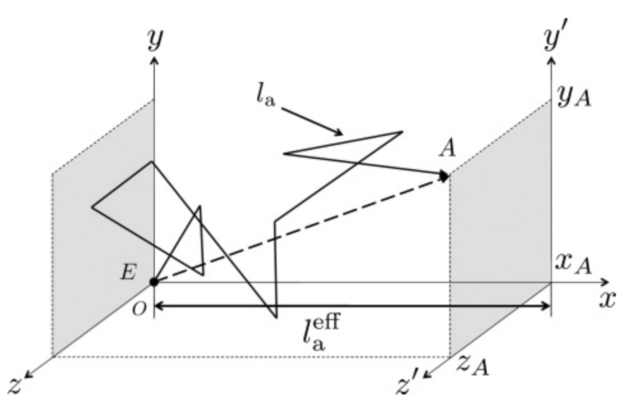

FIG. 1. Emitting, absorbing, and scattering medium; broken path of a power bundle from emission in E, up to absorption in A; l_{a}: total length of the path before absorption; $l_{\mathrm{a}}^{\mathrm{eff}}$: effective traveled distance along x axis.

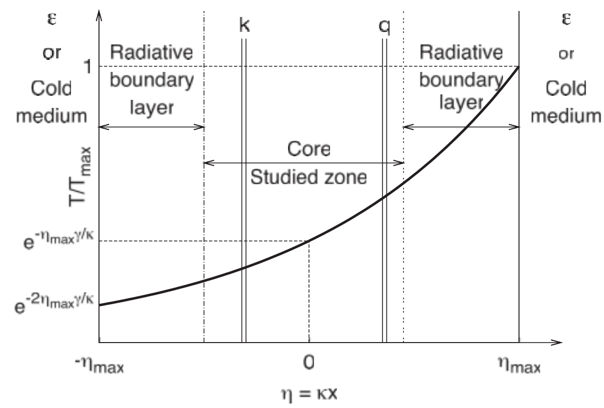

FIG. 2. Considered system and temperature profile. k and q denote the cell indexes.

On the basis of the previous analysis, we research, especially for a scattering medium, a quantitative criterion that generalizes Eq. (16), of the type

$$\frac{1}{\kappa^{\mathrm{eff}}}\frac{1}{T}\frac{dT}{dx} \ll 1, \tag{17}$$

where κ^{eff} has to be determined as a function of the absorption coefficient κ, the albedo ω, and the scattering phase function of the medium. More precisely, we search then to express $(P - P^{\mathrm{F}})/P$, the relative accuracy on the radiative power per unit volume P, as a function of the local parameter χ

$$\chi = \frac{1}{\kappa^{\mathrm{eff}}}\frac{1}{T}\frac{dT}{dx}. \tag{18}$$

If the previous physical analysis is true and κ^{eff} has a physical meaning, the quantitative validity criterion of the Fourier law should not vary in the core of any semitransparent medium characterized by a uniform value of the logarithmic derivative of the temperature field (i.e., by an exponential temperature profile). For this reason we study in the following, as shown in Fig. 2, the temperature fields of the type

$$\frac{T(x)}{T_{\max}} = \exp\left[\gamma\left(x - \frac{\eta_{\max}}{\kappa}\right)\right] \quad \text{with} \quad \gamma = \frac{1}{T}\frac{dT}{dx}, \tag{19}$$

in slabs of absorption optical thickness κL, equal to $2\eta_{\max}$, where L is the width of the slab; x varies in the range $[-\eta_{\max}/\kappa; \eta_{\max}/\kappa]$. It is worth noticing that for a given medium characterized by κ^{eff}, the exponential temperature profile characterized by γ and given by Eq. (19) is, at any point x, the limit associated with a given value μ of $(P - P^{\mathrm{F}})/P$. Real temperature profiles are not exponential. A temperature profile that verifies the criterion $(P - P^{\mathrm{F}})/P < \mu$ must exhibit, at any point x, a slope weaker than the one of this exponential profile.

III. REFERENCE AND APPROXIMATED MODELS

The aim of this section is to build an accurate reference database of the radiative power field within the considered slab in order to define the validity criterion of the radiative Fourier law from comparisons with results issued from this law. All of the analysis is based on Planck's average absorption coefficient; consequently, calculations will be carried out in the following by considering a gray slab characterized by an absorption coefficient κ, an albedo ω, and an asymmetry parameter g. For the sake of simplicity, the scattering phase function p is chosen of the Henyey-Greenstein type:

$$p(\theta_{\rm sc}) = \frac{1-g^2}{(1-2g\cos\theta_{\rm sc}+g^2)^{3/2}}, \tag{20}$$

where $\theta_{\rm sc}$ is the scattering angle. Consequently the study will be carried out with the temperature field, defined by Eq. (19) vs κ, ω, and g.

The slab of absorption optical thickness equal to $2\eta_{\max}$ is discretized in N volume elements i of width d equal to $2\eta_{\max}/(N\kappa)$. Since this study is focused to the core of the medium, we will not carry out an exhaustive study of the boundary conditions. Nevertheless, two types of radiative boundary conditions will be considered: (i) gray opaque walls of emissivity ε, characterized by a diffuse reflection law, in the restrictive case of a purely absorbing medium and (ii) in all cases, external cold and passive media (i.e., absorbing but nonemitting and nonscattering media). The temperature at the boundaries are deduced from Eq. (19), that is, $T_{\max}$ and $T_{\max}\exp(-2\eta_{\max}\gamma/\kappa)$.

A. Reciprocal Monte Carlo approach

The reference model for calculating the radiative power within the considered system is a fully reciprocal Monte Carlo transfer approach, based on a huge number of realizations of optical paths. The reciprocity principle is valid for an emitting, absorbing, and scattering medium with or without partially reflecting walls [23]. It states that the ratio of $\mathcal{P}^{\rm ea}_{kq}$, power emitted by a cell k (volume or surface element) and absorbed by another cell q, to $\mathcal{P}^{\rm ea}_{qk}$, power emitted by q and absorbed by k, is given, under the considered gray assumptions, by

$$\frac{\mathcal{P}^{\rm ea}_{kq}}{\mathcal{P}^{\rm ea}_{qk}} = \left(\frac{T_k}{T_q}\right)^4. \tag{21}$$

The radiative power in a cell q is then

$$\mathcal{P}_q = \sum_{k=1}^{N}\left(\mathcal{P}^{\rm ea}_{kq} - \mathcal{P}^{\rm ea}_{qk}\right). \tag{22}$$

In Eq. (22), the summation includes the two possible gray boundary walls; it is limited to the N volume elements, when the external medium is cold.

Any realization r is characterized by a developed broken path from a source point O. The probability of scattering between the curvilinear abscissa s and $s+ds$ is given by

$$d\Gamma_{\rm sc} = \exp\left[-\int_0^s \kappa(s')\,ds'\right] \times \left\{\exp\left[-\int_0^s \beta\omega(s')\,ds'\right]\beta\omega(s)\,ds\right\}. \tag{23}$$

The absorption and scattering events are statistically independent; therefore, absorption can be treated in a deterministic manner and scattering can be stochastically treated [22]. More precisely, for a given realization, a point where a scattering phenomenon occurs is stochastically determined by using the scattering distribution function $f_{\rm sc}(s)$; $f_{\rm sc}(s)\,ds$ is the last factor in braces of Eq. (23). The distance $l_{\rm sc}$ at which the next scattering phenomenon occurs is obtained by solving the equation $\int_0^{l_{\rm sc}} f_{\rm sc}(s)\,ds = a$, in which a is a randomly generated number in the range [0; 1], according to a uniform law [18,22]. Along a given path, multiple scattering phenomena can occur before total absorption. Any scattering direction $(\theta_{\rm sc},\varphi_{\rm sc})$ is also stochastically characterized from the scattering phase function, that is, in fact, a scattering probability. Consequently, the optical path associated with a realization r is characterized by a set of consecutive segments; each of them, from the second one, begins and ends with a reflection or a scattering phenomenon. Each segment is divided in elements associated with crossing of cells k, characterized by assumed uniform conditions; c stands for the segment index and F_c the point associated with an impact on the cell k or a scattering event in the cell k. $\tau_r(EF_c)$ is then the cumulated transmissivity from the emission point of the path E to the reflection or scattering point F_c. The quantity α_{kcr}, equal to $1-\exp(-\kappa l_{kc})$, is the absorptivity associated with a segment of length l_{kc}; l_{kc} is the distance within the cell k, from the point F_c, that is an entry point or a scattering point, to the next scattering point or exit point, as shown in Fig. 3.

In these conditions $\mathcal{P}^{\rm ea}_{qk}$, that appears in Eq. (21), is calculated by a stochastic approach, at the limit of a large number N_r of realizations. For any realization r, all the power $\mathcal{P}^{\rm e}_q$ is assumed emitted by a cell q, and we obtain

$$\mathcal{P}^{\rm ea}_{qk} = \frac{1}{N_r}\sum_{r=1}^{N_r}\mathcal{P}^{\rm ea}_{qkr}, \tag{24a}$$

$$\mathcal{P}^{\rm ea}_{qkr} = \mathcal{P}^{\rm e}_q\left(\sum_{c=1}^{M_c}\tau_r(EF_c)\alpha_{kcr}\right). \tag{24b}$$

It is worth noticing that the pseudorandom number generator used here is the Mersenne Twister algorithm (MT19937), based on the multiple-recursive matrix method and developed by Matsumoto and Nishimura [24]. This algorithm provides a huge period of 2^{19937} that is suitable for Monte Carlo simulations.

The previous calculations lead to the stochastic determination of $\mathcal{P}^{\rm ea}_{kq}$, and consequently of $\mathcal{P}^{\rm ea}_{qk}$, by using the reciprocity principle given by Eq. (21). In practice, when the external medium is cold and nonscattering ($T=0$), which corresponds

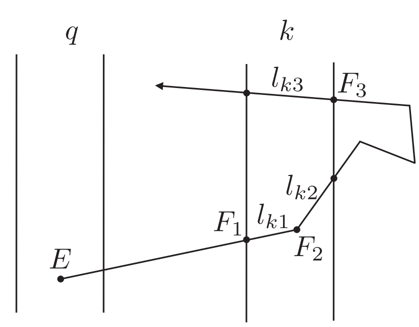

FIG. 3. Path of a power bundle with scattering.

to the strongest limitation of the Fourier law, a universal and symmetrical function $\mathcal{P}_{kq}^{ea}$ has been built with 8×10^6 realizations from the same cell, and cut for an absorption optical thickness η_{max} equal to 10. The relative standard deviation $\sigma(\mathcal{P}_{kq}^{ea})/\mathcal{P}_{kq}^{ea}$ is typically equal to 2×10^{-3} for a number of volume cell equal to 1000. When the boundaries are opaque gray walls, specific functions $\mathcal{P}_{kq}^{ea}$ have been calculated, that take into account the reflection by these walls.

Obviously, all the calculations have been carried out by using nondimensional quantities. T_{max} has been chosen as reference temperature. The dimensional reference for the flux in any boundary cell or the power per unit area is equal to $4\kappa d \sigma_S T_{max}^4$, where σ_S is the Stefan-Boltzmann constant. As the width d of any cell is equal to $2\eta_{max}/(N\kappa)$, this reference is $8\eta_{max}\sigma_S T_{max}^4/N$.

B. Reference model for an emitting and absorbing medium

The monodimensional radiative power field of an emitting, absorbing, nonscattering semitransparent medium slab, surrounded by gray opaque walls of emissivity ε, is given by an analytical expression [22]

$$P^{An} = 4\kappa\sigma_S T_{max}^4 \left\{ \frac{1}{2}E_2(\eta_{max}-\eta)\int_0^\infty \frac{\pi I_{1\nu}(\eta_{max})}{\sigma_S T_{max}^4}d\nu + \frac{1}{2}E_2(\eta)\int_0^\infty \frac{\pi I_{2\nu}(-\eta_{max})}{\sigma_S T_{max}^4}d\nu + \frac{1}{2}\int_{-\eta_{max}}^{\eta} E_1(\eta-\eta')\left[\frac{T(\eta')}{T_{max}}\right]^4 d\eta' + \frac{1}{2}\int_{\eta}^{\eta_{max}} E_1(\eta'-\eta)\left[\frac{T(\eta')}{T_{max}}\right]^4 d\eta' - \left[\frac{T(\eta)}{T_{max}}\right]^4 \right\}, \tag{25}$$

where $I_{1\nu}$ and $I_{2\nu}$ are the intensities leaving the boundaries 1 and 2, respectively; they are given by

$$\int_0^\infty \pi I_{j\nu}(\zeta_j)d\nu = \frac{\sigma_S}{1-[(1-\varepsilon)E_3(2\eta_{max})]^2}\left\{ \varepsilon T_j^4 + \varepsilon(1-\varepsilon)E_3(2\eta_{max})T_i^4 + 2\kappa(1-\varepsilon)^2E_3(2\eta_{max})\int_{-\eta_{max}}^{\eta_{max}} E_2(\eta'-\zeta_j)T^4(\eta')d\eta' + 2\kappa(1-\varepsilon)\int_{-\eta_{max}}^{\eta_{max}} E_2(\zeta_i-\eta')T^4(\eta')\,d\eta' \right\}, \tag{26}$$

where E_n are the n order exponential integral functions [25] and ζ_j corresponds to $+\eta_{max}$ or $-\eta_{max}$. In the case of cold nonscattering external media the two first terms within the largest brackets in Eq. (25) vanish.

C. Model based on the Fourier law

The power per unit volume P^F associated with the use of the Fourier law, given by Eq. (13), applied to the exponential temperature profile of Eq. (19), is simply

$$P^F = 4\sigma_S T_{max}^4 \frac{16\gamma^2(1-\omega)}{3\kappa(1-\omega g)}\left[\frac{T(x)}{T_{max}}\right]^4. \tag{27}$$

IV. RESULTS AND DISCUSSION

In the following we compare three kinds of solutions: (i) analytical, issued from Eq. (25) for nonscattering media; (ii) obtained by Monte Carlo simulation in Sec. III A, and (iii) issued from Eq. (27) (Fourier law), for scattering and nonscattering media.

A. Emitting and absorbing medium

An example of radiative power per unit volume P within a nonscattering slab of absorption optical thickness $\eta_{max} = 10$, surrounded by an external cold and nonscattering medium, is shown in Fig. 4. The reference Monte Carlo method of Sec. III A exhibits a good agreement with the analytical solution, deriving from Eq. (25); the relative standard deviation of Monte Carlo calculations is around 0.01%. The shapes of these curves in the radiative boundary layers $[-10; -4]$ and $[3; 10]$, where the Fourier law is not valid, are due in the considered case to the fact that the external medium does not emit.

The radiative power per unit volume, corresponding to cases of gray opaque boundaries of different emissivities, is also shown in Fig. 4. The Monte Carlo simulation, carried out for $\varepsilon = 0.8$, fits the analytical solution given by Eq. (25),

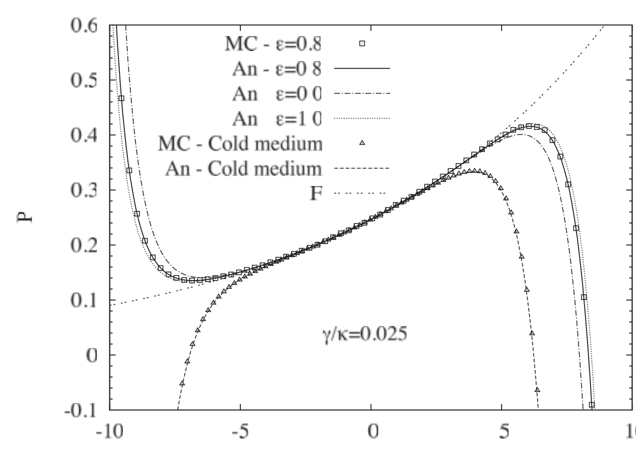

FIG. 4. Purely absorbing medium. Nondimensional radiative power $P/(80n^2\sigma_S T_{max}^4/N)$, vs κx, for an external cold medium ($N = 1000$) or for different values of wall emissivity ($N = 200$). $\gamma/\kappa = \kappa^{-1}(T^{-1}dT/dx) = 0.025$. Calculations from Monte Carlo approach (MC); analytical from Eq. (25) (An), and from the Fourier law (F).

with a relative standard deviation of about 0.005. Calculations based on the Fourier law are also in good agreement with this reference model in the core of the medium. It is worth noticing that the discrepancy from the Fourier law is practically independent of the emissivity value, for all the range [0; 1]. There is no difference between analytical calculations and the Monte Carlo calculations, if we account for the standard deviation of this Monte Carlo calculation of about 0.006, as shown in the case $\varepsilon = 0.8$. The failure of the conduction model in the radiative boundary layers is again highlighted here. The widths of the radiative boundary layers are much smaller in these last cases than in the case of a cold external medium that does not emit. Consequently we only consider, in the following, the more restrictive case of a cold external medium.

B. Emitting, absorbing, and scattering medium

In order to quantify the validity conditions of the radiative Fourier law, the relative discrepancy $(P - P^{\mathrm{F}})/P$ has been exhaustively calculated for an emitting, absorbing, and scattering medium, as a function of the albedo ω varying in the range [0; 0.99] and the asymmetry parameter g varying in the range [−0.99; 0.99]. For instance, $(P - P^{\mathrm{F}})/P$ is plotted as a function of ω in Fig. 5(a), for an isotropically scattering medium ($g = 0$). It clearly appears that $(P - P^{\mathrm{F}})/P$ is uniform in the core of the medium whatever the value of the albedo ω, when $1/T\,dT/dx$ is uniform. This fact is coherent with the criterion given by Eqs. (17) and (18) of Sec. II B. When ω increases, the Fourier law validity criterion is increasingly better verified; indeed, the scattering phenomena shorten the mean free effective path of absorption $\kappa^{\mathrm{eff}-1}$, as shown in Fig. 1; the effective absorption thickness of the medium increases for the same distance, projected in axis x. Therefore, the relative discrepancy profile is flat in this optical thickness range for high albedo values, and the influence of the boundaries on the power within the core of the medium strongly vanishes.

Then the calculations have been performed for an anisotropic scattering, characterized by the asymmetry parameter g. For instance, Fig. 5(b) shows the relative discrepancies as a function of g, for $\omega = 0.5$ and $\gamma = 0.025$. Here again the uniformity of $(P - P^{\mathrm{F}})/P$ in the core of the medium is coherent with the criterion of Sec. II B. The comparison of the different curves of Fig. 5(b) shows that the asymmetry parameter g also strongly influences the validity of the Fourier law. For given values of both the albedo ω and the temperature field parameter γ, the boundary influences are increasingly stronger in the case of forward scattering, when g increases in the range [0; 1]; indeed, the effective free mean path $\kappa^{\mathrm{eff}-1}$ increases vs g. For larger values of the asymmetry parameter g, the cosine of scattered angle θ'_{sc} in the local basis is frequently close to 1 and the absorption effective distance, taking into account scattering $\kappa^{\mathrm{eff}-1}$, tends to κ^{-1}. It is consistent with the fact that the quantity $\kappa + \sigma(1 - g)$ that appears in the radiative conductivity k tends to κ, when g is close to 1. On the contrary, in the case of backward scattering, the effect of boundaries vanishes when g decreases in the range [−1; 0]. Finally, the free mean path $\kappa^{\mathrm{eff}-1}$ is drastically reduced compared to κ^{-1}. It is worth noticing that when the porous medium is characterized by a small value of ω and a large value of g (i.e., g tends to one), the medium is weakly scattering, as illustrated in Tables I and II.

Figure 6(a) shows the relative discrepancy between radiative power calculated from the Fourier law and reference

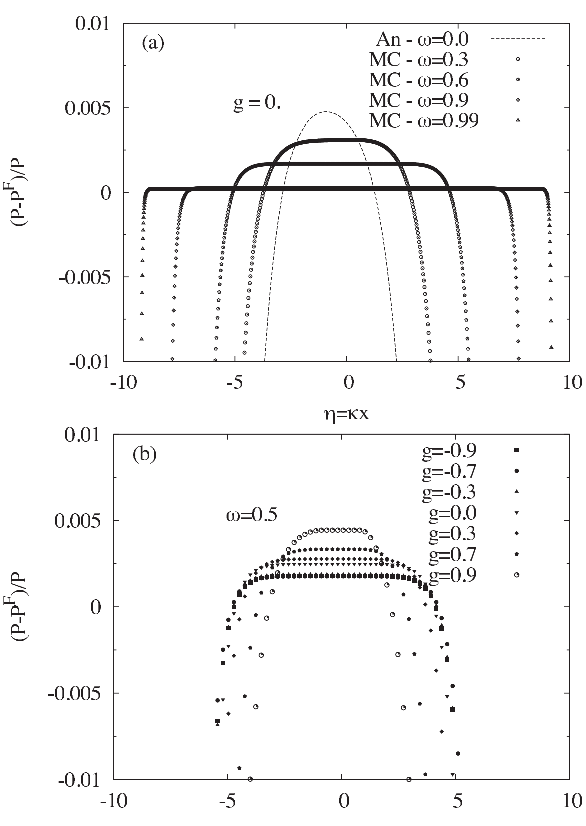

FIG. 5. External cold medium ($N = 1000$). $\gamma/\kappa = 0.025$. Reference calculations from Monte Carlo approach (MC) for a scattering medium, or analytical from Eq. (25) (An) for a nonscattering medium. (a) Relative discrepancy on radiative power vs ω. (b) Relative discrepancy on radiative power vs g.

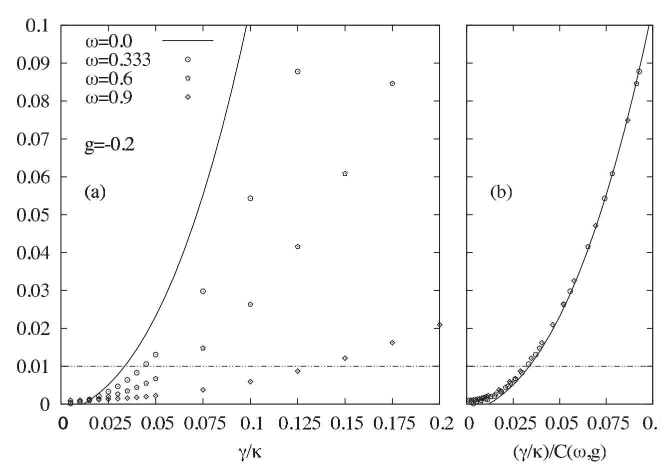

FIG. 6. External cold medium ($N = 1000$). $g = -0.2$. Reference calculations: analytical (nonscattering medium $\omega = 0$); from Monte Carlo approach (scattering medium $\omega \neq 0$). (a) Relative discrepancy on radiative power for different values of ω vs γ/κ. (b) Universal relative discrepancy on radiative power vs $(\gamma/\kappa)/C(\omega,g)$.

TABLE I. Ratio $\kappa^{\rm eff}/\kappa$ vs the scattering albedo ω and the asymmetry parameter of scattering g.

	ω											
g	0.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
−0.99	1.	1.10	1.21	1.35	1.51	1.71	1.98	2.36	2.99	4.45	6.61	16.7
−0.9	1.	1.10	1.21	1.35	1.52	1.74	2.05	2.48	3.23	5.00	7.52	18.0
−0.7	1.	1.10	1.21	1.35	1.53	1.76	2.08	2.54	3.32	5.06	7.46	17.2
−0.5	1.	1.09	1.21	1.34	1.52	1.74	2.05	2.49	3.23	4.85	7.10	16.3
−0.3	1.	1.09	1.19	1.32	1.49	1.70	1.98	2.40	3.08	4.58	6.65	15.1
−0.2	1.	1.08	1.18	1.31	1.46	1.66	1.94	2.33	2.98	4.43	6.41	14.6
−0.1	1.	1.08	1.18	1.29	1.44	1.63	1.89	2.26	2.89	4.26	6.15	13.9
0.0	1.	1.07	1.16	1.28	1.41	1.59	1.84	2.19	2.78	4.08	5.88	13.3
0.1	1.	1.07	1.15	1.26	1.38	1.55	1.78	2.11	2.67	3.89	5.59	12.6
0.2	1.	1.06	1.14	1.24	1.35	1.51	1.72	2.03	2.54	3.69	5.29	11.9
0.3	1.	1.06	1.13	1.21	1.32	1.46	1.66	1.94	2.41	3.47	4.97	11.2
0.5	1.	1.04	1.10	1.16	1.25	1.35	1.51	1.73	2.12	3.00	4.23	9.48
0.7	1.	1.03	1.06	1.11	1.16	1.23	1.34	1.49	1.77	2.41	3.35	7.38
0.9	1.	1.01	1.02	1.04	1.06	1.09	1.13	1.19	1.31	1.62	2.11	4.34
0.99	1.	1.01	1.01	1.01	1.01	1.01	1.02	1.03	1.04	1.08	1.16	1.68

Monte Carlo results, as a function of γ/κ [i.e., $(\kappa T)^{-1}dT/dx$] for different values of ω and $g = -0.2$, at the center of the whole medium ($\eta = 0$). The value $g = -0.2$ is a typical value of backward scattering in a porous medium. The use of the Fourier law is increasingly more valid, when the albedo increases.

The key result is that all of the curves related to scattering media characterized by ω and g, in the ranges [0; 0.99] and [−0.99; 0.99], respectively, are affine and merge with the curve associated with a nonscattering medium ($\omega = 0$), by replacing κ by an effective absorption coefficient $\kappa^{\rm eff} = C(\omega,g)\kappa$. $\kappa^{\rm eff}$ is, in particular, independent of γ, which characterizes the temperature profile. At this step, the analysis of Sec. II B is validated.

The set of values of $C(\omega,g)$ are given in Table I, versus the albedo ω and the asymmetry parameter g. An example is given in Fig. 6(b) for $g = -0.2$. The universal curve of Fig. 6(b) is of practical interest to determine a quantitative validity criterion of the radiative Fourier law. For instance, the qualitative criterion for a relative discrepancy on the radiative power per unit volume, due to the use of the Fourier law, smaller than 0.01, is simply

$$\gamma = \frac{1}{T}\frac{dT}{dx} < 0.033\,\kappa^{\rm eff}(\omega,g), \qquad (28)$$

where $\kappa^{\rm eff}$ is the Planck average of an effective absorption coefficient, which accounts for scattering phenomena (see Sec. II B). The constant of Eq. (28), here equal to 0.033, obviously depends on the required relative precision.

A real temperature profile is generally not exponential. It fulfills the criterion $(P - P^{\rm F})/P < 0.01$, if and only if (i) at any point, its slope is smaller than that of the exponential characterized by $\gamma = 0.033\ \kappa^{\rm eff}(\omega,g)$, (ii) the considered points do not belong to a radiative boundary layer. The width

TABLE II. Ratio $\kappa^{\rm eff}/[\kappa + \sigma(1-g)]$ vs the scattering albedo ω and the asymmetry parameter of scattering g.

	ω											
g	0.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
−0.99	1.	0.90	0.81	0.72	0.64	0.57	0.49	0.42	0.33	0.23	0.17	0.08
−0.9	1.	0.90	0.82	0.74	0.67	0.60	0.53	0.45	0.37	0.27	0.20	0.09
−0.7	1.	0.92	0.85	0.78	0.72	0.65	0.58	0.51	0.42	0.31	0.22	0.10
−0.5	1.	0.94	0.88	0.82	0.76	0.69	0.63	0.55	0.46	0.33	0.24	0.10
−0.3	1.	0.95	0.90	0.85	0.79	0.73	0.67	0.59	0.49	0.36	0.25	0.11
−0.2	1.	0.96	0.91	0.86	0.81	0.75	0.69	0.61	0.51	0.37	0.26	0.12
−0.1	1.	0.96	0.92	0.88	0.83	0.77	0.71	0.63	0.53	0.39	0.28	0.12
0.0	1.	0.97	0.93	0.89	0.85	0.79	0.73	0.65	0.55	0.40	0.29	0.13
0.1	1.	0.97	0.94	0.90	0.86	0.81	0.76	0.68	0.58	0.42	0.30	0.14
0.2	1.	0.98	0.95	0.92	0.88	0.84	0.78	0.70	0.60	0.45	0.32	0.14
0.3	1.	0.98	0.96	0.93	0.90	0.86	0.81	0.73	0.63	0.47	0.34	0.15
0.5	1.	0.99	0.98	0.96	0.93	0.90	0.86	0.80	0.70	0.54	0.40	0.18
0.7	1.	1.00	0.99	0.98	0.97	0.95	0.92	0.88	0.80	0.65	0.50	0.24
0.9	1.	1.00	1.00	1.00	0.99	0.99	0.98	0.97	0.94	0.85	0.72	0.39
0.99	1.	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.98	0.84

of the radiative boundary layer of the whole medium strongly depends on the environment and the type of radiative boundary condition. It can only be determined case by case, as has been done here for gray boundaries or an external cold and nonscattering medium.

It is worth noticing that the criterion of Eq. (28) must be verified at any spatial scale, in particular at scales smaller than $1/\beta$. For instance, for a foam of typical β value equal to 1 mm^{-1}, a typical asymmetry parameter of scattering $g = -0.2$ (see Ref. [26]) and an albedo equal to 0.99, κ^{eff} is equal to 14.6 mm^{-1} and the condition becomes, at 300 K, $dT/dx < 146\ \text{K}\,\text{mm}^{-1}$. More precisely, this condition must be verified at any scale, for instance a temperature variation smaller than 1.5 K for a distance 10 μm. The spatial scale at which temperature is defined in a porous medium could be another limitation.

For degraded rod bundles of a nuclear core at the beginning of a severe accident, the extinction coefficient is typically equal to $0.1\,\text{mm}^{-1}$, with $\omega = 0.2$ and $g = -0.2$. For these radiative conditions, the temperature gradient must not exceed $1550\,\text{K}\,\text{m}^{-1}$ and $6200\,\text{K}\,\text{m}^{-1}$, at 500 K and 2000 K, respectively, but also, at lower scale, $155\,\text{K}\,\text{cm}^{-1}$ and $62\,\text{K}\,\text{cm}^{-1}$, respectively.

It worth noticing, from Tables I and II, that the effective absorption coefficient κ^{eff} accounting for multiple scattering phenomena verifies the inequality

$$\kappa \leqslant \kappa^{\text{eff}} \leqslant \kappa + \sigma(1 - g), \tag{29}$$

where $[\kappa + \sigma(1 - g)]^{-1}$ is the characteristic length, for which a collimated ray becomes isotropic. In particular, for $g = 0$ in Table I, we obtain

$$\kappa \leqslant \kappa^{\text{eff}} \leqslant \beta. \tag{30}$$

Consequently, if $\text{Kn}^{\text{ext}} = (\beta\delta)^{-1}$ has been chosen as a perturbation parameter, $1/\beta$ is not the exact length scale associated with the Fourier law. Moreover, if the conductivity is expressed vs $\kappa + \sigma(1 - g)$ [see Eq. (11)], neither is $[\kappa + \sigma(1 - g)]^{-1}$ the pertinent scale.

V. CONCLUSION

An accurate validity criterion of the radiative Fourier law has been determined for both Beerian and non-Beerian media. It is only expressed versus the local temperature gradient at any point of the medium and an effective absorption coefficient, accounting for possible multiple scattering phenomena. This effective absorption coefficient κ^{eff} has been determined versus the real absorption coefficient κ, the albedo ω, and the scattering asymmetry parameter g of the medium. It is worth noticing that this effective absorption coefficient is larger than the real absorption coefficient and smaller than the quantity $\kappa + \sigma(1 - g)$ that appears in the Fourier law (i.e., smaller than the extinction coefficient β in the case of an isotropic scattering phase function). Neither extinction β nor $\kappa + \sigma(1 - g)$ are the pertinent parameters that govern the validity criterion of the Fourier law.

[1] S. Rosseland, *Theoretical Astrophysics* (Oxford University Press, London, 1936).

[2] D. Baillis and J. F. Sacadura, J. Quant. Spectrosc. Radiat. Transfer **67**, 327 (2000).

[3] D. Baillis, M. Arduini-Schuster, and J. F. Sacadura, J. Quant. Spectrosc. Radiat. Transfer **73**, 297 (2002).

[4] V. A. Petrov, Int. J. Heat Mass Transfer **40**, 2241 (1997).

[5] S.-Y. Zhao, B.-M. Zhang, and X.-D. He, Int. J. Therm. Sci. **48**, 440 (2009).

[6] A. V. Gusarov and E. P. Kovalev, Phys. Rev. B **80**, 024202 (2009).

[7] A. V. Gusarov, Phys. Rev. B **81**, 064202 (2010).

[8] S. A. Slutz, R. O. Gauntt, G. A. Harms, T. Latham, W. Roman, and R. J. Rodgers, J. Propul. Power **10**, 419 (1994).

[9] P. Rubiolo and J.-M. Gatt, Int. J. Therm. Sci. **41**, 401 (2002).

[10] F. Bellet, E. Chalopin, F. Fichot, E. Iacona, and J. Taine, Int. J. Heat Mass Transfer **52**, 1544 (2009).

[11] J. Taine, F. Bellet, V. Leroy, and E. Iacona, Int. J. Heat Mass Transfer **53**, 4071 (2010).

[12] M. Tancrez and J. Taine, Int. J. Heat Mass Transfer **47**, 373 (2004).

[13] K. Furutsu and Y. Yamada, Phys. Rev. E **50**, 3634 (1994).

[14] V. Venugopalan, J. S. You, and B. J. Tromberg, Phys. Rev. E **58**, 2395 (1998).

[15] B. Chen, K. Stamnes, and J. J. Stamnes, Appl. Opt. **40**, 6356 (2001).

[16] K. M. Yoo, F. Liu, and R. R. Alfano, Phys. Rev. Lett. **64**, 2647 (1990).

[17] J. F. Clouet, J. Quant. Spectrosc. Radiat. Transfer **58**, 33 (1997).

[18] R. Siegel, J. R. Howell, and M. P. Menguc, *Thermal Radiation Heat Transfer*, 5th ed. (Taylor and Francis, New York, 2010).

[19] R. G. Deissler, J. Heat Transfer **86C**, 240 (1964).

[20] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).

[21] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, England, 1970).

[22] J. Taine, E. Iacona, and J.-P. Petit, *Transferts Thermiques*, 4th ed. (Dunod, Paris, 2008).

[23] J.-J. Greffet and M. Nieto-Vesperinas, J. Opt. Soc. Am. A **15**, 2735 (1998).

[24] M. Matsumoto and T. Nishimura, in *ACM Transactions on Modeling and Computer Simulation*, edited by L. Devroye, M. G. Montréal (ACM, New York, 1998), Vol. 8, pp. 3–30.

[25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed. (Dover, New York, 1964).

[26] B. Zeghondy, E. Iacona, and J. Taine, Int. J. Heat Mass Transfer **49**, 370230 (2006).