

Partial annealing of a coupled mean-field spin-glass model with an embedded pattern

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A partially annealed mean-field spin-glass model with a locally embedded pattern is studied. The model consists of two dynamical variables, spins and interactions, that are in contact with thermal baths at temperatures T_S and T_J , respectively. Unlike the quenched system, characteristic correlations among the interactions are induced by the partial annealing. The model exhibits three phases: paramagnetic, ferromagnetic and spin-glass. In the ferromagnetic phase, the embedded pattern is stably realized. The phase diagram depends significantly on the ratio of the two temperatures, $n = T_S/T_J$. In particular, a reentrant transition from the embedded ferromagnetic to the spin-glass phase with T_S decreasing is found only below a certain value of n . This indicates that above the critical value n_c the embedded pattern is supported by a local field from a nonembedded region. Some equilibrium properties of the interactions in the partial annealing are also discussed in terms of frustration.

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I. INTRODUCTION

Spin-glass (SG) models [1,2] have been extensively applied to widespread fields for describing random or randomized spin systems. In such a system, there are two dynamical variables, fast and slow variables that are spins and interactions in a spin model, respectively. The latter is usually assumed to be quenched for simplicity and is distributed independently and identically according to a given distribution. This is based on the fact that the relaxation time of the interactions is considerably larger than that of the spins. As a result of the quenched randomness, the state of the interactions is not affected at all by the fluctuation of spins. The randomness of the interactions causes many competitions between the spins, and sometimes there is no way for eliminating them completely. This competition, called frustration, provides rich and interesting phenomena in random systems.

The quenched randomness is an appropriate assumption for a spin-glass problem as a magnetic material. In some interesting systems, however, the fluctuation of the spin variables has a large influence on the interaction variables. For example, the interactions of amino-acid sequences provide a globally stable state in the protein. The existence of such global attraction in the folding process was proposed as a consistency principle [3] or funnel landscape [4]. The characteristics of interactions are not expected in randomly constructed interactions. It is supposed that such characteristics are acquired through an evolutionary process under some fluctuations of the fast variables. In the case of a protein, the fluctuation is due to the folding dynamics of the amino-acid sequences. Similar properties have also been discovered in gene regulatory networks [5] and transcriptional networks [6]. The formation mechanism of the funnel landscape is still not fully understood.

Recently, an adiabatic two-temperature spin model has been studied as a model of evolution by using Monte Carlo

simulations [7,8]. In the model, interactions evolved so as to increase the probability of spins to find a specific pattern of local spins. Interestingly, adapted interactions that exhibit funnel-like dynamics with some robustness have been found only in an intermediate temperature region. This type of interactions has no frustration around the local spins and retains some frustrations in the rest, implying that the local spin pattern is stabilized in the adapted interactions by energetic and entropic effects. Note that there is no explicit driving force for constructing interactions outside the local spins. Unfortunately, since numerical simulations are often hampered by finite-size effects, and thermodynamic properties and phase structure have not yet been understood well. We consider it worthwhile to clarify the nature of such self-organized interactions with an local spin pattern under the thermal fluctuations from the viewpoint of statistical mechanics.

In contrast to the quenched system, feedback from the fast variables to the slow ones is taken into account for studying such systems. The feedback effect is essential for the emergence of some functional feature in the evolutionary process. Under the assumption of complete separation of time scales between the fast and slow variables, the adiabatic elimination of fast variables has been employed in nonequilibrium and nonlinear physics [9,10]. One such approaches is partial annealing [11], introduced by Penney et. al. [12], whereby a nonequilibrium system with the fast and slow variables is mapped onto a particular equilibrium statistical-mechanical system with two temperatures. This partial annealing approach has been applied to many areas: glasses [13], charged systems [14], spin-lattice gases [15], neural networks [16], protein-folding [17], liquid crystals [18], and evolution [19,20].

In this paper, we study a coupled mean-field spin-glass model [21] with partial annealing. In this model, two fully-connected mean-field systems are coupled to each other through spin-glass interactions. One of them is a ferromagnetic system regarded as a local region in which a ferromagnetic pattern is embedded. The other is a spin-glass system providing redundant interactions together with the coupled interactions. We formulate the mean-field theory for this model with partial

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annealing and discuss how the redundant interactions are modified by the thermal fluctuation of spins.

This paper is organized as follows. In Sec. II, we introduce the mean-field spin-glass model and give a theoretical formulation of the model with partial annealing by using the replica method. In Sec. III, we present a phase diagram of the model and discuss the equilibrium state of interactions through frustration. Finally, Sec. IV is devoted to conclusions and providing an outlook for further developments.

II. MODEL AND REPLICA METHOD

We study a coupled mean-field spin-glass model of two fully connected spin systems, which consists of N_1 Ising spins $\{S_{1,i}\} (i = 1, \dots, N_1)$ and N_2 Ising spins $\{S_{2,i}\} (i = 1, \dots, N_2)$. The spin variables of the two systems and the interactions between two spins are denoted by \mathbf{S} and \mathbf{J} for short, respectively. The Hamiltonian of the model is given by

$$H(\mathbf{S}|\mathbf{J}) = - \sum_{p=1}^2 \sum_{i < j} J_{p,ij} S_{p,i} S_{p,j} - \sum_{i,j} J_{3,ij} S_{1,i} S_{2,j}, \quad (1)$$

where the summation in the first term is over all pairs within each spin system and the summation in the second term is over all i and j . $\{J_{1,ij}\}$ and $\{J_{2,ij}\}$, denoted hereafter by \mathbf{J}_1 and \mathbf{J}_2 , respectively, are the intra-interactions within each system and $\{J_{3,ij}\}$, denoted by \mathbf{J}_3 , are inter-interactions between these two systems. A schematic picture of the model is shown in Fig. 1. In a quenched system, they are assumed to be independently and identically distributed according to the Gaussian distribution with mean $J_{0,p}$ and variance J_p ,

$$P_0(J_{p,ij}) = \sqrt{\frac{N_p}{2\pi(J_p)^2}} \exp \left\{ - \frac{N_p}{2(J_p)^2} \left(J_{p,ij} - \frac{J_{0,p}}{N_p} \right)^2 \right\}, \quad (2)$$

where the order of interactions $J_{p,ij}$ is scaled as $N_p^{1/2}$ for keeping the Hamiltonian extensive and N_3 is set to be the geometrical mean of N_1 and N_2 as $N_3 = \sqrt{N_1 N_2}$ [21].

In this paper, we consider that an ordering pattern is embedded in one system (system 1) and no *a priori* pattern is introduced in the other system (system 2). This is represented by a specific case of the model Hamiltonian, in which system 1 and system 2 are pure ferromagnetic and spin-glass systems,

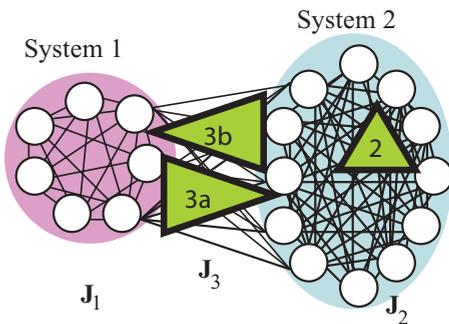


FIG. 1. (Color online) A schematic representation of a coupled mean-field spin-glass model. Spins and interactions are defined on vertexes denoted by circles and edges between vertexes, respectively. The three triangles represent index of frustration parameters, discussed in Sec. III C.

respectively, and they are coupled to each other through spin-glass interactions. Explicitly, the model is given as

$$\begin{aligned} J_{0,1} &= J_0(>0), & J_{0,2} &= J_{0,3} = 0, \\ J_1 &= 0, & J_2 &= J_3 = J(>0). \end{aligned} \quad (3)$$

Note that in the model with these parameters the interactions \mathbf{J}_2 and \mathbf{J}_3 are expected to be modified by the partial annealing while \mathbf{J}_1 is kept fixed to the pure ferromagnetic interaction. A size ratio r between the two systems is defined as

$$N_1/N_2 = 1/r^2, \quad (4)$$

yielding $N_1 = N/(1+r^2)$ and $N_2 = Nr^2/(1+r^2)$ with a total number of spins $N = N_1 + N_2$. The two limiting cases $r = 0$ and $r = \infty$ correspond to the Husimi-Temperley model and the Sherrington-Kirkpatrick model, respectively.

In the partial annealing system, the interactions \mathbf{J} as well as the spin variables \mathbf{S} are treated as dynamical variables. Time scales associated with \mathbf{J} are extremely slow and it is assumed that the time scale is separated from that of the spin variables. Then, the equilibrium distribution of the spins at an inverse temperature $\beta_S = 1/T_S$ is given by

$$P(\mathbf{S}|\mathbf{J}) = \frac{1}{Z(\mathbf{J})} \exp[-\beta_S H(\mathbf{S}|\mathbf{J})], \quad (5)$$

where $Z(\mathbf{J})$ is a partition function under a given \mathbf{J} . The distribution function of \mathbf{J} at an inverse temperature $\beta_J = 1/T_J$, different from β_S , is given by

$$P(\mathbf{J}) = \frac{1}{Z} \exp(-\beta_J H_J), \quad (6)$$

where H_J is a Hamiltonian of \mathbf{J} and Z is the total partition function. The Hamiltonian of \mathbf{J} is generally expressed in terms of equilibrium quantities of \mathbf{S} and the bare distribution of Eq. (2). Although the explicit form of H_J can be arbitrarily chosen, in this study, as in [11,12], we set it as

$$H_J(\mathbf{J}) = F(\mathbf{J}) - T_J \log P_0(\mathbf{J}), \quad (7)$$

where $F(\mathbf{J})$ is the free energy defined by $F(\mathbf{J}) = -T_S \log Z(\mathbf{J})$ and $P_0(\mathbf{J}) = \prod_{p=1}^3 \prod_{i < j} P_0(J_{p,ij})$.

Then, the equilibrium distribution $P(\mathbf{J})$ and the total partition function are rewritten as

$$P(\mathbf{J}) = \frac{1}{Z} P_0(\mathbf{J}) Z^n(\mathbf{J}), \quad (8)$$

$$Z = [Z^n(\mathbf{J})]_0, \quad (9)$$

where n is the ratio between two temperatures, $n = T_S/T_J$, and $[\dots]_0$ means the average over \mathbf{J} according to the bare distribution $P_0(\mathbf{J})$. When $n = 0$, the distribution $P(\mathbf{J})$ is identical to $P_0(\mathbf{J})$ and the system corresponds to the quenched one. For finite n and β_J , the interactions \mathbf{J} with a lower free energy likely occur.

In the quench limit, the model is reduced to that studied by Takayama [21,22]. The coupled mean-field model has been introduced in order to study inhomogeneity of interactions between spins in real SG materials [21,22]. Thus, the previous studies focused attention only on the quenched system. In this work, we rather pay attention to the system with partial annealing, in which the interactions except for the embedded one in system 1 are adiabatically affected by fluctuations of

spins. Our main purpose is to study the partial annealing effect on the stability of the embedded ferromagnetic ordering in system 1 and to clarify characteristics of the resultant interactions by partial annealing.

The total free energy per spin f at two inverse temperatures β_S and β_J can be written as

$$f(T_S, T_J) = -\frac{1}{\beta_S} \lim_{N \rightarrow \infty} \frac{1}{N} \log[Z^n(\mathbf{J})]_0. \quad (10)$$

Following the standard procedure of the replica method [1], the quantity $[Z^n]_0$ is calculated for a positive integer n and an analytic continuation to a real value given by two temperatures is taken after the calculation. Within the assumption of replica symmetry (RS), the free-energy density is described in terms of order parameters q_1, q_2 , and m_1 , and their conjugate parameters \hat{q}_1, \hat{q}_2 , and \hat{m}_1 , as

$$\begin{aligned} f(T_S, T_J) = & -T_S \left[-\frac{n(n-1)}{2} \sum_{p=1}^2 n_p q_p \hat{q}_p - n_1 n m_1 \hat{m}_1 \right. \\ & + \frac{J_0 \beta_S n_1 n}{2} m_1^2 + \frac{J^2 \beta_S^2 n_2 n (n-1)}{4} q_2^2 + \frac{J^2 \beta_S^2 n_2}{4} \\ & + \frac{\hat{n}_3 J^2 \beta_S^2 n (n-1)}{2} q_1 q_2 + \frac{J^2 \beta_S^2 n \hat{n}_3}{2} \\ & + n_1 \log \int Dz 2 \cosh^n(\sqrt{\hat{q}_1 z} + \hat{m}_1) - \frac{\hat{q}_1 n n_1}{2} \\ & \left. + n_2 \log \int Dz 2 \cosh^n(\sqrt{\hat{q}_2 z}) - \frac{\hat{q}_2 n n_2}{2} \right], \quad (11) \end{aligned}$$

where $n_p = N_p/N$ ($p = 1, 2$), $\hat{n}_3 = N_3/N$, and $\int Dz = \int_{-\infty}^{+\infty} dz e^{-z^2/2} / \sqrt{2\pi}$. The order parameters follow the self-consistent equations,

$$q_p(T_S, n) = \frac{\int Dz \tanh^2 \Xi_p \cosh^n \Xi_p}{\int Dz \cosh^n \Xi_p} \quad (12)$$

and

$$m_1(T_S, n) = \frac{\int Dz \tanh \Xi_1 \cosh^n \Xi_1}{\int Dz \cosh^n \Xi_1}, \quad (13)$$

where $\Xi_1 = \sqrt{\hat{q}_1 z} + \hat{m}_1$ and $\Xi_2 = \sqrt{\hat{q}_2 z}$. The conjugate parameters, \hat{q}_1, \hat{q}_2 , and \hat{m}_1 , are then given by

$$\hat{q}_1 = \beta_S^2 J^2 (r q_2), \quad \hat{q}_2 = \beta_S^2 J^2 (q_1/r + q_2), \quad \hat{m}_1 = \beta_S J_0 m_1. \quad (14)$$

By solving the self-consistent equations we have the following solutions: a paramagnetic solution ($q_1 = q_2 = m_1 = 0$), a ferromagnetic one ($q_1 > 0, q_2 > 0, m_1 > 0$), and a spin-glass one ($q_1 > 0, q_2 > 0, m_1 = 0$). The transition temperatures between two of the phases corresponding to these solutions are given by

$$T_S^{\text{FM-SG}} = J_0 [1 + (n-1)q_1], \quad (15)$$

$$T_S^{\text{PM-SG}} = \sqrt{(1 + \sqrt{5})/2} J \equiv \sqrt{\alpha} J, \quad (16)$$

$$T_S^{\text{PM-FM}} = J_0, \quad (17)$$

where PM, FM, and SG mean paramagnetic, ferromagnetic, and spin-glass phase, respectively. The phase boundary

between the paramagnetic phase and the spin-glass one and that between the paramagnetic phase and the ferromagnetic one are independent of r and n . Thus, the multicritical point located at $(J_0/J, T_S/J) = (\sqrt{\alpha}, \sqrt{\alpha})$ in the $J_0/J - T_S/J$ plane is independent of r and n . This is consistent with the results of previous work in the quench limit [21].

By following the stability analysis of the RS solutions of de Almeida and Thouless (AT) [23], the stability condition is derived as

$$1 - \beta_S^2 J^2 (1 - 2q_2 + r_2) - \beta_S^4 J^4 (1 - 2q_1 + r_1) \times (1 - 2q_2 + r_2) > 0, \quad (18)$$

where

$$r_p = \frac{\int Dz \tanh^4 \Xi_p \cosh^n \Xi_p}{\int Dz \cosh^n \Xi_p} \quad (p = 1, 2). \quad (19)$$

III. RESULTS

A. Phase diagram

As shown in the previous section, the transition temperature $T_S^{\text{FM-SG}}$ between the ferromagnetic and spin-glass phases depends on the characteristic parameters r and n of our model. Here we carefully discuss the r and n dependence of the phase boundaries. To completely obtain the phase boundary of $T_S^{\text{FM-SG}}$, we should numerically solve the self-consistent equations (12) and (13). Some limiting cases around the multicritical point and near $T_S = 0$ can be argued by an expansion of the order parameters.

At sufficiently low T_S , the spin-glass order parameters q_p for $p = 1$ and 2 behave as

$$q_p(n) \simeq \begin{cases} 1 - 4 \exp[-2\beta_S^2 J^2 c_p (n-1)], & \text{for } n > 1, \\ 1 - \frac{\exp(-\beta_S^2 J^2 n^2 c_p/2)}{(1-n)\beta_S J \sqrt{c_p}} & \text{for } n < 1, \end{cases} \quad (20)$$

where $c_1 = r$ and $c_2 = 1 + r^{-1}$. They decrease from 1 exponentially in β_S for $n > 1$ and linearly in T_S for $n < 1$. Substituting them into Eq. (15), we find that $T_S^{\text{FM-SG}} = nJ_0$ around $J_0 = T_S = 0$, irrespective of r , and that the RS solution at $T_S = 0$ always satisfies the stability condition (18). Therefore, at $T_S = 0$ the ferromagnetic phase stably exists for any J_0 and $n > 0$ and it vanishes at $n = 0$. The latter is recovered in the quench limit studied in [21]. At sufficiently low T_S , the partial annealing effect yields ferromagnetic ordering even for weak J_0 and for small ratio r by appropriately selecting \mathbf{J}_2 and \mathbf{J}_3 . The stability of the ferromagnetic phase near $T_S = 0$ is a particular feature of the partial annealing system of the coupled mean-field model.

Near the multicritical point, the spin-glass order parameters can be expressed as

$$q_1(n) \simeq \begin{cases} \frac{2(2\alpha-1)}{(2-n)(\alpha-1)(\alpha c_2-1)} (T_S - \sqrt{\alpha}) & \text{for } n < 2, \\ \sqrt{\frac{6r^2(2\alpha-1)}{r^2(\alpha-1)+\alpha^3}} (T_S - \sqrt{\alpha})^{1/2} & \text{for } n = 2, \end{cases} \quad (21)$$

$$q_2(n) \simeq \frac{q_1(n)\alpha}{r}. \quad (22)$$

The critical exponent of the spin-glass order parameter is 1 at $n < 2$ and 1/2 at $n = 2$. For $n > 2$, the order parameter is

difficult to obtain by an expansion because the transition is of first order. The phase boundary around the multicritical point significantly depends on r , in contrast to that around $T_S = 0$.

The phase diagrams at $n = 2$ for $r = 1/3$ and $r = 3$ are shown in Fig. 2 as an example for $n > 1$. The obtained phase diagram weakly depends on r ; for $J_0/J > \sqrt{\alpha}$, as T_S decreases the transition from the paramagnetic to the ferromagnetic phase occurs at $T_S^{\text{PM-FM}}$, and for $J_0/J < \sqrt{\alpha}$, the spin-glass phase appears at $T_S^{\text{FM-SG}} < T_S < T_S^{\text{PM-SG}}$. All the phases found for $n > 1$ fulfill the stability condition (18). In particular, the obtained spin-glass phase is correctly described by the RS solution. It is interesting to see that the region of the spin-glass phase for $r = 3$ is reduced as compared with that for $r = 1/3$. More concretely, the transition temperature $T_S^{\text{FM-SG}}$ around the multicritical point for $r = 1/3$ is lower than that for $r = 3$. This is a counterintuitive result because in the case with $r = 3$ the majority spins in system 2 connect with each other through the spin-glass interactions while the majority spins are ferromagnetically coupled in system 1 in the case with $r = 1/3$. We shall discuss this point later.

In addition to the phases found for $n > 1$, other phases appear for $n < 1$: These are a mixed phase characterized by the AT instability with $m_1 > 0$ and a spin-glass phase with replica symmetry broken (RSB). The phase diagram at $n = 0.1$ as a typical example for $n < 1$ is shown for $r = 1/3$ and $r = 3$ in Fig. 3. The mixed phase is found between the ferromagnetic and spin-glass phases and the region of the mixed phase enlarges with increasing r . The region of the

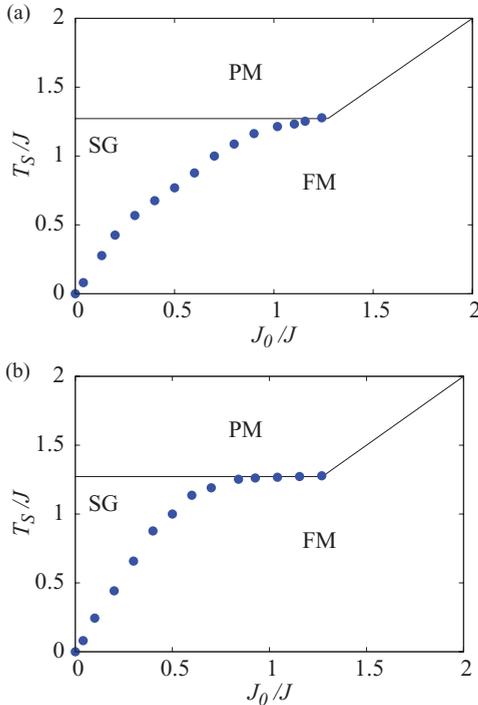


FIG. 2. (Color online) Phase diagram of a coupled mean-field model for (a) $r = 1/3$ and (b) $r = 3$ with the ratio of two temperatures being $n = 2$. The notations PM, FM, and SG mean the paramagnetic, ferromagnetic, and spin-glass phases, respectively. The points \bullet are the transition temperature $T_S^{\text{FM-SG}}$ obtained by solving the saddle-point equations numerically.

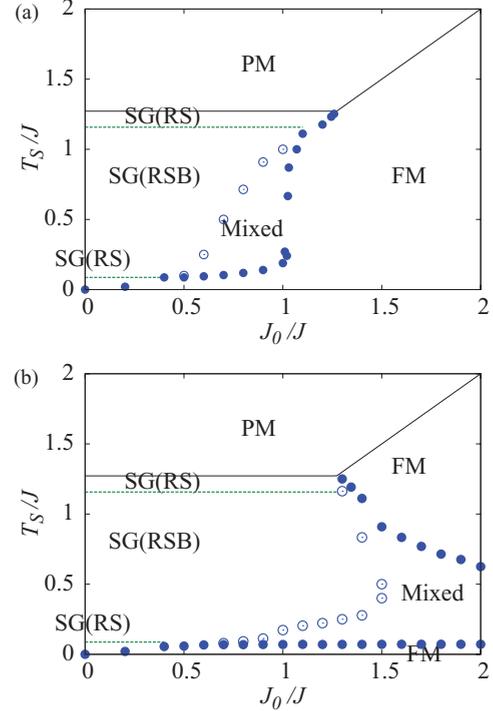


FIG. 3. (Color online) Phase diagram of a coupled mean-field model with $n = 0.1$ for (a) $r = 1/3$ and (b) $r = 3$. The open and closed circles represent the phase boundary between RSB SG and mixed phases, and between FM and mixed phases, respectively. The dashed line is a boundary between RS SG and RSB SG phases.

spin-glass phase also enlarges with r and a broken replica symmetric phase exists in the intermediate- T_S region in the spin-glass phase, in contrast to that observed for $n = 2$. For $r = 3$, a reentrant transition (PM \rightarrow FM \rightarrow Mixed) occurs at $J_0/J > \sqrt{\alpha}$. Furthermore, the transition from the spin-glass or mixed phase to the ferromagnetic phase occurs at lower T_S , because the ferromagnetic phase is always stable at $T_S = 0$ in our model for any finite n . Hence, we can see three successive transitions, PM \rightarrow FM \rightarrow Mixed \rightarrow FM for $J_0/J > \sqrt{\alpha}$.

B. Reentrant transition

While a characteristic feature of our model is found near $T_S = 0$, another feature is present around the multicritical point in the phase diagram. As seen in Fig. 3(b), a reentrant transition from the ferromagnetic to mixed and spin-glass phases occurs near the multicritical point for $J_0/J > \sqrt{\alpha}$. This means that the embedded ferromagnetic ordering of system 1 described by the RS solution becomes unstable as T_S decreases. Such a reentrant transition has already been reported for the quenched system with $n = 0$ [21]. In this work, we study a partial annealing effect, namely with finite n , on the stability of the embedded ferromagnetic ordering. The gradient of $T_S^{\text{FM-SG}}$ at the multicritical point is regarded as an indicator of the ‘‘reentrant transition.’’ Namely, the negative slope of $T_S^{\text{FM-SG}}$ at the multicritical point implies the existence of the reentrant transition, although the existence of a reentrant transition at a temperature lower than the multicritical point is not completely ruled out even in the case with positive slope of $T_S^{\text{FM-SG}}$.

The reentrant transition can occur when the condition

$$\left. \frac{dJ_0^c(T_S)}{dT_S} \right|_{\text{MCP}} < 0 \quad (23)$$

is satisfied; here $J_0^c(T_S) = T_S/[1 + (n-1)q_1(T_S)]$ and ‘‘MCP’’ means the multicritical point. Note that the order parameter q_1 is a function of T_S . The RS solution for q_1 is sufficient for evaluating the boundary, because the RS solution is always stable at the multicritical point. From Eq. (23), the region of r where the reentrant transition occurs is derived as

$$r > r_c(n), \quad \text{for } n < n_c \simeq 0.8396 \dots, \quad (24)$$

where the critical ratio $r_c(n)$ is given by

$$r_c(n) = \frac{(2-n)(1+\alpha)}{2(1-n)(2\alpha-1) - (2-n)(\alpha-1)}. \quad (25)$$

For $n > n_c$, the reentrant transition occurs at $r < r_c(n)$ with $r_c(n)$ being a negative value; hence it is unphysical. The derivation of $r_c(n)$ is based on the assumption that the transition is of second order. Certainly the transition is of second order for $n \leq 2$, which is larger than n_c . Therefore, the reentrant transition occurs only for $n < n_c$. We show the n dependence of $r_c(n)$ in Fig. 4. At smaller n , the reentrant transition occurs at smaller r . Eventually, r_c reaches $1.618 \dots$ in the quench limit $n = 0$, which is consistent with the results in Ref. [21]. Furthermore, it is found that the gradient $dJ_0^c(T_S)/dT_S$ at the multicritical point is a monotonically increasing function of r for $n > n_c$, and this fact yields the shrinking of the spin-glass phase as r increases, as shown in Fig. 2. Thus, the stability of the embedded ferromagnetic ordering of system 1 depends quantitatively on the value of the relative temperature $n = \beta_J/\beta_S$.

Here, we argue a typical feature of the partially annealed interaction \mathbf{J}_3 from the obtained phase diagram and the critical ratio $r_c(n)$. In equilibrium of the partial annealing system, the interactions J_{ij} are in general proportional to $n\langle S_i S_j \rangle$ given as a most probable value in $P(\mathbf{J})$. Supposing that the order parameter q_2 of system 2 has a finite value at sufficiently low temperature T_S , the spins in system 1 are subjected to an effective field $r\sqrt{q_2}$ caused by system 2 through the interactions \mathbf{J}_3 . For $n < n_c$, the interactions \mathbf{J}_3 are considered to be almost random according to the bare distribution $P(\mathbf{J}_0)$, and hence an effective field is a random field for system 1. This random effective field also does not favor the ferromagnetic

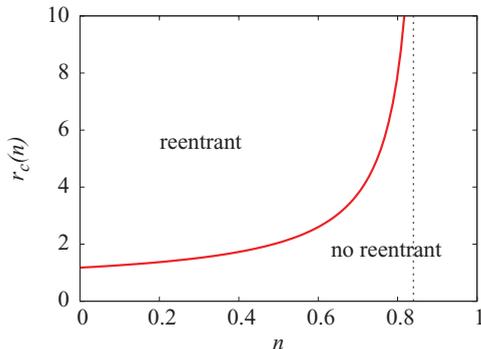


FIG. 4. (Color online) n dependence of the critical ratio r_c above which the reentrant transition can occur. The ratio r_c diverges at $n_c \simeq 0.8396 \dots$, which is shown by the broken line, with increasing n .

order of system 1, and the spin glass phase is enhanced as r increases. As a consequence of the competition between r and J_0 , the reentrant transition at $J_0/J > \sqrt{\alpha}$ is found at r_c . For $n > n_c$, the interactions \mathbf{J}_3 likely have the same sign as $\langle S_{1,i} S_{2,j} \rangle$. Then, the effective field from system 2 to system 1 is not random but it supports the ferromagnetic order in system 1. Therefore, the ferromagnetic ordering is stabilized even as r increases and system 2 with the spin-glass couplings dominates. The reentrant transition does not occur for any r for $n > n_c$.

C. Local structures of frustration

In this section, we discuss equilibrium properties of the coupled mean-field model from the viewpoint of interactions \mathbf{J} . Frustration is a key quantity that characterizes the structure of the interactions in spin glasses and related random spin models. It is defined as a product of J_{ij} s along a minimal loop, whose length is three in the fully-connected model studied in this work. If the interactions among the three spins satisfy the condition $J_{ij}J_{jk}J_{ki} < 0$, then the three terms of the local energy cannot be minimized simultaneously. Such interactions are said to have frustration [24]. Meanwhile, interactions satisfying $J_{ij}J_{jk}J_{ki} > 0$ do not have frustration and the energy of the spins attains a global minimum value although the relative directions of the spins are not aligned totally. This type of interaction is called a Mattis model [25] and all of the interaction sets can be reduced to the pure ferromagnetic model by local gauge transformation [2].

In the coupled mean-field model, we should consider three distinct frustration parameters originating from three types of interactions, as shown in Fig. 1:

$$\Phi_{(2)} = \sum_{i < j < k} J_{2,ij} J_{2,jk} J_{2,ki}, \quad (26)$$

$$\Phi_{(3a)} = \sum_{i < j, k} J_{1,ij} J_{3,jk} J_{3,ki}, \quad (27)$$

$$\Phi_{(3b)} = \sum_{i, j < k} J_{3,ij} J_{2,jk} J_{3,ki}. \quad (28)$$

In equilibrium, these parameters are to be taken as averages over the equilibrium distribution $P(\mathbf{J})$ in Eq. (6) as $[\Phi_{(2)}]_n$, $[\Phi_{(3a)}]_n$ and $[\Phi_{(3b)}]_n$ with $[\dots]_n$ being the average with respect to \mathbf{J} . For the limit $n \rightarrow 0$ with β_S kept finite, the distribution $P(\mathbf{J})$ is identical to the bare distribution $P_0(\mathbf{J})$ and the average $[\dots]_n$ is reduced to $[\dots]_0$. In this limit, the frustration parameters become zero. If the frustration parameters take a positive finite value at finite n , this indicates that the frustration is decreased as a consequence of the correlation of \mathbf{J} .

When the bare distribution of the interactions is Gaussian, the averaged frustration parameters are expressed in terms of the order parameters [11,26]. In this model, the averaged frustration parameters under the RS ansatz are described as follows:

$$[\Phi_{(2)}]_n = \beta_S^3 \{ \lambda_2^3 + (n-1)\mu_2^3 \}, \quad (29)$$

$$[\Phi_{(3a)}]_n = \beta_S^2 n J_0 \lambda_2 m_1^2, \quad (30)$$

$$[\Phi_{(3b)}]_n = \beta_S^3 \{ \lambda_1 \lambda_2^2 + (n-1)\mu_1 \mu_2^2 \}, \quad (31)$$

where $\lambda_p = 1 + (n-1)q_p$ and $\mu_p = 1 - q_p$ are the eigenvalues of the $n \times n$ matrix Q_p whose diagonal components are

1 and off-diagonal components are q_p . We can also derive the frustration parameter with a multiple-step RSB, which does not yield a significant quantitative change [26].

At finite n , $[\Phi_{(2)}]_n$ and $[\Phi_{(3b)}]_n$ take a finite value depending on β_S , while $[\Phi_{(3a)}]_n$ is zero in the paramagnetic phase. This moderate decrease of the frustration is considered to be due to an emergence of “short-range” correlation of \mathbf{J} induced by partial annealing. A considerably qualitative change of the frustration parameters is accompanied by phase transitions. In the spin-glass phase with $q_1 > 0$, $q_2 > 0$, and $m_1 = 0$, the frustration parameters $[\Phi_{(2)}]_n$ and $[\Phi_{(3b)}]_n$ largely increase, but $[\Phi_{(3a)}]_n$ is still zero. Therefore, the frustration is nonuniformly distributed in the system and a “local” structure of frustration is formed; the frustrations in system 2 and a part of \mathbf{J}_3 decrease, but a remaining part of \mathbf{J}_3 has frustration as high as for the randomly constituted interactions. The local structure prefers to decrease selectively the frustration of system 2 in the spin-glass phase, and it does not cooperatively support the ferromagnetic state of system 1. Meanwhile, all of the frustration parameters take a positive value in the ferromagnetic phase, because in this phase all order parameters are finite. In this case, all interactions \mathbf{J}_2 and \mathbf{J}_3 decrease the energy of the ferromagnetic state of system 1, meaning that the effective field from system 2 energetically supports the ferromagnetic state of system 1. This type of interactions is similar to that of the Mattis states [25]. These observations of the frustration parameters certify the validity of the argument of $r_c(n)$ and n_c in the previous section.

Before closing this section, we discuss the order of the frustration parameters. The interactions are of order of $N^{-1/2}$ in the bare distribution. When the frustrations completely vanish when keeping the order of J_{ij} , the order of the frustration parameters become $O(N^{3/2})$ by definition. However, the averaged frustration parameters should be $O(1)$ quantities for any T_S , as seen from the expressions Eqs. (29), (30) and (31). This means that the order of J_{ij} s is appropriately modified to $O(N^{-1})$ through the partial annealing, and the extensivity of thermodynamic quantities such as energy and free energy is maintained throughout the entire temperature region. This is a characteristic of partial annealing with the bare distribution function of \mathbf{J} being Gaussian. When the bare distribution is bimodal and the allowed value of \mathbf{J} is restricted to $\pm J$, namely $\pm \mathbf{J}$ model, the order of J_{ij} cannot be changed and the energy becomes $O(N^{3/2})$ in the resultant ferromagnetic phase, which shows the lack of extensivity. We should be careful in studying the thermodynamics in such models.

IV. SUMMARY AND DISCUSSION

We have studied equilibrium properties of a coupled mean-field model in a partial annealing system, in which system 1 with an embedded ferromagnetic ordering and system 2 with no embedded pattern are coupled with spin-glass interactions. In this model, the interactions as well as spins are regarded as dynamical variables, but their time scales are completely separated from each other. The spins \mathbf{S} and interactions \mathbf{J} touch to their own heat baths with different temperatures T_S and T_J . By using the replica method, the free energy of the system is derived as functions of the two temperatures, and the phase diagram is obtained

in the two-temperature plane. The phase boundary between the ferromagnetic phase and the spin-glass phase interestingly depends on the model parameters, the size ratio r of system 2 to system 1, and the ratio n between the two temperatures.

We carefully studied $T_S^{\text{FM-SG}}$ around the multicritical point. It is found that there exists a critical value n_c in the ratio n which characterizes whether the reentrant transition occurs around the multicritical point. For $n < n_c$, the partial annealing effect on the intercoupling \mathbf{J}_3 is not significant enough to enhance the embedded ferromagnetic ordering in system 1. The effective field from system 2 to system 1 does not support the ferromagnetic ordering in system 1. Therefore for a fixed $n < n_c$ as the ratio r increases, in other words the number of spins in system 2 increases relatively, a competition involving the ferromagnetic order is expected; the random field destabilizes the ferromagnetic order and the coupling J_0 stabilizes it. As a result of this competition, the ferromagnetic order is destabilized eventually. This can be seen as the reentrant transition in the phase diagram. Meanwhile, for $n > n_c$ the effective field supports achieving a ferromagnetic ordering in system 1 irrespective of the value of r , and the spin-glass region is narrowed monotonically as r increases.

We introduced three frustration parameters, $\Phi_{(2)}$, $\Phi_{(3a)}$ and $\Phi_{(3b)}$, for characterizing partially annealed interactions of the coupled mean-field model. They are expressed in terms of the order parameters of the spin system in equilibrium. Using the parameters, we classify the type of interactions in this model into three categories, each of which corresponds to a phase: interactions with a local structure of frustration in the spin-glass phase, Mattis-like interactions in the ferromagnetic phase, and randomly constructed interactions in the paramagnetic phase. The characteristic interactions found in the spin-glass phase do not cooperatively support the ferromagnetic ordering of system 1; that is to say, the frustration is eliminated from system 2 independently of the ferromagnetic interactions in system 1.

This is quite different from the results obtained in a related model previously studied [7,8], in which a locally embedded pattern is supported by the interactions surrounding the pattern and the frustration in the rest of the interactions still remains extensive. Although both models have the same structure, in that an ordering pattern is embedded in a part of the system, the partial annealing leads to the different property of interactions; the interactions in the previously studied model support the embedded pattern, and those in this study disturb it. One reason of the difference may originate from the different choice of H_J . In the present work, $H_J(\mathbf{J})$ contains the spin free energy, while $H_J(\mathbf{J})$ in [7,8] has a local order parameter. The results suggest that an explicit form of $H_J(\mathbf{J})$ strongly affects the construction of the interaction \mathbf{J} and the ordering of the spin \mathbf{S} ; in particular, an entropic effect of \mathbf{J} is nontrivial. For a proper understanding of the constructed interactions in partial annealing, we have to pursue some variant models with a general type of H_J . However, analytical studies of partial annealing at this time heavily relies on the replica method, which is applicable only to systems with the spin free energy as H_J . The partially annealed system with H_J that does not contain the spin free energy has yet to be studied in terms of spin-glass theory. An extended formalism of the partially annealed system is

required for further applications to biological or engineering models.

In this work, we focus our attention on the mean-field analysis based on fully connected spin models. When we introduce a diluted model such as the Viana-Bray model [27] in partial annealing, the geometric structure of the partially annealed \mathbf{J} may play an important role in cooperative phenomenon. Furthermore, a decoupling transition of two degrees of freedom might occur in the diluted models, while the transition of frustration is completely correlated to the transition of the spin variables in our model. Some work in this direction is in progress.

We end with an account of a perspective of the partially annealed system. In most the studies of partial annealing, the free energy is used for the Hamiltonian $H_J(\mathbf{J})$ of the slow variables \mathbf{J} . This yields a replicated system with a finite replica number determined by the ratio of two temperatures in the partially annealed system. The partial annealing can be considered to give a physical meaning to the replica approach before taking the limit of the replica number to

zero [12]. Meanwhile, the large deviations of the free energy of mean-field spin glasses are studied by the replica method with a finite replica number [28]. A mechanism of replica symmetry breaking is studied through a phase transition as the replica number is taken to zero [29]. Thus, the study of a finitely replicated system can provide new insights into spin-glass theory. In this work, we discussed a phase transition as a cooperative phenomena of the fast and slow variables with the replica number varying; this might give a different viewpoint on finitely replicated systems. It would be interesting to classify possible universality classes of phase transitions of finitely replicated systems, particularly in finite dimensions.

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