Exact soliton solutions and their stability control in the nonlinear Schrödinger equation with spatiotemporally modulated nonlinearity

Qing Tian,[1] Lei Wu,[2] Jie-Fang Zhang,[1,3,4] Boris A. Malomed,[5] D. Mihalache,[6] and W. M. Liu[7]
[1]*School of Physics Science and Technology, Soochow University, Suzhou, Jiangsu 215006, China*
[2]*Department of Mechanical Engineering, University of Strathclyde, Glasgow, G11XJ, United Kingdom*
[3]*Zhejiang University of Media and Communications, Hangzhou 310018, Zhejiang, China*
[4]*Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China*
[5]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[6]*Horia Hulubei National Institute for Physics and Nuclear Engineering (IFIN-HH), 407 Atomistilor, Magurele-Bucharest, 077125, Romania*
[7]*Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
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We put forward a generic transformation which helps to find exact soliton solutions of the nonlinear Schrödinger equation with a spatiotemporal modulation of the nonlinearity and external potentials. As an example, we construct exact solitons for the defocusing nonlinearity and harmonic potential. When the soliton's eigenvalue is fixed, the number of exact solutions is determined by energy levels of the linear harmonic oscillator. In addition to the stable fundamental solitons, stable higher-order modes, describing array of dark solitons nested in a finite-width background, are constructed too. We also show how to control the instability domain of the nonstationary solitons.

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I. INTRODUCTION

Solitons are the subject of intensive theoretical and experimental studies in various fields, such as hydrodynamics, plasma physics, nonlinear optics, and Bose-Einstein condensates (BECs). Many nonlinear partial differential and difference equations, which model these physical media, give rise to soliton solutions. Among them, the nonlinear Schrödinger equation (NLSE) and its variants have been the most thoroughly investigated models, especially in nonlinear optics, where many species of solitons have been predicted [1], and in BEC, where, in particular, the dynamical management of solitons may be used in applications such as matter-wave interferometry and gyroscopes [2].

The construction and stabilization of exact soliton modes have always been important topics, because exact stable solutions are of great significance to both the understanding of the nonlinear dynamics and its applications. In the one-dimensional (1D) NLSE, exact stable bright solitons exist if the nonlinearity is self-focusing. In higher dimensions, bright solitons are unstable, as they are subject to the collapse. Nevertheless, multidimensional solitons can be stabilized by varying the nonlinearity temporally [3,4] or spatially [5–7] or by introducing the nonlocal [8] or saturable [1] nonlinearity. The spatiotemporal nonlinearity management is also useful in generating solitons with novel properties [4]. In particular, the time-periodic variation of the nonlinearity can create robust breathers [9] and discrete compactons [10] and can enhance the mobility of discrete solitons [11], while the spatially periodic nonlinearity modulation supports persistent Bloch oscillations [12] and may break the stability limit imposed by the Vakhitov-Kolokolov criterion [7,13,14]. Further, using *self-focusing* spatiotemporally modulated nonlinearity, it is possible to construct an infinite number of exact solitons, such as resonant and breathing ones [15]. However, only the fundamental soliton is stable, and for self-defocusing spatially modulated nonlinearities, no exact localized solitons have been reported as yet.

Here we aim to propose a scheme for constructing exact soliton solutions to the 1D NLSE with a general form of the spatial modulation of the cubic *self-defocusing* nonlinearity. In this way, we construct *stable* higher-order modes built as arrays of dark solitons nested in a finite-width background. We also show that the instability of those solitons which are not stable can be controlled by varying the nonlinearity's strength in time. The stabilization mechanism resembles the classical effect of the parametric stabilization, such as the inverted pendulum, which can be sustained by an oscillating pivot.

II. MODEL AND GENERIC TRANSFORMATION FOR STATIONARY SOLUTIONS

We start with the normalized NLSE,

$$i\psi_\xi = -\psi_{xx} + V(\xi,x)\psi + g(\xi,x)|\psi|^2\psi, \qquad (1)$$

where ψ is the macroscopic wave function for the cigar-shaped BEC if ξ stands for time, V is the axial potential, and g is proportional to the interatomic s-wave scattering length. The spatiotemporal modulation of $g(\xi,x)$ can be induced by means of the Feshbach-resonance technique [4,7,9,16]. On the other hand, Eq. (1) with $\xi = z$ governs the propagation of an optical beam in a planar waveguide along the z direction, the external potential being generated by a modulation of the local refractive index [17], while various forms of $g(z,x)$ may be realized via accordingly designed distributions of nonlinearity-enhancing dopants [7,18].

We first address stationary soliton solutions, when V and g are functions of x only:

$$\psi(x,\xi) = \rho(x)U[X(x)]\exp(-i\mu\xi), \qquad (2)$$

where eigenvalue μ (chemical potential, in the case of BEC) is a real constant, $X(x) \equiv \int_{-\infty}^{x} \rho^{-2}(s)\,ds$, and $\rho(x)$ is sought for as

$$\rho(x) = \left(\alpha\varphi_1^2 + 2\beta\varphi_1\varphi_2 + \gamma\varphi_2^2\right)^{1/2}, \qquad (3)$$

with $\varphi_1(x)$ and $\varphi_2(x)$ being two linearly independent solutions of $\varphi_{xx} + [\mu - V(x)]\varphi = 0$, where α, β, and γ are real constants (cf. Ref. [15]). Further, setting

$$g(x) = [\rho(x)]^{-6} F(U), \tag{4}$$

we conclude that $U(X)$ satisfies the ordinary differential equation

$$EU = -U_{XX} + F(U)U^3, \tag{5}$$

where $E \equiv (\alpha\gamma - \beta^2)W^2$, with constant Wronskian $W = \varphi_1\varphi_{2x} - \varphi_2\varphi_{1x}$. We are interested in the case of $E > 0$, when Eq. (3) guarantees that $\rho(x)$ does not vanish; hence the nonlinearity coefficient (4) does not have singularities.

Exact soliton solutions of Eq. (1) are generated by exact solutions to Eq. (5), which may be obtained with many forms of $F(U)$. The widely used choice is that with $F(U)$ being a constant [15,19], whereas the sign of the nonlinearity coefficient $g(x)$ is either focusing (negative) or defocusing (positive) in the whole spatial domain. A more versatile choice is $F(U) = g_3 + g_5U^2$, which transforms Eq. (5) into a solvable cubic-quintic equation. The addition of the term g_5U^2 can generate a larger variety of exact soliton solutions to Eq. (1). Note that the respective cubic-nonlinearity coefficient may even be made sign-changing, for $g_3g_5 < 0$ [20]. Yet more exact solitons can be generated by choosing other forms of $F(U)$. We will construct exact solutions taking

$$F(U) = U^{-3}[EU - \sin(EU)], \tag{6}$$

which casts Eq. (5) into the stationary sine-Gordon equation, $U_{XX} + \sin(EU) = 0$. This equation with $E > 0$ describes the motion of a pendulum, i.e., either oscillations, with U being a periodic function of X, or the rotation, with U varying monotonically. The localized soliton requires U to be a periodical function; thus

$$U(X) = 2E^{-1}\arcsin[q\,\mathrm{sn}(\sqrt{E}X,q)], \tag{7}$$

where sn is the Jacobi's elliptic function with modulus q. The period of $U(X)$ in Eq. (7) is $4K(q)/\sqrt{E}$, where $K(q)$ is the complete elliptic integral of the first kind.

The cubic nonlinearity given by Eqs. (4) and (6) has the defocusing sign, i.e., $g(x) > 0$. Therefore, an external potential is needed to confine the system. Here we consider the most physically relevant case of the harmonic potential, $V(x) = \omega_0^2x^2$. In this case, solutions $\varphi_1(x)$ and $\varphi_2(x)$ may be expressed in terms of the M and W functions of Whittaker and Watson [21] and Gamma function Γ: $\varphi_1(x) = \omega_0^{-3/4}\mathrm{sgn}(x)|x|^{-1/2}M$ and $\varphi_2(x) = \Gamma[(3\omega_0 - \mu)/(4\omega_0)]\,\omega_0^{-1/4}(\pi|x|)^{-1/2}\,\{W + 2\sqrt{\pi}\Gamma[(3\omega_0-\mu)/(4\omega_0)]^{-1}M\}$, when $\mu \neq \widetilde{\mu}^{(\ell)} \equiv (2\ell+1)\omega_0$, $\ell = 0,1,2,\ldots,$ with $\widetilde{\mu}^{(\ell)}$ being the ℓth energy eigenvalue of the harmonic oscillator. Since $\varphi_1(0) = \varphi_{2x}(0) = 0$ and $\varphi_{1x}(0) = \varphi_2(0) = 1$, the Wronskian is $W = -1$. As $E > 0$, $\rho(x) \neq 0$ holds when $\alpha,\gamma > 0$.

We focus on the symmetric profiles of the nonlinearity modulation and the corresponding solitons by setting $\beta = 0$ in Eq. (3). To produce a typical example, we set $\omega_0 = 0.1$, $\alpha = 6^{1/3}$, and $\gamma = 1$, which makes $g(x) \sim 1$ around the center. Then, for given μ, $\rho(x)$ and $X(x)$ can be calculated. To meet the zero boundary condition at $x = \pm\infty$, modulus q must satisfy the constraint, $2nK(q) = \sqrt{E}R$, where n is a positive integer and $R \equiv \int_{-\infty}^{+\infty} ds/\rho^2(s)$. Since $K(q) > \pi/2$, it follows from here that $n < \sqrt{E}R/\pi$. This clearly means that there is only a *finite* number of the exact solutions for given μ. Integer n is the order of the soliton mode, which features $n-1$ density nodes. Further analysis demonstrates that $\sqrt{E}R/\pi$ increases monotonically as μ increases, while other parameters are fixed, and n takes values up to $\ell+1$ when $\widetilde{\mu}^{(\ell)} < \mu < \widetilde{\mu}^{(\ell+1)}$; when $\mu < \widetilde{\mu}^{(0)}$, there are no solutions (see Fig. 1). An explanation for this is that the defocusing nonlinearity pushes the energy levels up relative to the harmonic oscillator. Thus, the fundamental solitons ($n = 1$) exist when $\mu > \widetilde{\mu}^{(0)}$, first-order excited solitons ($n = 2$) exist when $\mu > \widetilde{\mu}^{(1)}$, etc. This conclusion coincides with the findings of Ref. [22] for the spatially homogeneous defocusing nonlinearity. A similar result holds for gap solitons in self-defocusing media: only first n families of the solitons exist when μ lies in the nth optical-lattice-induced band gap [23].

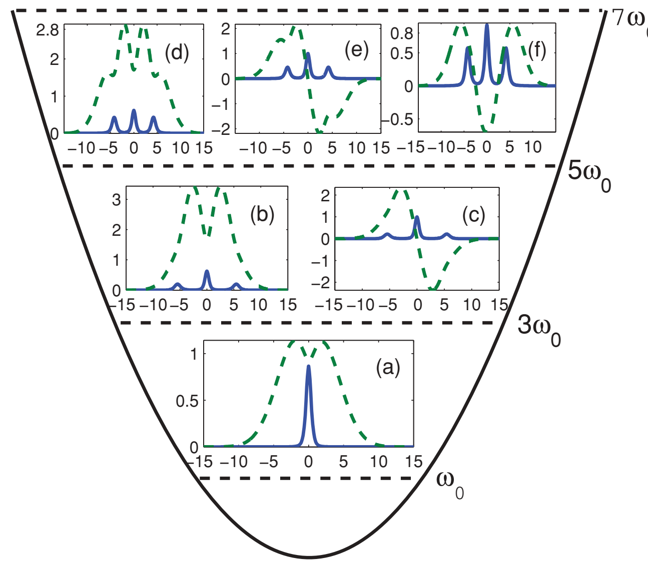

FIG. 1. (Color online) The relation between the number of exact soliton solutions and the discrete energy levels of the harmonic-oscillator potential (the underlying parabola). (a) Modulation profiles of the defocusing nonlinearity $g(x)$ (solid line), and the shape of the exact soliton (dashed line), when $\mu = 0.2$, $n = 1$, and $q = 0.7458$. The horizontal axis is coordinate x. Other insets display the same for (b) $\mu = 0.4, n = 1, q = 0.9984$, (c) $\mu = 0.4, n = 2, q = 0.8586$, (d) $\mu = 0.6, n = 1, q = 0.9998$, (e) $\mu = 0.6, n = 2, q = 0.9530$, and (f) $\mu = 0.6$, $n = 3$, $q = 0.5933$. The other parameters are $\alpha = 6^{1/3}$, $\beta = 0$, $\gamma = 1$, and $\omega_0 = 0.1$.

In keeping with the preceding analysis, Fig. 1(a) shows that only one exact soliton exists when $\mu = 0.2$. The corresponding single-humped nonlinearity-modulation profile is localized near $x = 0$, and the exact soliton has a small dip, because of the self-repulsive sign of the nonlinearity. We checked the stability of the soliton against various perturbations by means of direct simulations of Eq. (1). In particular, if kicked with initial velocity 0.1, it remains stable, featuring periodic oscillations [Fig. 2(a)].

Using the numerical integration in imaginary time, we have found that this and other fundamental solitons in the present model represent the ground state, which explains their

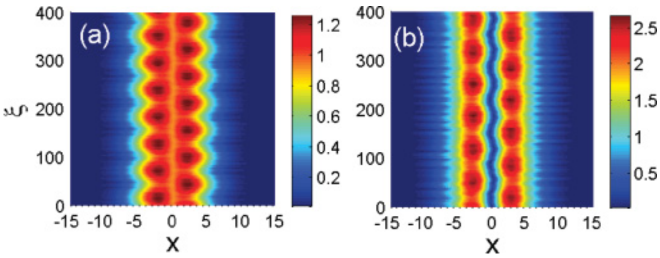

FIG. 2. (Color online) (a) Oscillations of the soliton from Fig. 1(a) which was left-kicked with velocity 0.1, the corresponding initial condition for Eq. (1) being $\rho(x)U[X(x)]\exp(-0.05ix)$. (b) Oscillations of the soliton from Fig. 1(c) which was left-kicked with initial velocity 0.05.

robustness. As μ increases, more and more exact solitons appear, while the nonlinearity profile develops several peaks. In particular, Figs. 1(b) and 1(c) demonstrate that two exact solitons exist in the case of $\mu = 0.4$. Besides the stable fundamental soliton, the first-excited-state soliton, which looks like a dark soliton nested in a finite-width background, is stable too [see Fig. 2(b)]. However, at $\mu = 0.6$, the exact first- and second-excited-state solitons are not stable; see Fig. 3(b) for the former one. While it is not easy to analyze the stability of all the solitons corresponding to the excited states, at larger values of μ it is possible to generate arrays of nested dark solitons which maintain their stability—in particular, if they are kicked with small spatially homogeneous [Fig. 4(b)] or inhomogeneous [Fig. 4(c)] velocities. This finding suggests a new approach to constructing the Newton's cradle and exciting supersolitons in systems described by the scalar NLSE [24], where the nested dark chains may be generated by the phase-imprinting method [25].

Note that, for a fixed chemical potential, there is one-to-one correspondence between the *exact* soliton and the nonlinearity

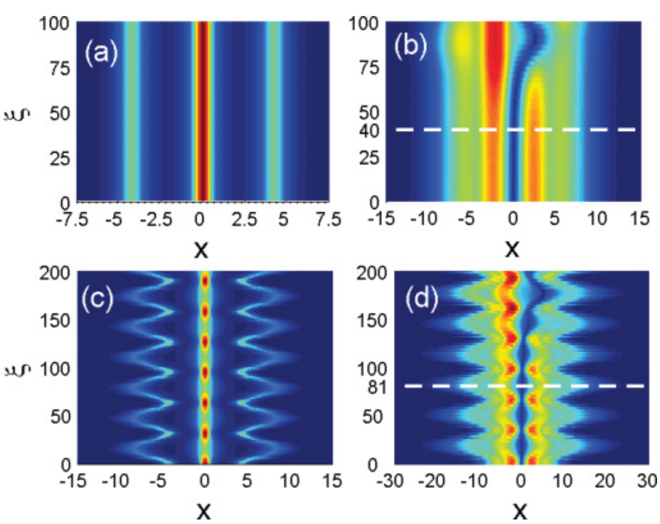

FIG. 3. (Color online) (a) The profile of the spatially modulated nonlinearity and (b) the unstable evolution of the soliton from Fig. 1(e). (c) The spatiotemporal nonlinearity modulation and (d) the unstable evolution of the soliton, for the initial condition the same as in panel (b), but with the nonlinearity taken as per Eq. (9) and half the trapping frequency, $\omega = \omega_0/2$.

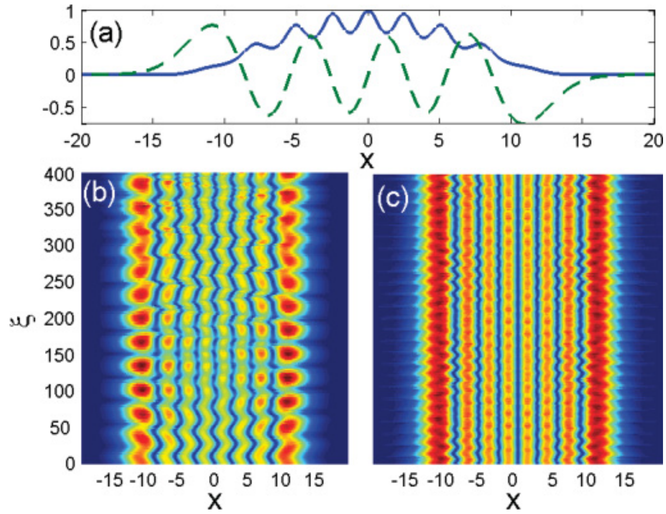

FIG. 4. (Color online) (a) The modulation profile of the defocusing nonlinearity and the corresponding exact soliton (solid and dashed lines) for $\mu = 1.6$, $n = 8$, and $q = 0.4752$. (b,c) The quasistable evolution of the soliton. The initial condition in panel (b) for Eq. (1) is $\rho(x)U[X(x)]\exp(-0.05ix)$, while that in panel (c) is $\rho(x)U[X(x)]\exp(-0.05i|x|)$.

profile $g(x)$. On the other hand, for fixed $g(x)$ one can find other soliton solutions numerically, by varying μ and using the relaxation method. We have checked that, if the exact soliton is stable, then its counterparts numerically found for the same $g(x)$ are stable too, provided that the chemical potential does not vary too much. Further, it was checked that these solitons remain stable when the nonlinearity profiles were taken somewhat different from the special form (4). Thus, higher-order solitons are physically meaningful objects.

III. STABILITY CONTROL OF NONAUTONOMOUS SOLITONS

With g and V made functions of x and ξ, Eq. (1) turns into a nonautonomous NLSE [26]. In particular, it is known that, for $g = g(\xi)$ and $V = \omega(\xi)x^2$, exact solitons can be constructed by dint of a self-similar transformation which reduces the original NLSE to a known solvable form [27]. Inspired by this finding, we here seek a possibility to construct exact nonstationary solitons for Eq. (1) with the spatiotemporal modulation of the nonlinearity and harmonic external potential, transforming it into the NLSE with time-independent coefficients,

$$i\Psi_T = -\Psi_{YY} + \omega_0^2 Y^2 \Psi + g_2(Y)|\Psi|^2\Psi. \tag{8}$$

The reduction can be implemented if the nonlinearity coefficient and external potential in Eq. (1) are of the form

$$g(\xi,x) = g_1(\xi)g_2[x/W(\xi)], \quad V(\xi,x) = \omega(\xi)x^2, \tag{9}$$

where $W(\xi)$ is the width of the nonlinearity modulation, which also determine the soliton's width. Thus, we introduce the self-similar transformation (cf. Ref. [27])

$$\psi(\xi,x) = \Psi(T,Y)W^{-1/2}\exp(iW_\xi x^2/4W), \tag{10}$$

with $T \equiv \int_0^\xi ds/W^2(s)$ and $Y \equiv xW^{-1}(\xi)$. It can be checked that Eq. (8) can be recovered, in this way, only for $g_1(\xi) = 1/W(\xi)$, with $W(\xi)$ satisfying $W_{\xi\xi} + 4\omega^2(\xi)W = 4\omega_0^2 W^{-3}$.

The exact stationary soliton solutions displayed in Fig. 1 can be borrowed to produce solutions to Eq. (8), replacing Y by x, g_2 by g, T by ξ, and Ψ by ψ. Moreover, given the functional form of $\omega(\xi)$, such as $a_0 + a_1\cos(\xi)$, the equation for $W(\xi)$ can be solved. Then, one can construct exact nonstationary soliton solutions of Eq. (1) as per Eq. (10).

It is worthy to note that the stability of the nonstationary soliton is related to that of the corresponding stationary solution and the functional form of $W(\xi)$. If the stationary soliton of Eq. (8) is stable, the respective nonstationary soliton of Eq. (1) is stable too. However, if the stationary soliton is unstable, with the instability setting in at, say, $T = T_0$, the instability of the nonstationary soliton may be delayed or even suppressed, choosing an appropriate form of $W(\xi)$. Namely, if $T = \int_0^\xi W^{-2}(s)\,ds$ always remains smaller than T_0, the instability is completely prevented, because the system does not have enough time (propagation distance) to reach the instability threshold. On the other hand, the onset of the instability will be delayed if equation $T_0 = \int_0^\xi W^{-2}(s)\,ds$ yields $\xi > T_0$.

As a typical example, we take the exact soliton corresponding to Fig. 1(e). Without the temporal nonlinearity modulation, its instability starts at $\xi \approx 40$ [Fig. 3(b)]. If, at $\xi = 0$, the trap frequency is abruptly reduced to a half, and after that the nonlinearity is varied as per Eq. (9) with $W(\xi) = \sqrt{1 + 3\sin(\omega_0\xi)^2}$ [Fig.3(c)], we find that the soliton first experiences the exact self-similar evolution in accordance with Eq. (10), and then it starts to develop the instability at $\xi \approx 81$, according to equality $40 = \int_0^\xi W^{-2}(s)ds$ [Fig. 3(d)]. Clearly, the instability onset is delayed. It can be delayed further if one decreases the trapping frequency more, varying the nonlinearity accordingly.

IV. CONCLUSION

To conclude, we have proposed a transformation for finding exact soliton solutions to the nonlinear Schrödinger equation with general forms of the spatially or spatiotemporally modulated cubic nonlinearity and external potential. We have found that the number of solitons is determined by the soliton's chemical potential and discrete energy levels of the trapping harmonic-oscillator potential. We have also found stable higher-order modes, built as arrays of dark solitons embedded into a finite-width background. Finally, we have shown how one can control the instability of nonstationary solitons. The method can be also applied to finding exact vortex solitons for two-dimensional nonlinear Schrödinger equations.

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[20] Take the simplest example with $E,\mu,V = 0$, $\rho = 1$, and $g_3 > 0$, such that $U(X) = (g_3 X^2/2 - 2g_5/3g_3)^{-1/2}$ solves Eq. (5). Then, $g(x) = g_3 + g_5(g_3 x^2/2 - 2g_5/3g_3)^{-1}$ is a sign-changing function, rising from $-g_3/2$ to g_3 as x goes from 0 to $+\infty$.

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