Spanning trees in a fractal scale-free lattice

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Spanning trees provide crucial insight into the origin of fractality in fractal scale-free networks. In this paper, we present the number of spanning trees in a particular fractal scale-free lattice (network). We first study analytically the topological characteristics of the lattice and show that it is simultaneously scale-free, highly clustered, "large-world," fractal, and disassortative. Any previous model does not have all the properties as the studied one. Then, by using the renormalization group technique we derive analytically the number of spanning trees in the network under consideration, based on which we also determine the entropy for the spanning trees of the network. These results shed light on understanding the structural characteristics of and dynamical processes on scale-free networks with fractality. Moreover, our method and process for employing the decimation technique to enumerate spanning trees are general and can be easily extended to other deterministic media with self-similarity.

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I. INTRODUCTION

Subgraphs,especially those of recurring elementary interaction patterns in real-life networks, are central to characterizing network structures and understanding dynamical processes occurring in them, since they carry important information about overall organization of networks and their function. Therefore, detecting relevant subgraphs has received considerable interest from the scientific community, because it is the first step for further understanding the inherent relation between the structure and function of complex networks. In recent years, a lot of effort has been devoted to the study of subgraphs, including motifs [1,2], communities [3,4], loops [5], cliques [6], and so on.

Among a variety of subgraphs, spanning trees are one of the most important and fundamental categories. They are relevant to diverse aspects of networks, including reliability [7,8], transport [9], self-organized criticality [10], loop-erased random walks [11], resistor networks, and standard random walks [11], to name just a few. For example, the number of spanning trees in a network is closely related to the effective resistance between two nodes in the network [12], which in turn determines the mean first-passage time between the two nodes, a fundamental quantity for random walks that have found a wide range of distinct applications in various theoretical and applied fields, such as physics, chemistry, biology, and computer science, among others [13–15]. Particularly, it was shown that the number of spanning trees corresponds to the partition function of the q-state Potts model in a peculiar case of q approaching zero [16]. Recently, spanning trees in networks have been a focus of much research [17–20]. Among many contexts, counting the number of spanning trees in specific networks has been studied, e.g., regular lattices [21,22], the Sierpinski gaskets [23], the Erdsö-Rényi random graphs [24], and the pseudofractal scale-free web [25]. These investigations have unveiled the different influences of other network structures on spanning trees in networks.

As known to us all, most real networks in nature and society are scale-free [26]. In addition, it is generally accepted that many real systems are synchronously fractal [27]. Frequently cited examples [28] include the World Wide Web, metabolic networks, and yeast protein interaction networks. In view of the abundance of the fractal property in scale-free networks, it is of exceptional importance to understand its origins and mechanisms [29]. It was reported [30] that the origin of fractality of a scale-free network can be understood from a perspective of the skeleton of the network, which is a special type of spanning tree formed by edges with the highest betweenness or loads, i.e., the communication backbone of the underlying network [19]. Despite the significance of spanning trees in scale-free networks, work about spanning trees in fractal scale-free networks is still lacking.

In this paper, we present a first study of spanning trees in fractal scale-free networks. For this purpose, we first study the properties of a deterministic lattice (network) [31–33] from the viewpoint of complex networks [34,35]. We determine exactly the topological characteristics of the lattice and show that it is simultaneously scale-free and fractal. We also show that the network is highly clustered and disassortative, but lacks the small-world property. We then derive the number of spanning trees in the fractal scale-free lattice by using a decimation procedure analogous to but distinct from that in [11], based on which we determine the entropy of its spanning trees. The research provides useful insight into the structure of fractal scale-free networks and will be helpful for better understanding dynamical processes defined in them.

II. NETWORK CONSTRUCTION AND ITS STRUCTURAL TOPOLOGIES

This section is devoted to the construction and the relevant structural properties of the studied network, such as degree distribution, clustering coefficient, average path length (APL), fractality, and correlations.

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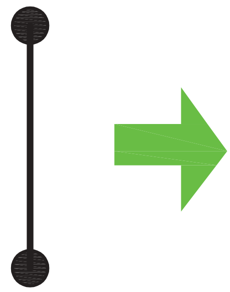

FIG. 1. (Color online) Iterative construction method of the network. Each iterative link is replaced by a connected cluster on the right-hand side of the arrow. The dashed blue link is a noniterated link.

A. Construction algorithm

The addressed network is constructed in an iterative way [31–33] as shown in Fig. 1. Let F_t ($t \geqslant 0$) denote the network after t iterations. Then the network is generated as follows: For $t = 0$, F_0 is an iterative edge connecting two nodes. For $t \geqslant 1$, F_t is obtained from F_{t-1}. We replace each existing iterative edge in F_{t-1} by a connected cluster of edges on the right-hand side of the arrow in Fig. 1. The growing process is repeated t times, with the fractal scale-free network obtained in the limit $t \to \infty$. Figure 2 shows the growing process of the network.

Next we compute the numbers of total nodes (vertices) and links (edges) in F_t. Notice that there are two types of links (i.e., iterative links and noniterated links) in the network. Let $L_v(t)$, $L_i(t)$, and $L_n(t)$ be the numbers of new vertices, iterative links, and noniterated links created at step t, respectively. Since all old iterative links are not preserved in the growing process, $L_i(t)$ is in fact the total number of iterative links at time t. Note that each of the existing iterative links yields two nodes connected by one noniterated link, and the addition of each new node leads to two iterative links. By construction, for $t \geqslant 1$, we have $L_i(t) = 4\,L_i(t-1)$, $L_v(t) = 2\,L_i(t-1)$, and $L_n(t) = L_i(t-1)$. Considering the initial condition $L_v(0) = 2$, $L_i(0) = 1$, and $L_n(0) = 0$, it follows that $L_v(t) = 2 \times 4^{t-1}$, $L_i(t) = 4^t$, and $L_n(t) = 4^{t-1}$.

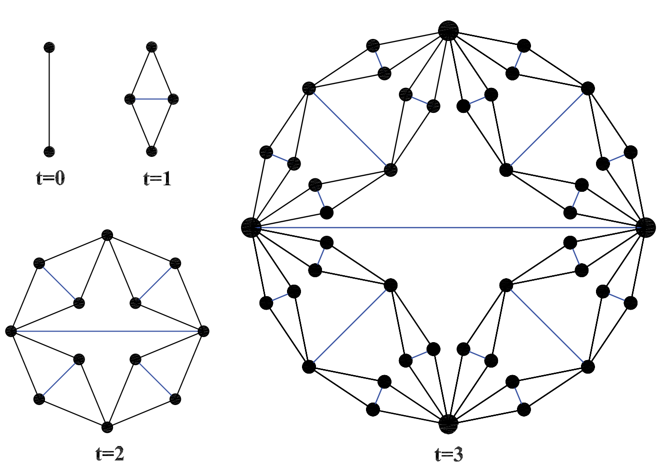

FIG. 2. (Color online) Scheme of the growth for the network. Only the first three iterative processes are shown.

Thus the numbers of total nodes N_t and edges E_t present at step t are

$$N_t = \sum_{t_i=0}^{t} L_v(t_i) = \frac{2 \times 4^t + 4}{3} \tag{1}$$

and

$$E_t = L_i(t) + \sum_{t_i=1}^{t} L_n(t_i) = \frac{4^{t+1} - 1}{3}, \tag{2}$$

respectively. And the average degree after t iterations is $\langle k \rangle_t = \frac{2\,E_t}{N_t}$, which approaches 4 in the infinite t limit.

B. Degree distribution

When a new node u is added to the network at step t_u ($t_u \geqslant 1$), it has three links, among which two are iterative links and one is a noniterated link. Let $L_i(u,t)$ be the number of iterative links at step t that will create new nodes connected to the node u at step $t+1$. Then at step t_u, $L_i(u,t_u) = 2$. From the iterative generation process of the network, one can see that at any subsequent step each iterative link of u is broken and generates two new iterative links connected to u. We define $k_u(t)$ as the degree of node u at time t, then the relation between $k_u(t)$ and $L_i(u,t)$ satisfies

$$k_u(t) = L_i(u,t) + 1, \tag{3}$$

where the last term 1 represents the only noniterated link of node u.

Now we compute $L_i(u,t)$. By construction, $L_i(u,t) = 2\,L_i(u,t-1)$. Considering the initial condition $L_i(u,t_u) = 2$, we can derive $L_i(u,t) = 2^{t-t_u+1}$. Then at time t, the degree of vertex u becomes

$$k_u(t) = 2^{t-t_u+1} + 1. \tag{4}$$

It should be mentioned that the initial two nodes created at step 0 have a little different evolution process from that of other nodes. Since the initial two nodes have no noniterated link, we can easily obtain that at step t, for either of the initial two nodes, its degree just equals the number of iterative links connecting it, both of which are 2^t.

Equation (4) shows that the degree spectrum of the network is discrete. It follows that the cumulative degree distribution [35] is given by

$$P_{\mathrm{cum}}(k) = \sum_{\tau \leqslant t_u} \frac{L_v(\tau)}{N_t} = \frac{2 \times 4^{t_u} + 4}{2 \times 4^t + 4}. \tag{5}$$

Substituting for t_u in this expression using $t_u = t + 1 - \frac{\ln(k-1)}{\ln 2}$ gives

$$P_{\mathrm{cum}}(k) = \frac{2 \times 4^t \times 4(k-1)^{-(\ln 4/\ln 2)} + 4}{2 \times 4^t + 4}. \tag{6}$$

When t is large enough, one can obtain

$$P_{\mathrm{cum}}(k) = 4\,(k-1)^{-2}. \tag{7}$$

So the degree distribution follows a power-law form with the exponent $\gamma = 3$.

C. Clustering coefficient

The clustering coefficient [36] of a node u with degree k_u is given by $C_u = 2e_u/[k_u(k_u - 1)]$, where e_u is the number of existing links among the k_u neighbors. Using the construction rule, it is straightforward to calculate analytically the clustering coefficient $C(k)$ for a single node with degree k. For the initial two nodes born at step 0, their degree is $k = 2^t$, and the existing links among these neighbors is $\frac{k}{2}$, all of which are noniterated links. For those nodes created at step ϕ $(0 < \phi < t)$, there are only $\frac{k-1}{2}$ links that actually exist among the neighbor nodes. Finally, for the smallest nodes created at step t, each has a degree of $k = 3$, and the existing number of links between the neighbors of each node is 2. Thus, there is a one-to-one correspondence between the clustering coefficient $C(k)$ of the node and its degree k:

$$C(k) = \begin{cases} 1/(k-1) & \text{for } k = 2^t, \\ 1/k & \text{for } k = 2^m + 1 \quad (2 \leqslant m \leqslant t), \\ 2/k & \text{for } k = 2^1 + 1, \end{cases} \quad (8)$$

which is inversely proportional to k in the limit of large k. The scaling of $C(k) \sim k^{-1}$ has been observed in many real-world scale-free networks [37].

Using Eq. (8), we can obtain the clustering C_t of the whole network at step t, which is defined as the average clustering coefficient of all individual nodes. Then we have

$$C_t = \frac{1}{N_t}\left[\frac{L_v(0)}{K_0 - 1} + \sum_{r=1}^{t-1} \frac{L_v(r)}{K_r} + \frac{2\,L_v(t)}{K_t}\right], \quad (9)$$

where K_r is the degree of a node at time t, which was created at step r [see Eq. (4)]. In the infinite network order limit $(N_t \to \infty)$, Eq. (9) converges to a nonzero value $\bar{C} = 0.5435$. Therefore, the average clustering coefficient of the network is very high.

D. Fractal dimension

As a matter of fact, the fractal lattice grows as an inverse renormalization procedure; see Fig. 2 in reverse order. To find the fractal dimension, we follow the mathematical framework presented in [29]. By construction, in the infinite t limit, the different quantities of the network grow as

$$\begin{cases} N_t \simeq 4\,N_{t-1}, \\ k_u(t) \simeq 2\,k_u(t-1), \\ L_t = 2\,L_{t-1}, \end{cases} \quad (10)$$

where the third equation describes the change of the diameter L_t of the graph F_t, where L_t is defined as the longest shortest path between all pairs of nodes in F_t.

From the relations provided by Eq. (10), it is clear that the quantities N_t, $k_u(t)$, and L_t increase by a factor of $f_N = 4$, $f_k = 2$, and $f_L = 2$, respectively. Then between any two times t_1 and t_2 $(t_1 < t_2)$, we can easily obtain the following relation:

$$\begin{cases} L_{t_2} = 2^{t_2 - t_1}\,L_{t_1}, \\ N_{t_2} = 4^{t_2 - t_1}\,N_{t_1}, \\ k_u(t_2) = 2^{t_2 - t_1}\,k_u(t_1). \end{cases} \quad (11)$$

From Eq. (11), we can derive the scaling exponents in terms of the microscopic parameters [29]: the fractal dimension is $d_B = \frac{\ln f_N}{\ln f_L} = 2$, and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_L} = 1$. The exponent of the degree distribution satisfies $\gamma = 1 + \frac{d_B}{d_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution [see Eq. (7)].

E. Degree correlation

Degree correlation in a network can be measured by means of the quantity, called *average nearest-neighbor degree* (ANND) and denoted as $k_{nn}(k)$, which is a function of node degree, and is more convenient and practical in characterizing degree correlation [38]. For the fractal graph considered here, one can exactly calculate $k_{\rm nn}(k)$. By construction, all neighbors of the initial two nodes have the same degree 3, while for each other node with degree greater than 3, only one of its neighbor has the same degree as itself, all the other neighbors have degree 3. Then we have

$$\begin{cases} k_{\rm nn}(k) = 3 & \text{for } k = 2^t, \\ k_{\rm nn}(k) = 4 - \dfrac{3}{k} & \text{for } k = 2^m + 1 \quad (m = 2,3\ldots t). \end{cases} \quad (12)$$

For those nodes with degree 3, it is easily to obtain

$$\begin{aligned} k_{\rm nn}(3) &= \frac{2 \times (2^t)^2 + \sum_{\tau=1}^{\tau=t-1}[L_v(\tau)k(\tau,t)\,k(\tau,t) - 1)]}{3L_v(t)} + 1 \\ &= \frac{4}{3}t + \frac{5}{3} - \frac{4}{3} \times \frac{1}{2^t}, \end{aligned} \quad (13)$$

where $k(\tau,t)$ is the degree of a node at time t that was born at step τ. Thus $k_{\rm nn}(3)$ grows linearly with time for large t. Equations (12) and (13) show the network is disassortative.

F. Average path length

We represent all the shortest path lengths of F_t as a matrix in which the entry d_{ij} is the shortest path from node i to j. A measure of the typical separation between two nodes in F_t is given by the APL $\langle d \rangle_t$, also known as characteristic path length, defined as the mean of geodesic lengths over all couples of nodes. One can compute analytically the APL for F_t by using a recursive technique [39,40]. The analytic expression for $\langle d \rangle_t$ reads

$$\langle d \rangle_t = \frac{(16 \times 2^t + 21)16^t + (21t - 27)8^t + 75 \times 4^t + 119 \times 2^t - 15}{21(2 + 5 \times 4^t + 2 \times 16^t)}. \quad (14)$$

For large t, $\langle d\rangle_t \to \frac{8}{21}\times 2^t$. Note that in the infinite t limit, $N_t \sim 4^t$, so the APL scales as $\langle d\rangle_t \sim (N_t)^{1/2}$, which indicates that the network is not a small world. Note that the recently observed global network of avian influenza outbreaks [41,42] is also large-world like the model addressed here.

Thus, we have shown that $\langle d\rangle_t$ has the power-law scaling behavior of the number of nodes N_t, which is similar to that of the two-dimensional regular lattice [43]. This phenomenon is not hard to understand. Let us look at the scheme of the network growth. Each next step in the growth of F_t doubles the APL between a fixed pair of nodes (except the small number of pairs directly connected by a noniterated link), while the total number of nodes increases fourfold (asymptotically, in the infinite limit of t); see Eq. (1). Thus the APL $\langle d\rangle_t$ of F_t grows as a square power of the node number in the network.

III. SPANNING TREES ON THE NETWORK

In the previous section, we have shown that the network exhibits many interesting properties, i.e., it is simultaneously scale-free, highly clustered, large-world, fractal and disassortative, which are not observed in other networks. Thus, the network is unique within the class of networks. Next, we proceed to investigate spanning trees in the fractal scale-free network. Our aim is to derive the exact formula for the number of spanning trees and determine its entropy.

To facilitate the description of the computation, we give the following definitions. Let W_t and X_t express the two nodes in F_t that are generated at step 0 and thus called initial nodes. Similarly, we use Y_t and Z_t to denote the two nodes in F_t, created at iteration 1. Since Y_t and Z_t have the highest degree in F_t, they are named hub nodes. Then, the network can be also built in the following way. Given the iteration t, F_{t+1} may be obtained by joining at the initial nodes four replicas of F_t (see Fig. 3). Obviously, the network is self-similar.

A. Recursion relations for related quantities

By using the self-similar property of the network, we can count the number of spanning trees by using a decimation procedure similar to but different from that in [11], both of which are based on the idea of renormalization. Let s_t denote the number of spanning trees in network F_t. To find s_t, we also define some quantities to be used. Let w_t denote the number of spanning subgraphs of F_t consisting of two trees such that one initial node, W_t, belongs to one tree, while the other initial node, X_t, and the two hub nodes (i.e., Y_t and Z_t) are in the other tree. In addition, let y_t denote the number of spanning subgraphs of F_t consisting of two trees such that nodes W_t and Y_t belong to one tree, while X_t and Z_t are in the other tree. Similarly, we can define quantities x_t and z_t (see Fig. 4). By symmetry, we have $w_t = x_t$ and $y_t = z_t$. Then, the total number of spanning subgraphs in F_t composed of two trees with the two initial nodes W_t and X_t belonging to the two different trees is

$$g_t = w_t + x_t + y_t + z_t = 2(w_t + y_t). \tag{15}$$

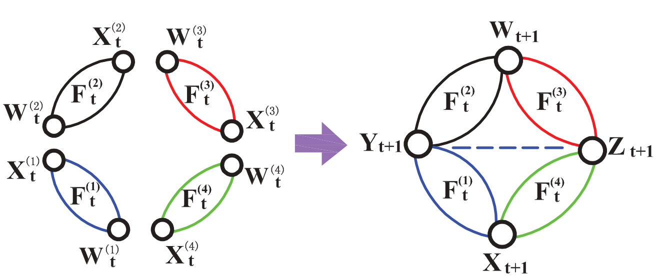

FIG. 3. (Color online) Alternative construction method of the network. F_{t+1} can be obtained by joining four copies of F_t denoted as $F_t^{(\eta)}$ ($\eta = 1,2,3,4$), the initial nodes of which are represented by $W_t^{(\eta)}$ and $X_t^{(\eta)}$, respectively. In the merging process, $X_t^{(2)}$ (respectively, $W_t^{(1)}$) and $W_t^{(3)}$ (respectively, $X_t^{(4)}$) are identified as an initial node, W_{t+1} (respectively, X_{t+1}), in F_{t+1}. Analogously, $X_t^{(1)}$ (respectively, $X_t^{(3)}$) and $W_t^{(2)}$ (respectively, $W_t^{(4)}$) are identified as a hub node, Y_{t+1} (respectively, Z_{t+1}), in F_{t+1}. The dashed edge expresses the noniterated one connecting two hub nodes, Y_{t+1} and Z_{t+1}, in network F_{t+1}.

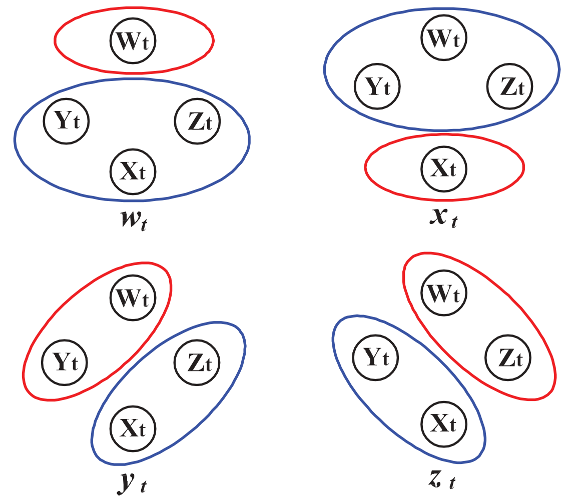

FIG. 4. (Color online) Illustrative definitions for the spanning subgraphs of F_t. The initial nodes and hub nodes in one ellipse (two ellipses) belong to one tree (two trees).

We continue to provide the recursion relations for the above-defined quantities. We first focus on w_{t+1}. We distinguish two cases. The first case is that the two hub nodes Y_{t+1} and Z_{t+1} are not directly connected in the spanning subgraph. The other case is that nodes Y_{t+1} and Z_{t+1} are linked to each other by a bond. Then w_{t+1} can be written as

$$w_{t+1} = w_{t+1}^{(1)} + w_{t+1}^{(2)}, \tag{16}$$

where $w_{t+1}^{(1)}$ and $w_{t+1}^{(2)}$ denote the respective numbers of spanning subgraphs of the two cases. Below we calculate the two quantities $w_{t+1}^{(1)}$ and $w_{t+1}^{(2)}$.

Figure 3 shows that network F_{t+1} consists of four copies of F_t, viz., $F_t^{(\eta)}$ ($\eta = 1,2,3,4$), with four couples of nodes identified. According to this self-similarity, $w_{t+1}^{(1)}$ can be obtained with appropriate configurations of the four components $F_t^{(\eta)}$ as shown in Fig. 5. In this case, to assure that nodes X_{t+1}, Y_{t+1}, and Z_{t+1} are in one tree and that node W_{t+1} is in another tree, all nodes in $F_t^{(2)}$ must be in a subgraph including two trees with the initial two nodes $W_t^{(2)}$ and $X_t^{(2)}$ in separate trees, and the number of these possible subgraphs is g_t. It is the same with nodes in $F_t^{(3)}$. In contrast, all nodes in $F_t^{(1)}$ are in one

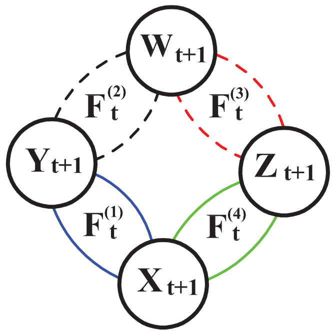

FIG. 5. (Color online) Illustration of the expression for the number of spanning subgraphs $w_{t+1}^{(1)}$ corresponding to network F_{t+1}. Two nodes connected by solid lines (dotted lines) are in one tree (two trees).

single tree. The number of such configurations is s_t. This also holds for nodes in $F_t^{(4)}$. Thus,

$$w_{t+1}^{(1)} = (s_t)^2 (g_t)^2. \tag{17}$$

After deriving $w_{t+1}^{(1)}$, we proceed to express $w_{t+1}^{(2)}$ as a function of s_t and g_t (see Fig. 6). Since in this case, Y_{t+1} and Z_{t+1} are directly linked to by an edge, it is a little different from the first case: nodes in $F_t^{(4)}$ (or $F_t^{(1)}$) are in two trees that contain all nodes in $F_t^{(4)}$ (or $F_t^{(1)}$) and are subgraphs of $F_t^{(4)}$ (or $F_t^{(1)}$), with $W_t^{(4)}$ (or $W_t^{(1)}$) and $X_t^{(4)}$ (or $X_t^{(1)}$) falling into separated trees, while nodes in $F_t^{(1)}$ (or $F_t^{(4)}$), $F_t^{(2)}$, and $F_t^{(3)}$ are the same as those corresponding to the first case. Then, we have

$$w_{t+1}^{(2)} = 2s_t(g_t)^3. \tag{18}$$

Plugging Eqs. (17) and (18) into Eq. (16) leads to

$$w_{t+1} = (s_t)^2 (g_t)^2 + 2s_t(g_t)^3. \tag{19}$$

Analogously, we can write y_{t+1} in terms of s_t and g_t as

$$y_{t+1} = (s_t)^2 (g_t)^2, \tag{20}$$

which can be understood based on Fig. 7. Then, we have the following relation:

$$g_{t+1} = 2(w_{t+1} + y_{t+1}) = 4(s_t)^2(g_t)^2 + 4s_t(g_t)^3. \tag{21}$$

Now we begin to give the derivative process of the recursion relation for s_{t+1}. Analogously, according to whether or not Y_{t+1} and Z_{t+1} are connected directly by a link, we distinguish two cases. For the first case (Y_{t+1} and Z_{t+1} are not directly

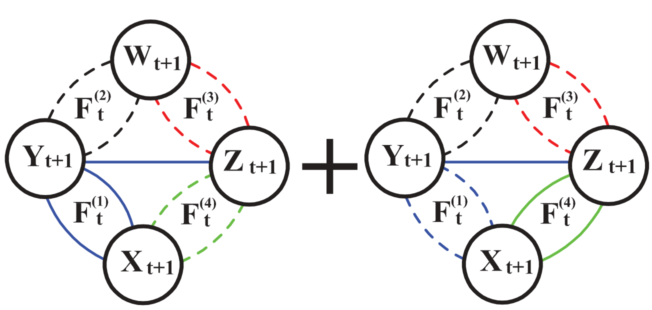

FIG. 6. (Color online) Illustration of the expression for the number of spanning subgraphs $w_{t+1}^{(2)}$ in network F_{t+1}.

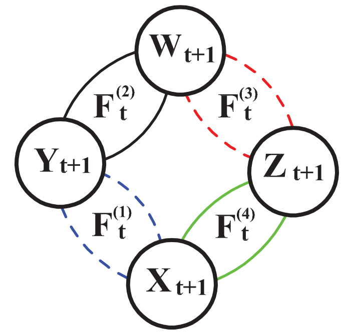

FIG. 7. (Color online) Illustration of the recursive expression for the number of spanning subgraphs y_{t+1} corresponding to network F_{t+1}. In the figure, two nodes connected by solid lines (dotted lines) belong to one tree (two trees).

connected), we denote the number of spanning trees by $s_{t+1}^{(1)}$, while for the second case, the number of spanning trees is represented by $s_{t+1}^{(2)}$. Then, $s_{t+1} = s_{t+1}^{(1)} + s_{t+1}^{(2)}$. From Figs. 8 and 9, using an analysis similar to that of w_{t+1}, we can obtain the following recursive relations:

$$s_{t+1}^{(1)} = 4g_t(s_t)^3 \tag{22}$$

and

$$s_{t+1}^{(2)} = 4(s_t)^2 (g_t)^2, \tag{23}$$

for $s_{t+1}^{(1)}$ and $s_{t+1}^{(2)}$, respectively. Therefore,

$$s_{t+1} = 4g_t(s_t)^3 + 4(s_t)^2 (g_t)^2. \tag{24}$$

All these obtained relations are useful for deriving the explicit formula for the number of spanning trees in F_t.

B. Exact solutions to the number and entropy of spanning trees

Based on the above-obtained recursive relations, we can compute the number of spanning trees for the fractal

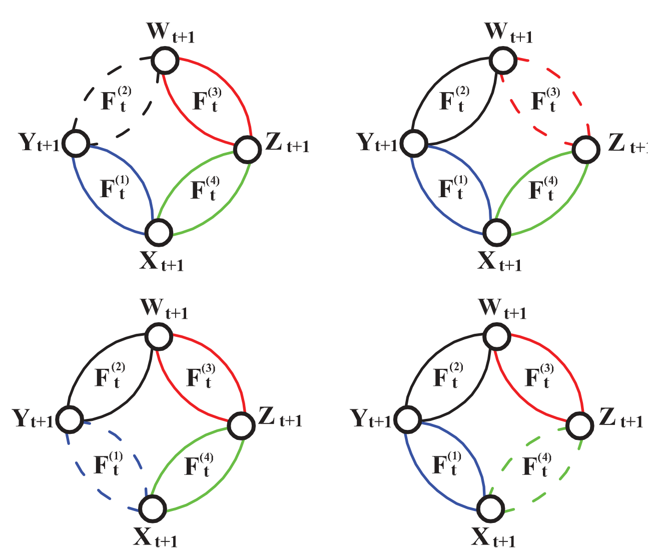

FIG. 8. (Color online) Illustration for the recursion expression for the number of spanning trees $s_{t+1}^{(1)}$ in F_{t+1}.

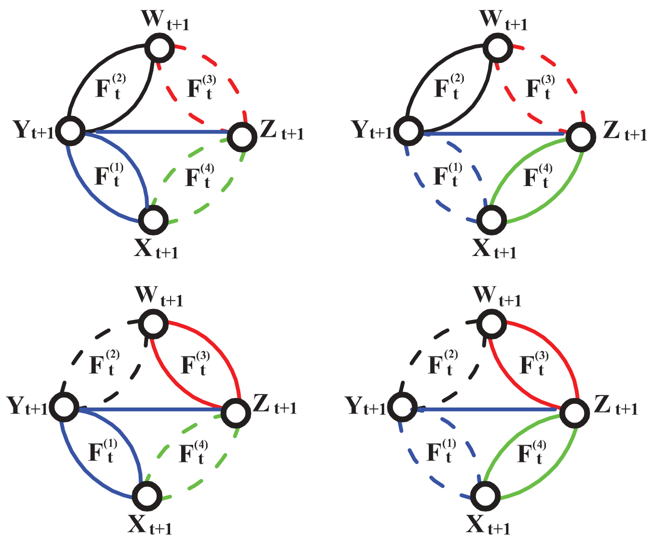

FIG. 9. (Color online) Illustration for the recursion expression for the number of spanning trees $s_{t+1}^{(2)}$ in F_{t+1}.

scale-free lattice F_t. To gain this aim, we define a new quantity $q_t = s_t - g_t$. Equation (24) minus Eq. (21) yields

$$q_{t+1} = 4s_t g_t (s_t + g_t)(s_t - g_t) = 4s_t g_t (s_t + g_t) q_t. \qquad (25)$$

Considering $s_1 = 8$ and $g_1 = 2(w_1 + y_1) = 2(3 + 1) = 8$, we have $q_1 = 0$. Thus,

$$q_t \equiv 0, \qquad (26)$$

implying

$$s_t = g_t. \qquad (27)$$

Replacing g_t by s_t, Eq. (24) becomes

$$s_{t+1} = 8(s_t)^4. \qquad (28)$$

Applying $s_1 = 8$, Eq. (28) is solved inductively to obtain the closed-form expression for s_t:

$$s_t = 2^{4^t - 1}. \qquad (29)$$

Equation (29), together with Eq. (1), shows that s_t approximately increases exponentially in network order N_t, which permits one to determine the entropy of spanning trees—an important quantity characterizing network structure—for F_t as the limiting value [44,45]

$$E_{F_t} = \lim_{N_t \to \infty} \frac{\ln s_t}{N_t} = \lim_{t \to \infty} \frac{\ln s_t}{N_t} = \frac{3}{2} \ln 2 \qquad (30)$$

that approaches to a constant value 1.0397, a finite number larger than 1.

The obtained entropy of spanning trees in F_t can be compared to those found for other networks with the same average degree of nodes. In the pseudofractal fractal web [25], the entropy is 0.8959, a value less than 1; while for the square lattice and the two-dimensional Sierpinski gasket, their entropy of spanning trees is 1.16624 [21] and 1.0486 [23], respectively, both of which are greater than 1.0397.

We note that the fractal network considered here, in fact, can be obtained from the $q = 1$ case of the fractal network family $H(q,t)$ previously studied [46] by adding to it the noniterated links [33], while the latter is exactly the (2,2)-flower that belongs to a more general class of hierarchical networks (including both fractal and nonfractal networks) initially introduced in [47]. Then, it is natural to expect that the above analytical approach for determining spanning trees can be applicable to other self-similar media. In the Appendix, we show how to use the above technique to compute the number of spanning trees in the family of fractal scale-free networks $H(q,t)$ and show how the entropy varies with the parameter q.

C. Numerical solution

In order to confirm the analytical solution given by Eq. (28), we have compared it with numerical results. According to the well-known result [48], we can get numerically but exactly the number of spanning trees, represented by $N_{\rm ST}(t)$, in network F_t, by computing the product of all nonzero eigenvalues of the Laplacian matrix corresponding to F_t as

$$N_{\rm ST}(t) = \frac{1}{N_t} \prod_{i=1}^{i=N_t-1} \lambda_i(t), \qquad (31)$$

where $\lambda_i(t)$ $(i = 1,2,\ldots,N_t - 1)$ are the $N_t - 1$ nonzero eigenvalues of the Laplacian matrix, denoted by $\mathbf{M}_t$, for F_t. The nondiagonal element m_{ij} $(i \neq j)$ of $\mathbf{M}_t$ is -1 (or 0) if nodes i and j are (or not) directly connected by a link, while the diagonal entry m_{ii} is exactly the degree of node i. Making use of Eq. (31), we compute the numerical values of $N_{\rm ST}(t)$ up to $t = 8$. For all cases of $1 \leqslant t \leqslant 8$, the obtained numerical results are completely consistent with those provided by Eq. (28), indicating that the analytical formula given by Eq. (28) is valid.

IV. CONCLUSIONS AND DISCUSSION

To conclude, we have investigated a lattice model from the viewpoint of complex networks. Its deterministic self-similar construction allows us to derive analytical exact expressions for the relevant features. We have shown that the graph simultaneously exhibits many interesting structural characteristics: power-law degree distribution, large clustering coefficient, large-world phenomenon, fractal similar structure, and negative degree correlations. The simultaneous existence of scale-free, high clustering, and large-world behaviors is compared with previous network models.

In addition, we have presented how to enumerate spanning trees in the fractal scale-free lattice under consideration. Based on a decimation procedure, we have given some useful recursive relations for some spanning subgraphs, from which we determined exactly the number and entropy of spanning trees in the network. In general, the number of spanning trees in a network can be obtained by directly calculating a related determinant corresponding to the network. However, this universal method is not acceptable for large graphs. For this reason, it is interesting to develop techniques to derive closed-form and simple formulas for special classes of graphs. In this context, our work provides a detailed analysis for determining spanning trees in fractal scale-free networks and presents a new perspective from which to study spanning trees in other media with self-similar structure [46,47], which is also a promising avenue in the research of enumeration problems on networks, e.g., spanning forests [49], dimer statistics [50], and so on.

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APPENDIX: SPANNING TREES IN THE FRACTAL SCALE-FREE NETWORKS $H(q,t)$

The family of scale-free fractal lattices (networks) $H(q,t)$ ($q \geqslant 2$) investigated in [46] is constructed in an iterative way as shown in Fig. 10. Initially ($t = 0$), $H(q,0)$ is a link connecting two nodes. For $t \geqslant 1$, $H(q,t)$ is obtained from $H(q,t-1)$ by replacing each edge in $H(q,t-1)$ by the connected clusters on the right-hand side of the arrow in Fig. 10. When $q = 2$, $H(q,t)$ is reduced to the (2,2)-flower, a particular case of the (x,y)-flowers ($x \geqslant 1$, $y \geqslant 2$) presented in [47], which may be either fractal or nonfractal.

Most of the topological properties of $H(q,t)$ can be determined exactly [46]. The numbers of nodes and edges in $H(q,t)$ are

$$N_{q,t} = \frac{q(2q)^t + 3q - 2}{2q-1} \tag{A1}$$

and

$$E_{q,t} = (2q)^t, \tag{A2}$$

respectively. For large networks, the average node degree is $4 - 2/q$. The network family is scale-free with the exponent γ of degree distribution equal to $2 + \ln 2/\ln q$. In addition, the networks are fractal with the fractal dimension being $1 + \ln q/\ln 2$.

After introducing the construction and features of the networks $H(q,t)$, we next derive explicitly the number of spanning trees in $H(q,t)$ by using a method similar to that applied in the text.

We define P_t as the number of spanning trees in $H(q,t)$, and let Q_t denote the number of spanning subgraphs consisting of two trees such that the two initial nodes in $H(q,0)$ belong to separated trees. Since the networks $H(q,t)$ have a similar structure, using a derivation process similar to that described in the text, we can obtain the following exact expressions governing the recurrence relations between P_t and Q_t:

$$P_{t+1} = q(P_t)^2(2P_tQ_t)^{q-1}, \tag{A3}$$

$$Q_{t+1} = (2P_tQ_t)^q. \tag{A4}$$

Then we have

$$\frac{P_{t+1}}{Q_{t+1}} = \frac{q}{2}\frac{P_t}{Q_t}, \tag{A5}$$

which together with the initial condition $\frac{P_0}{Q_0} = 1$ leads to

$$\frac{P_t}{Q_t} = \left(\frac{q}{2}\right)^t, \tag{A6}$$

namely,

$$P_t = \left(\frac{q}{2}\right)^t Q_t. \tag{A7}$$

Inserting Eq. (A7) into Eq. (A4), we obtain

$$Q_{t+1} = 2^q\left(\frac{q}{2}\right)^{qt}(Q_t)^{2q}. \tag{A8}$$

Considering $Q_0 = 1$, Eq. (A8) is solved to yield

$$Q_t = 2^{\frac{2q-tq-2q^2+2tq^2-2^{t+1}q^{t+1}+2^{t+1}q^{t+2}}{(2q-1)^2}}\, q^{\frac{-q+tq-2tq^2+2^tq^{t+1}}{(2q-1)^2}}. \tag{A9}$$

Thus, according to Eq. (A6) we have the following expression for the number of spanning trees in $H(q,t)$:

$$P_t = 2^{\frac{2q-tq-2q^2+2tq^2-2^{t+1}q^{t+1}+2^{t+1}q^{t+2}}{(2q-1)^2}-t}\, q^{\frac{-q+tq-2tq^2+2^tq^{t+1}}{(2q-1)^2}+t}. \tag{A10}$$

Then, the entropy of spanning trees for $H(q,t)$ is

$$E_{H(q,t)} = \lim_{t\to\infty}\frac{\ln P_t}{N_{q,t}} = \frac{2q-2}{2q-1}\ln 2 + \frac{1}{2q-1}\ln q$$
$$= \ln 2 + \frac{1}{2q-1}(\ln q - \ln 2). \tag{A11}$$

In the specific case of $q = 1$, the average degree, the exponent of degree distribution, and the entropy of spanning trees in this particular network are 3, 3, and $\ln 2$, repressively. The entropy is larger than that corresponding to the nonfractal scale-free Koch network [51] with the same average degree and exponent of degree distribution, the entropy of which is $\frac{1}{2}\ln 3$ [52]. This again shows that the fractality in scale-free networks significantly increases the number of spanning trees in the networks.

Finally, it deserves to be mentioned that the decimation method for enumerating spanning trees in self-similar networks is universal. In addition to the fractal scale-free network family $H(q,t)$ considered above, it also applies to nonfractal networks. For example, we have made use of this technique to compute the number of spanning trees in the $(1,y)$-flowers [47] and recovered the result previously obtained in [25], which is actually the (1,2)-flower. Considering that the main topic of this paper is focused on fractal scale-free networks, here we omit the detailed derivation process.

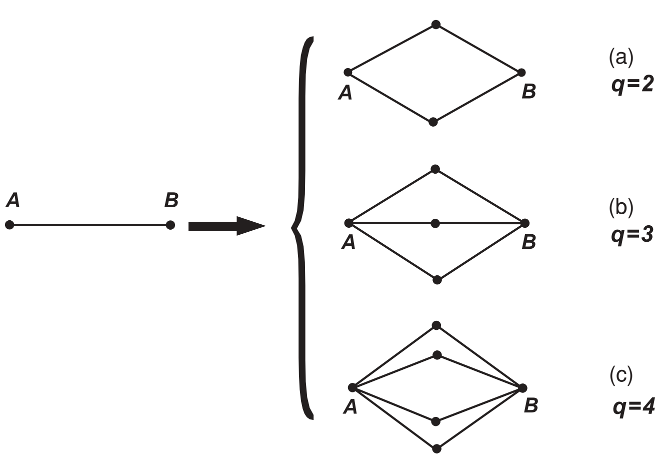

FIG. 10. Iterative construction method of the hierarchical lattices for $q = 2$, $q = 3$, and $q = 4$, respectively.

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